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4
Appendix to Optimal Fiscal and Monetary Policy under Sectorial Heterogeneity

4.1
Appendix A - The Firms' Problem

Noting that $\theta > 1$, FOC from firms' optimization problem is given by:

$$E_t \sum_{j=t}^{\infty} \alpha_k^{j-t} \Theta_{t,j} \frac{\partial \Psi_j \left(p_{k,t} \left(z \right), . \right)}{\partial p_{k,t} \left(z \right)} = 0.$$

Taking derivatives and isolating terms $p_{k,t}(z)/P_{k,t}$, yields:

$$\frac{p_{k,t}(z)}{P_{k,t}}^{1+\theta\nu} = \frac{\frac{\theta\lambda}{\theta-1} m_k^{-\nu} E_t \sum_{j=t}^{\infty} \left(\alpha_k \beta\right)^{j-t} \mu_{k,j}^{w} \frac{P_{k,j}}{P_{k,t}}^{\theta(\nu+1)} \frac{Y_{k,j}}{a_{k,j}}^{\nu+1}}{E_t \sum_{j=t}^{\infty} \left(\alpha_k \beta\right)^{j-t} \left(1 - \tau_{k,j}\right) C_j^{-\sigma^{-1}} \frac{P_{k,j}}{P_{k,t}}^{\theta-1} p_{k,j} Y_{k,j}} \tag{4-1}$$

4.2
Appendix B - Steady State

The government budget constraint in steady state is given by:

$$(1 - \beta)\bar{b}^* = \sum_{k=1}^{K} \bar{\tau}_k \bar{Y}_k - \bar{G}. \tag{4-2}$$

Assuming debt and government expenses are non-zero in steady state imply $\bar{\tau}_k > 0$, for some k. Also, given $p_{k,-1} = 1$ and zero inflation, all k. It is clear that , $\bar{Y}_k = m_k \bar{Y}$, which imply (5-3) becomes

$$(1 - \beta)\bar{b}^* + \bar{G} = \bar{\tau}\bar{Y}, \tag{4-3}$$

where $\bar{\tau} = \sum_{k=1}^{K} m_k \bar{\tau}_k$. From the firms maximization problem:

$$\bar{K}_k = \bar{F}_k.$$

Using definitions for both terms:

$$\frac{\theta\lambda}{\theta - 1} \bar{\mu}_k^w m_k^{-\nu} \bar{Y}_k^{\nu} = (1 - \bar{\tau}_k) \left(\bar{C}\right)^{-\sigma}, \tag{4-4}$$

which implies that sectorial tax rate is given by

$$\bar{\tau}_k = 1 - \frac{\theta\lambda}{\theta - 1}\bar{\mu}_k^w \left(\bar{C}\right)^{\sigma} \bar{Y}^{\nu}, \tag{4-5}$$

which only depends of aggregate variables and sector specific parameter $\bar{\mu}_k^w$. We assume that steady state wage markup is the same across sectors, that is $\bar{\mu}_k^w = \bar{\mu}^w$, all k. In this case, steady state distortive tax rates are the same across sectors, that is

$$\bar{\tau}_k = \bar{\tau}, \forall k \tag{4-6}$$

Once one considers an always-possible normalization $\bar{Y} = 1$, the only restriction made is that the level of consumption over GDP should not be too high in order to tax rates to be positive. Equations

$$\frac{\theta\lambda}{\theta - 1}\bar{\mu}^w \bar{Y}^{\nu} = (1 - \bar{\tau})\left(\bar{Y} - \bar{G}\right)^{-\sigma} \tag{4-7}$$

and (4-3) define the aggregate output level in steady state as well as the aggregate tax rate, as in Benigno and Woodford (2003).

Define the set of commitments $X_t = \{K_{k,t},\ F_{k,t},\ W_t\}$, all k, and let X_0 be the set of initial commitments that make policy optimal form a timeless perspective. The centralized policy maker chooses a sequence of $\mathcal{X}_t = \{\Pi_t,\ \Pi_{k,t},\ Y_t,\ Y_{k,t},\ F_{k,t},\ K_{k,t},\ W_t,\ \Delta_{k,t},\ \tau_{k,t},\ b_t^*,\ p_{k,t}\}$, all k, for $t \geq t_0$ in order to maximize the representative consumer's utility subject to the constraints given in the main text and taking as given the initial commitments X_0 and the initial conditions $\mathcal{I}_{-1} = \{b_{-1}^*,\ \Delta_{k,-1},\ p_{k,-1}\}$ for every k and $t \geq t_0$. In order to impose constant commitments $X_0 = \bar{X}$ we consider additional restrictions such as the first order conditions for the problem in $t = t_0$ are equivalent to the first order conditions for a generic $t > 0$. Consider the set of Lagrange multipliers corresponding to equations in the main text. In order to complete the proof, we need to show that first order conditions for the indicated steady state are satisfied for time-invariant Lagrange multipliers. After taking FOCs from maximization problem, it is possible to show that the system of steady state variables and time-invariant multipliers is just-identified. Complete proof is given in the Technical Appendix.

4.3 Appendix C - Second Order Approximation to Utility Function

4.3.1 Second Order Approximation of Utility Function

We start with a second order Taylor expansion of the representative consumer's welfare function where ξ_t refers to the full vector of random disturbances, as in Benigno and Woodford (2003). We start by working with $u(Y_t, \xi_t)$. Define hereafter, for any variable X_t, $\tilde{X}_t \equiv \frac{X_t - \bar{X}}{\bar{X}}$, $\hat{X}_t \equiv \log \frac{X_t}{\bar{X}}$

It is know that the following relation holds up to second order:

$$\tilde{X}_t \simeq \hat{X}_t + \frac{1}{2}\hat{X}_t^2. \tag{4-8}$$

Given the functional form assumed, we have:

$$u(Y_t, \xi_t) = \bar{C}^{-\sigma}\bar{Y}[\tilde{Y}_t - \frac{\sigma}{2}\frac{\bar{Y}}{\bar{C}}\tilde{Y}_t^2 + \sigma\frac{\bar{Y}}{\bar{C}}\tilde{Y}_t\tilde{G}_t] + tips + O_p^3, \tag{4-9}$$

where $\tilde{G}_t$ represents the absolute deviation *over GDP*. Defining $s_C = \frac{\bar{C}}{\bar{Y}}$, yields

$$u(Y_t, \xi_t) = \bar{C}^{-\sigma}\bar{Y}[\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2(1 - \sigma s_C^{-1}) + \sigma s_C^{-1}\hat{Y}_t\hat{G}_t] + tips + O_p^3. \tag{4-10}$$

A second order Taylor expansion of $v(Y_{k,t}, \xi_t)\Delta_{k.t}$ around steady state values yields

$$\begin{aligned} v(Y_{k,t}, \xi_t)\Delta_{k.t} \;=\; & v\left(\bar{Y}_k, \bar{\xi}\right)\tilde{\Delta}_{k.t} + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t} + \frac{1}{2}\hat{Y}_{k,t}^2) + \\ & + \frac{1}{2}v_{Y_kY_k}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k^2(\hat{Y}_{k,t}^2) + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t})\tilde{\Delta}_{k.t} + \\ & + v_{Y_k\xi}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t}\hat{a}_{k,t}) + v_{\xi}\left(\bar{Y}_k, \bar{\xi}\right)\tilde{\Delta}_{k.t}(\hat{a}_{k,t}) + \\ & + tips + O_p^3. \end{aligned} \tag{4-11}$$

Using the definition for $\Delta_{k,t}$ one can show that $\tilde{\Delta}_{k,t}$ is a term of second order. In this sense, interactions between $\tilde{\Delta}_{k,t}$ and $\hat{a}_{k,t}$ or $\tilde{\Delta}_{k,t}$ and $\hat{Y}_{k,t}$ can be ignored up to second order. Hence, expression (5-9) simplifies to

$$v(Y_{k,t}, \xi_t)\Delta_{k.t} = \lambda\left[\frac{\bar{Y}_{k,t}}{m_k}\right]^{1+\nu}\{\frac{\hat{\Delta}_{k.t}}{1+\nu} + \hat{Y}_{k,t} + \frac{1+\nu}{2}\hat{Y}_{k,t}^2 - (1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}\} + tips + O_p^3, \tag{4-12}$$

once one notice that $\hat{\Delta}_{k,t}^2$ is of higher order than O_p^2. Using a second order Taylor expansion over the law of motion for sectorial price dispersion given by

$$\Delta_{k.t} = \alpha_k\Pi_{k,t}^{\theta(1+\nu)}\Delta_{k.t-1} + (1-\alpha_k)\left(\frac{1 - \alpha_k\Pi_{k,t}^{\theta-1}}{(1-\alpha_k)}\right)^{\frac{\theta(1+\nu)}{\theta-1}} \tag{4-13}$$

yields

$$\hat{\Delta}_{k.t} = \alpha_k\hat{\Delta}_{k.t-1} + \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\pi_{k,t}^2 + O_p^3, \tag{4-14}$$

once we used the relation $\hat{\Pi}_{k,t} = \pi_{k,t} + \frac{1}{2}\pi_{k,t}^2$, where $\pi_{k,t}$ is the percent variation of sectorial price level $\pi_{k,t} = \log P_{k,t}/P_{k,t-1}$. Iterating backwards yields

$$\hat{\Delta}_{k.t} = \alpha_k^{t-1}\hat{\Delta}_{k.-1} + \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\sum_{j=0}^{t}\alpha_k^{t-j}\pi_{k,j}^2 + O_p^3. \quad (4\text{-}15)$$

Here we consider the sectorial price dispersion in the remote past as a "term independent of policy". Further considering that it is possible to change positions of sums over t and k on (5-10), and re-ordering the terms:

$$\sum_{t=0}^{\infty}\beta^t\hat{\Delta}_{k.t} = \frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)(1-\alpha_k\beta)}\theta(1+\nu)(1+\theta\nu)\sum_{t=0}^{\infty}\beta^t\pi_{k,t}^2 + tips + O_p^3. \quad (4\text{-}16)$$

Substituting (5-11) over (5-10) yields

$$\begin{aligned} \upsilon\left(Y_{k,t},\xi_t\right)\Delta_{k.t} &= \lambda\left[\frac{\bar{Y}_{k,t}}{m_k}\right]^{1+\nu}\{\frac{1}{2}\frac{\alpha_k\theta(1+\theta\nu)}{(1-\alpha_k)(1-\alpha_k\beta)}\pi_{k,t}^2 + \hat{Y}_{k,t} + \\ &\quad + \frac{1+\nu}{2}\hat{Y}_{k,t}^2 - (1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}\} + tips + O_p^3. \end{aligned}$$

This way, we can approximate the representative consumer utility up to second order by the following expression:

$$\begin{aligned} U_{t_0} &= \Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\hat{Y}_t + \frac{(1-\tilde{\sigma})}{2}\hat{Y}_t^2 + \tilde{\sigma}\hat{Y}_t\hat{G}_t + \\ &\quad - \sum_{k=1}^{K} m_k(1-\Phi)[\frac{\theta}{\kappa_k}\frac{\pi_{k,t}^2}{2} + \hat{Y}_{k,t} + \frac{1+\nu}{2}\hat{Y}_{k,t}^2 + \\ &\quad -(1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}]\} + tips + O_p^3, \end{aligned} \quad (4\text{-}17)$$

where

$$\Omega \equiv \bar{C}^{-\sigma}\bar{Y}, \quad (4\text{-}18)$$

$$\kappa_k \equiv \frac{(1-\alpha_k)(1-\alpha_k\beta)}{(1+\theta\nu)\alpha_k}, \quad (4\text{-}19)$$

$$\tilde{\sigma} \equiv \sigma s_C^{-1} \quad (4\text{-}20)$$

and

$$(1-\Phi) \equiv \frac{\theta-1}{\theta}\frac{(1-\bar{\tau})}{\bar{\mu}^w}. \quad (4\text{-}21)$$

4.3.2
Second Order Approximation to AS Equation

The starting point is the expression for the sectorial non-linear Phillips Curve, given by:

$$\left(\frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{(1-\alpha_k)}\right)^{\frac{1+\theta\nu}{\theta-1}} = \frac{F_{k,t}}{K_{k,t}}. \tag{4-22}$$

We define $V_{k,t}$ as

$$V_{k,t} = \frac{1-\alpha_k \Pi_{k,t}^{\theta-1}}{(1-\alpha_k)}. \tag{4-23}$$

Using a second order Taylor expansion on $\hat{V}_{k,t}$:

$$\hat{V}_{k.t} = -\frac{\alpha_k(\theta-1)}{(1-\alpha_k)}\left[\pi_{k,t} + \frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2\right] + O_p^3. \tag{4-24}$$

Considering the expression for $K_{k,t}$ define $\Pi_{k,t,s} = P_{k,s}/P_{k,t}$, where $s \geq t$ is some date in the future and $P_{k,t}$ the aggregate price level in sector k in period t. We use a second order Taylor expansion:

$$\tilde{K}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{k}_{k,j} + \frac{1}{2}\hat{k}_{k,j}^2\} + O_p^3, \tag{4-25}$$

where the term $\hat{k}_{k,t}$ can be defined as $\hat{k}_{k,j} = \theta(1+\nu)\pi_{k,t,j} + (1+\nu)\hat{Y}_{k,j} - (1+\nu)\hat{a}_{k,j}$.

Taking the expression in the text for $F_{k,t}$ given by (6-22), we define the net revenue factor as $\Gamma_{k,t} \equiv 1-\tau_{k,t}$, and taking second-order Taylor expansion:

$$\tilde{F}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{f}_{k,j} + \frac{1}{2}\hat{f}_{k,j}^2\} + O_p^3. \tag{4-26}$$

where we define $\hat{f}_{k,j} = \hat{\Gamma}_{k,j} - \sigma\hat{C}_j + \hat{Y}_{k,j} + \hat{p}_{k,j} + (\theta-1)\pi_{k,t,j}$.

Using $\tilde{F}_{k,t}$, $\tilde{K}_{k,t}$, as well as $\hat{V}_{k.t}$, $\hat{F}_{k,t}$ and $\hat{K}_{k,t}$,after some algebra, we get:

$$\begin{aligned}\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t} &= (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{[z_{k,j} - (1+\theta\nu)\pi_{k,t,j}] + \\ &+\frac{1}{2}[z_{k,j} - (1+\theta\nu)\pi_{k,t,j}][\hat{X}_{k,j} + [(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}]\} \\ -\frac{1}{2}\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t}&(1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{X}_{k,j} + [(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\} + O_p^3,\end{aligned}$$

where

$$\hat{X}_{k,j} \equiv \hat{\Gamma}_{k,j} - \sigma\hat{C}_j + (2+\nu)\hat{Y}_{k,j} + \hat{p}_{k,j} - (1+\nu)\hat{a}_{k,j} + \hat{\mu}_{k,t}^w, \tag{4-27}$$

$$\hat{f}_{k,j} - \hat{k}_{k,j} = z_{k,j} - (1+\theta\nu)\pi_{k,t,j} \tag{4-28}$$

and

$$z_{k,j} = \hat{\Gamma}_{k,j} - \sigma\hat{C}_j - \nu\hat{Y}_{k,j} + \hat{p}_{k,j} + (1+\nu)\hat{a}_{k,j} - \hat{\mu}^w_{k,t}. \tag{4-29}$$

Define

$$Z_{k,t} \equiv E_t \sum_{j=t}^{\infty} (\alpha_k\beta)^{j-t} \left\{ \left[\hat{X}_{k,j} + [(\theta-1) + \theta(1+\nu)]\pi_{k,t,j} \right] \right\} \tag{4-30}$$

We can replace in the expression above and after some algebra we get:

$$\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\hat{V}_{k,t}(\pi_{k,t+1}) = (\pi_{k,t+1})E_t \sum_{j=t+1}^{\infty} (\alpha_k\beta)^{j-t-1} \{z_{k,j} - (1+\theta\nu)(\pi_{k,t,j})\} + O_p^3 \tag{4-31}$$

We can use the definition for $\hat{V}_{k,t}$ and replace above, also ignoring the terms O_p^3 or of higher order:

$$\begin{aligned}
-\kappa_k^{-1}\left[\pi_{k,t} + \frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2 - \alpha_k\beta E_t\pi_{k,t+1} - \frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\alpha_k\beta E_t\pi_{k,t+1}^2\right] = \\
z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t} - (1+\theta\nu)\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi_{k,t+1} + \\
-\frac{1}{2}[(\theta-1)+\theta(1+\nu)]\frac{\beta}{\kappa_k}E_t\pi_{k,t+1}^2 + \\
-\frac{1}{2}(1+\theta\nu)(\alpha_k\beta)E_t[\pi_{k,t+1}Z_{k,t+1}] + \\
+\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}[\pi_{k,t}Z_{k,t} - \alpha_k\beta E_t[\pi_{k,t+1}Z_{k,t+1}]] + O_p^3,
\end{aligned}$$

where we have defined κ_k elsewhere.

Further simplification yields

$$\begin{aligned}
-\kappa_k^{-1}\pi_{k,t} - \frac{1}{2}\kappa_k^{-1}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2 - \frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\pi_{k,t}Z_{k,t} \\
= z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t} - \kappa_k^{-1}\beta E_t\pi_{k,t+1} \\
-\frac{1}{2}\kappa_k^{-1}\left\{\frac{(\theta-1)}{(1-\alpha_k)} + \theta(1+\nu)\right\}\beta E_t\pi_{k,t+1}^2 \\
-\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\beta E_t[\pi_{k,t+1}Z_{k,t+1}] + O_p^3.
\end{aligned}$$

Multiplying both sides for $-\kappa_k$ allow us to write above expression as

$$\mathcal{V}_{k,t} = -\kappa_k\{z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t}\} + \frac{\theta(1+\nu)}{2}\pi^2_{k,t} + \beta E_t\mathcal{V}_{k,t+1} + O^3_p, \quad (4\text{-}32)$$

where:

$$\mathcal{V}_{k,t} = \pi_{k,t} + \frac{1}{2}\{\frac{(\theta-1)}{(1-\alpha_k)} + \theta(1+\nu)\}\pi^2_{k,t} + \frac{1}{2}\frac{\kappa_k\alpha_k}{(1-\alpha_k)}[\pi_{k,t}Z_{k,t}]. \quad (4\text{-}33)$$

A second order Taylor expansion of $\log(1-\tau_{k,t})$ yields

$$\log(1-\tau_{k,t}) = \log(1-\bar{\tau}) - \frac{\bar{\tau}}{1-\bar{\tau}}\tilde{\tau}_{k,t} - \frac{1}{2}\frac{\bar{\tau}^2}{(1-\bar{\tau})^2}\tilde{\tau}^2_{k,t} + O^3_p,$$

which can be recast as

$$\hat{\Gamma}_{k,t} = -\delta\hat{\tau}_{k,t} - \frac{\delta}{(1-\bar{\tau})}\frac{1}{2}\hat{\tau}^2_{k,t} + O^3_p.$$

Log-approximation on consumption as a function of aggregate output and government expenses yields:

$$\hat{C}_t = s_C^{-1}\hat{Y}_t - s_C^{-1}\hat{G}_t + \frac{1}{2}s_C^{-1}(1-s_C^{-1})\hat{Y}_t^2 - \frac{1}{2}s_C^{-1}(1+s_C^{-1})\hat{G}_t^2 + s_C^{-2}\hat{Y}_t\hat{G}_t + O^3_p \quad (4\text{-}34)$$

Using both results, one can be generally express (5-26) as

$$\mathcal{V}_{k,t} = E_{t_0}\sum_{j=t}^{\infty}\beta^{j-t}\{-\kappa_k[z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t}] + \frac{\theta(1+\nu)}{2}\pi^2_{k,t}\} + tips + O^3_p. \quad (4\text{-}35)$$

One could finally note that a first order approximation to (5-29) yields the known Phillips Curve of the form:

$$\begin{aligned}\pi_{k,t} \quad = \quad & \kappa_k\{(\tilde{\sigma}-\eta^{-1})\hat{Y}_t + (\nu+\eta^{-1})\hat{Y}_{k,t} + \delta\hat{\tau}_{k,t} \\ & -\tilde{\sigma}\hat{G}_t - (1+\nu)\hat{a}_{k,t} + \hat{\mu}^w_{k,t}\} + \beta E_t\pi_{k,t+1} + O^2_p.\end{aligned} \quad (4\text{-}36)$$

4.3.3
Second Order Approximation to the Budget Constraint

We approximate the intertemporal government budget restriction by a second order Taylor expansion. Taking the definitions of the intertemporal government budget constraint and primary surplus and making a second-order approximation, we get:

$$\tilde{W}_t = (1-\beta)E_t \sum_{j=t}^{\infty} \beta^{j-t}\{-\sigma\tilde{C}_t + \tilde{s}_t + \frac{1}{2}\sigma(\sigma+1)\tilde{C}_t^2 - \sigma\tilde{C}_t\tilde{s}_t\} + O_p^3. \quad (4\text{-}37)$$

It is also easy to show that $\hat{W}_t = \hat{b}^*_{t-1} - \sigma\hat{C}_t - \pi_t$ and $\tilde{W} = \hat{W} + \frac{1}{2}\hat{W} + O_p^3$. Then, we can re-write $\tilde{W}_t$ as:

$$\tilde{W}_t = \hat{b}^*_{t-1} - \sigma\hat{C}_t - \pi_t + \frac{1}{2}(\hat{b}^*_{t-1} - \sigma\hat{C}_t - \pi_t)^2 + O_p^3. \quad (4\text{-}38)$$

The approximation to the primary surplus is

$$s_d\tilde{s}_t = \sum_{k=1}^{K} m_k\bar{\tau}[(\hat{\tau}_k + \hat{p}_{k,t} + \hat{Y}_{k,t}) + \frac{1}{2}(\hat{\tau}_k + \hat{p}_{k,t} + \hat{Y}_{k,t})^2] - \hat{G}_t - \frac{1}{2}\hat{G}_t^2 + O_p^3, \quad (4\text{-}39)$$

where $s_d \equiv \frac{\bar{s}}{\bar{Y}}$ and $\bar{s} = \sum_{k=1}^{K} \bar{\tau}\bar{Y}_k - \bar{G} = \bar{\tau}\bar{Y} - \bar{G}$.

Hence, the second order approximation for the intertemporal budget constraint can be obtained from the above expressions. One can notice that a first order approximation yields:

$$\begin{aligned} \hat{b}^*_{t-1} - \tilde{\sigma}(\hat{Y}_t - \hat{G}_t) - \pi_t = \\ (1-\beta)E_t \sum_{j=t}^{\infty} \beta^{j-t}\{s_d^{-1} \sum_{k=1}^{K} m_k\bar{\tau}[\hat{\tau}_k + \hat{p}_{k,t} + \hat{Y}_{k,t}] + \\ +(\tilde{\sigma} - s_d^{-1})\hat{G}_t - \tilde{\sigma}\hat{Y}_t\} + tips + O_p^2 \end{aligned} \quad (4\text{-}40)$$

where $\hat{p}_{k,t}$ is a function of sectorial and overall outputs.

4.3.4
Aggregate and Sectorial Output Relation

Sectorial demand expressed is $p_{k,t}^{\eta} = m_k Y_t / Y_{k,t}$ and log-linearized as

$$\hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}), \quad (4\text{-}41)$$

which establishes an exact (inverse) relation between sector relative price and sector relative product. Also, using $p_{k,t} = \frac{\Pi_{k,t}}{\Pi_t} p_{k,t-1}$ and $\Pi_t^{1-\eta} \equiv \sum_{k=1}^{K} m_k \left(\Pi_{k,t} p_{k,t-1}\right)^{1-\eta}$ one gets

$$Y_t^{(\eta-1)/\eta} = \sum_{k=1}^{K} m_k^{1/\eta} Y_{k,t}^{(\eta-1)/\eta}, \quad (4\text{-}42)$$

which relates aggregate and sectorial outputs. Log-linearization of (5-32) yields

$$\hat{Y}_t + \frac{1}{2}(1-\eta^{-1})\hat{Y}_t^2 = \sum_{k=1}^{K} m_k \hat{Y}_{k,t} + \frac{1}{2}(1-\eta^{-1}) \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 + O_p^3. \quad (4\text{-}43)$$

4.4
Appendix D - Elimination of Linear Terms

4.4.1
Matrix Notation

We invite the reader to check out the complete version is available in the Technical Appendix. We start by defining

$$x_t' = \left[\begin{array}{ccccccccc} \hat{Y}_t & \hat{Y}_{1,t} & \ldots & \hat{Y}_{K,t} & \pi_{1,t} & \ldots & \pi_{K,t} & \hat{\tau}_{1,t} & \ldots & \hat{\tau}_{K,t} \end{array}\right], \quad (4\text{-}44)$$

$$\xi_t' = \left[\begin{array}{ccccccc} \hat{G}_t & \hat{a}_{1,t} & \ldots & \hat{a}_{K,t} & \hat{\mu}_{1,t}^w & \ldots & \hat{\mu}_{K,t}^w \end{array}\right]. \quad (4\text{-}45)$$

For notational convenience, we also define the following terms: $\upsilon \equiv 1+\nu$, $\omega_\eta \equiv 1-\eta^{-1}$, $\chi \equiv \nu+\eta^{-1}$, $\tilde{\sigma} \equiv \sigma s_C^{-1}$, $\varsigma \equiv \tilde{\sigma}-\eta^{-1}$, $\delta \equiv \frac{\bar{\tau}}{1-\bar{\tau}}$ and $-\omega_C \equiv 1-s_C^{-1}$.

Using the definitions above, expression (5-12) can be written in matrix notation as

$$U_{t_0} = \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{A_x' x_t - \frac{1}{2} x_t' A_{xx} x_t - x_t' A_\xi \xi_t\} + tips + O_p^3 \quad (4\text{-}46)$$

where A_x, A_{xx}, and A_ξ are, respectively, $(3K+1)\times 1$, $(3K+1)\times(3K+1)$ and $(3K+1)\times(2K+1)$ matrices.

The Sectorial Phillips Curve expressed in (5-29) can also be written in matrix notation. We start by substituting expressions for $\hat{p}_{k,t}$ into definitions for $z_{k,t}$ and $\hat{X}_{k,t}$, underlined in (5-23) and (5-22). Our aim is to separate quadratic and linear terms. Quadratic and linear terms of random disturbances are placed into *tips*. After some manipulation one obtains:

$$\mathcal{V}_{k,t_0} = E_{t_0} \sum_{j=t_0}^{\infty} \beta^{j-t_0} \{C_{x,k}' x_t + \frac{1}{2} x_t' C_{xx,k} x_t + x_t' C_{\xi,k} \xi_t\} + tips + O_p^3 \quad (4\text{-}47)$$

for a generic sector k. As in (5-44), matrices $C_{x,k}$, $C_{xx,k}$, and $C_{\xi,k}$ have, respectively, dimension $(3K+1)\times 1$, $(3K+1)\times(3K+1)$ and $(3K+1)\times(2K+1)$.

The government budget constraint can also be simplified in matrix notation. Taking expression given in (4-37), we eliminate references for $\hat{p}_{k,t}$, and replace $\hat{C}_t$ and $\tilde{s}_t$ for their expressions in terms of endogenous variables x_t and exogenous processes ξ_t. Grouping linear and quadratic terms, yields:

$$\tilde{\mathcal{W}}_{t_0} = (1-\beta)E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}\{B_x' x_t + \frac{1}{2}x_t' B_{xx} x_t + x_t' B_\xi \xi_t\} + tips + O_p^3 \quad (4\text{-}48)$$

where, as in (5-44) and (5-48), matrices B_x, B_{xx}, and B_ξ are, respectively, of dimensions $(3K+1)\times 1$, $(3K+1)\times(3K+1)$ and $(3K+1)\times(K+1)$.

Finally, (4-43) can be expressed in matrix notation as

$$0 = \sum_{j=t}^{\infty}\beta^{j-t}\{H_x' x_t + \frac{1}{2}x_t' H_{xx} x_t\} + O_p^3 \quad (4\text{-}49)$$

where we have used the fact that the definition for aggregate output in terms of its sectorial counterparts expressed in (4-43) is valid at all dates. Matrices H_x and H_{xx} have, respectively, dimension $(3K+1)\times 1$ and $(3K+1)\times(3K+1)$.

4.4.2 Elimination of Linear Terms

In order to eliminate linear terms in (5-44), we need to find a set a multipliers ϑ_C^1, ..., ϑ_C^K, ϑ_B, ϑ_H, such as

$$\vartheta_C^1 C_x^{1\prime} + ... + \vartheta_C^K C_x^{K\prime} + \vartheta_B B_x' + \vartheta_H H_x' = A_x' \quad (4\text{-}50)$$

By solving the linear system of equations, one gets the following set of solution: $\vartheta_B = -\frac{\Phi}{\Upsilon}, \vartheta_H = 1 - \Xi\frac{\Phi}{\Upsilon}$ and, for every k, $\vartheta_C^k = \frac{m_k(1-\bar{\tau})}{\kappa_k}\frac{\Phi}{\Upsilon}$, where we have used the fact that $\bar{\tau} = \bar{\tau}_k$, all k, and defined: $\Upsilon \equiv (\varsigma + \chi)(1-\bar{\tau}) + \tilde{\sigma}s_d - \bar{\tau}$ and $\Xi \equiv \varsigma(1-\bar{\tau}) + \tilde{\sigma}s_d - \bar{\tau}\eta^{-1}$.

Hence, using relations (5-44), (5-48),(4-48), (5-52), and (5-55) one can write:

$$\begin{aligned} E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}A_x' x_t \quad &= \quad E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}[\sum_{k=1}^{K}\vartheta_C^k C_x^{k\prime} + \vartheta_B B_x' + \vartheta_H H_x']x_t \\ &= -E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}\{\frac{1}{2}x_t' D_{xx} x_t + x_t' D_\xi \xi_t\} + \\ &\sum_{k=1}^{K}\vartheta_C^k \mathcal{V}_{k,t_0} + \frac{\vartheta_B \tilde{W}_{t_0}}{(1-\beta)}, \end{aligned} \quad (4\text{-}51)$$

where $D_{xx} = \sum_{k=1}^{K}\vartheta_C^k C_{xx,k} + \vartheta_B B_{xx} + \vartheta_H H_{xx}$ and $D_\xi = \sum_{k=1}^{K}\vartheta_C^k C_\xi^k + \vartheta_B B_\xi$.

We use this last relations in order to rewrite (5-44)

$$U_{t_0} = -\Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \frac{1}{2} x_t' Q_{xx} x_t + x_t' Q_\xi \xi_t \} + T_{t_0} + tips + O_p^3, \tag{4-52}$$

where

$$T_{t_0} = \Omega \{ \sum_{k=1}^{K} \vartheta_C^k \mathcal{V}_{k,t_0} + \frac{\vartheta_B \tilde{\mathcal{W}}_{t_0}}{(1-\beta)} \}, \tag{4-53}$$

is a vector of predetermined variables. Definitions of Q_{xx} and Q_ξ in terms of parameters of the economy defined in the Technical Appendix. As in Benigno and Woodford (2003) and Ferrero (2005), references to sector tax rates have been eliminated. Only references to sectorial inflation measures, sectorial and aggregate outputs remain, which imply (5-59) can be simplified further by getting rid-off tax rates references and by separating terms referring to sectorial and overall outputs from references to sectorial inflation. Proceeding in such fashion yields

$$U_{t_0} = -\frac{\Omega}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ x_{y,t}' \tilde{Q}_y x_{y,t} + 2 x_{y,t}' \tilde{Q}_\xi \xi_t + x_{\pi,t}' \tilde{Q}_\pi x_{\pi,t} \} + T_{t_0} + tips + O_p^3, \tag{4-54}$$

where $x_{y,t}$ is a $K+1 \times 1$ vector containing only references to aggregate and sectorial outputs measures, and $x_{\pi,t}$ is a $K \times 1$ vector containing only sectorial inflation measures and $\tilde{Q}_y$, $\tilde{Q}_\xi$ and $\tilde{Q}_\pi$ are matrices of coefficients. From (5-63), we now focus on the term:

$$x_{y,t}' \tilde{Q}_y x_{y,t} = q_y Y_t^2 + \sum_{k=1}^{K} m_k q_{y_k} Y_{k,t}^2 + 2 \sum_{k=1}^{K} m_k q_{y,y_k} Y_t Y_{k,t}, \tag{4-55}$$

where q terms are combinations of the parameters of the economy defined in the Technical Appendix. Under the assumption that wage markups is steady state as well as markups over marginal costs are the same across sectors ($\bar{\mu}_k^w = \bar{\mu}^w$ and $\theta_k = \theta$) , q coefficients are all independent of k. We use the following lemmas in order to simplify (5-64) further:

Lemma 17 *The following expression relating sum of sectorial output variances and covariances of sectorial outputs and aggregate output is of third order.*

$$\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3 \tag{4-56}$$

We present the proof in the Technical Appendix that can be downloaded in the Internet.

Lemma 18 *The following expression is, at least, of second order:*

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = O_p^2. \tag{4-57}$$

The proof follows directly from (4-43).

Lemma 19 *The following expression holds:*

$$[\hat{Y}_t - \sum_{k=1}^{K} m_k Y_{k,t}]\hat{G}_t = O_p^3. \tag{4-58}$$

The proof follows from proposition above plus the fact that all exogenous processes are O_p^1.

Lemma 20 *The following expression is of third order:*

$$\hat{Y}_t^2 - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3. \tag{4-59}$$

The proof follows from the Technical Appendix.

From (5-64), and using (5-80) and (4-59) one gets:

$$x'_{y,t}\tilde{Q}_y x_{y,t} = \lambda_{y_k} \sum_{k=1}^{K} m_k Y_{k,t}^2 + O_p^3, \tag{4-60}$$

where

$$\lambda_{y_k} = q_{y_k} + 2q_{y,y_k} + q_y. \tag{4-61}$$

From (5-63), we focus on the term:

$$x'_{y,t}\tilde{Q}_\xi \xi_t = q_{yG}\hat{Y}_t\hat{G}_t + q_{y_kG}\sum_{k=1}^{K} m_k Y_{k,t}\hat{G}_t + \sum_{k=1}^{K} m_k \hat{Y}_{k,t}[q_{y_k a_k}\hat{a}_{k,t} + q_{y_k\mu_k}\hat{\mu}_{k,t}]. \tag{4-62}$$

where q- coefficients are defined in the Technical Appendix. Using (4-58) we get:

$$x'_{y,t}\tilde{Q}_\xi \xi_t = \sum_{k=1}^{K} m_k Y_{k,t}[q'_{y_kG}\hat{G}_t + q_{y_k a_k}\hat{a}_{k,t} + q_{y_k\mu_k}\hat{\mu}_{k,t}] + O_p^3, \tag{4-63}$$

where

$$q'_{y_kG} = q_{yG} + q_{y_kG}.$$

Replacing (5-79) and (5-83) over (5-63) yields the expression for the second order approximation for the utility function:

$$U_{t_0} = -\frac{\Omega}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \lambda_{y_k} \sum_{k=1}^{K} m_k y_{k,t}^2 + \sum_{k=1}^{K} m_k \lambda_{k,\pi} \pi_{k,t}^2 \} + T_{t_0} + tips + O_p^3, \quad (4\text{-}64)$$

where

$$y_{k,t} = \hat{Y}_{k,t} - \hat{Y}_{k,t}^* \quad (4\text{-}65)$$

and

$$-\hat{Y}_{k,t}^* = \lambda_{y_k}^{-1} [(q_{yG} + q_{y_k G}) \hat{G}_t + q_{y_k a_k} \hat{a}_{k,t} + q_{y_k \mu_k} \hat{\mu}_{k,t}] \quad (4\text{-}66)$$

all k, and, most importantly, λ_{y_k} and $\lambda_{k,\pi}$ provide the weigh of each of these terms in the welfare-based criteria. Besides the complete definition of such terms, the Technical Appendix addresses the conditions for concavity.

4.5 Appendix E - Definitions of Homogeneous Case

The typical policy restrictions are given by a new keynesian Phillips Curve and a government budget constraint, respectively:

$$\pi_t = \kappa[(\tilde{\sigma} + \nu) y_t + \delta(\hat{\tau}_t - \hat{\tau}_t^*)] + \beta E_t \pi_{t+1} + u_t \quad (4\text{-}67)$$

$$\hat{b}_{t-1}^* - \tilde{\sigma} y_t - \pi_t = (1-\beta)[(\hat{\tau}_t - \hat{\tau}_t^*) + b_y y_t] + \beta E_t [\hat{b}_t^* - \tilde{\sigma} y_{t+1} - \pi_{t+1}] + \zeta_t \quad (4\text{-}68)$$

where $\hat{b}_t^*$ is defined as the debt at maturity at date t, or $\hat{b}_t^* = \hat{b}_t + \hat{R}_t$, where $\hat{R}_t$ is the gross interest rate. Other variables are defined according to the notation of our model given in Section 2: $\tilde{\sigma} \equiv \sigma s_C^{-1}$, $s_C^{-1} \equiv \bar{Y}/\bar{C}$, $b_y \equiv \bar{\tau}\bar{Y}/(\bar{\tau}\bar{Y} - \bar{G}) - \tilde{\sigma}$, and $\delta \equiv \bar{\tau}/(1-\bar{\tau})$. Hat-variables are defined as steady state levels. The shock terms such as ζ_t and u_t are linear functions of aggregate government expenses, productivity and wage markup shocks. Finally, $\hat{\tau}_t^*$ is the *tax rate target,* also defined as a linear combination of exogenous shocks.[1]

Definitions of the coefficient of the optimal targeting rules are given, in terms parameter in the model presented at Section 2, as

$$\omega_\varphi = -\lambda_\pi^{-1} [(1-\beta) s_d^{-1} (1-\bar{\tau}) \kappa^{-1} + 1] \quad (4\text{-}69)$$

$$n_\varphi = -\lambda_y^{-1} \tilde{\sigma} \quad (4\text{-}70)$$

$$m_\varphi = -\lambda_y^{-1} [(s_d^{-1}(1-\bar{\tau}) + b_y)(1-\beta) + \tilde{\sigma}] \quad (4\text{-}71)$$

[1]Details in Benigno and Woodford (2003).

4.6 Appendix F - Log-linear Approximation of Restrictions

4.6.1 Definition of Target Variables

Explicitly using the assumption that sector specific tax rates as well as wage markups in steady state are the same across sectors, we can define the target level of aggregate output using (5-84):

$$-\hat{Y}_t^* = \lambda_{y_k}^{-1}[(q_{yG} + q_{y_kG})\hat{G}_t + q_{y_k a_k}\hat{a}_t + q_{y_k\mu_k}\hat{\mu}_t] \tag{4-72}$$

where q- coefficients are defined in terms of the structural parameters of the economy and $\hat{a}_t$ and $\hat{\mu}_t$ are respectively defined as: $\hat{a}_t = \sum_{k=1}^{K} m_k \hat{a}_{k,t}$ and $\hat{\mu}_t^w = \sum_{k=1}^{K} m_k \hat{\mu}_{k,t}^w$.

4.6.2 Aggregate supply and cost-push disturbance term

Adding and subtracting, respectively, the terms referring to overall and sectorial output targets with the appropriate coefficients yield over first order approximation of AS equation yields

$$\pi_{k,t} = \kappa_k\{(\tilde{\sigma} - \eta^{-1})y_t + (\nu + \eta^{-1})y_{k,t} + \delta(\hat{\tau}_{k,t} - \hat{\tau}_{k,t}^*)\} + \beta E_t \pi_{k,t+1} + u_{k,t}, \tag{4-73}$$

for every k, where the definition for the cost-push term $u_{k,t}$ is a function of sectorial wage markup shocks:

$$u_{k,t} = \kappa_k[1 - (\nu + \eta^{-1})\lambda_{y_k}^{-1} q_{y_k\mu_k}]\hat{\mu}_{k,t}^w \tag{4-74}$$

and

$$\begin{aligned} -\delta\hat{\tau}_{k,t}^* \quad = \quad & -[(\tilde{\sigma} + \nu)\lambda_{y_k}^{-1}(q_{yG} + q_{y_kG}) + \tilde{\sigma}]\hat{G}_t - (\tilde{\sigma} - \eta^{-1})\lambda_{y_k}^{-1} q_{y_k\mu_k}\hat{\mu}_t^w \\ & -(\tilde{\sigma} - \eta^{-1})\lambda_{y_k}^{-1} q_{y_k a_k}\hat{a}_t - [(\nu + \eta^{-1})\lambda_{y_k}^{-1} q_{y_k a_k} + (1 + \nu)]\hat{a}_{k,t}. \end{aligned} \tag{4-75}$$

can be understood as the target level for distortive taxation in sector k and q-coefficients are defined in terms of the structural parameters of the economy. Averaging across sectors allows us to determine the generalized aggregate first order approximation for the AS equation in (1-34), similar to Carvalho (2006).

4.6.3 Budget Constraint and fiscal disturbance term

We start by taking a first order approximation to expression (4-37), yielding

$$\hat{b}^*_{t-1} - \tilde{\sigma}(\hat{Y}_t - \hat{G}_t) - \pi_t = (1-\beta)\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{b_y\hat{Y}_t + \bar{\tau}s_d^{-1}\sum_{k=1}^{K}m_k[\hat{\tau}_k + \omega_\eta\hat{Y}_{k,t}] + b_G\hat{G}_t\}, \tag{4-76}$$

where we have defined for convenience the terms s_d^{-1}, b_y and b_G, respectively, as $s_d^{-1} \equiv \frac{\bar{Y}}{\bar{\tau}\bar{Y}-\bar{G}}$, $b_y \equiv s_d^{-1}\bar{\tau}\eta^{-1} - \tilde{\sigma}$, and $b_G \equiv \tilde{\sigma} - s_d^{-1}$. Expression (4-76) can be written in recursive terms. Using the definition for aggregate output in terms of sectorial outputs and the definitions for target variables given in (5-84) and (4-75), we get:

$$\hat{b}^*_{t-1} - \tilde{b}_y y_t - \pi_t + \zeta_t = (1-\beta)\bar{\tau}s_d^{-1}\sum_{k=1}^{K}m_k(\hat{\tau}_{k,t} - \hat{\tau}^*_{k,t}) + \beta E_t[\hat{b}^*_t - \tilde{\sigma}y_{t+1} - \pi_{t+1}], \tag{4-77}$$

where $\tilde{b}_y \equiv \tilde{\sigma} + (1-\beta)(b_y + \bar{\tau}\omega_\eta s_d^{-1})$ and

$$\zeta_t = \omega_1^G\hat{G}_t + \omega_1^a\hat{a}_t + \omega_1^\mu\hat{\mu}_t^w - \omega_2^G E_t\hat{G}_{t+1} - \omega_2^a E_t\hat{a}_{t+1} - \omega_2^\mu E_t\hat{\mu}^w_{t+1}, \tag{4-78}$$

where ω_1^G, ω_2^G, ω_1^a, ω_2^a, ω_1^μ and ω_2^μ are defined in terms of the structural parameters of the economy.

4.6.4 Aggregate and Sectorial Output Relation

First order approximation to (4-43) combined with the redefinition in terms of deviation from aggregate and sectorial output targets, yields

$$y_t = \sum_{k=1}^{K}m_k y_{k,t}. \tag{4-79}$$

4.7 Appendix G - Optimal Solution with Commitment

For simplicity, define: $\check{\tau}_{k,t} \equiv \hat{\tau}_{k,t} - \hat{\tau}^*_{k,t}$. Setting up the Lagrangian:

$$\max_{\{\pi_t,\pi_{1,t},...,\pi_{1,t},y_t,y_{1,t},...,y_{K,t},\check{\tau}_{1,t},...,\check{\tau}_{K,t},\hat{b}_t^*\}} \frac{1}{2}E_{t_0}\{\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\lambda_{y_k}\sum_{k=1}^{K}m_k y_{k,t}^2+\sum_{k=1}^{K}m_k\lambda_{k,\pi}\pi_{k,t}^2$$
$$+2m_1M_{1,t}^{\pi}\{\pi_{1,t}-\kappa_1[(\tilde{\sigma}-\eta^{-1})y_t+(\nu+\eta^{-1})y_{1,t}+\delta\check{\tau}_{1,t}]-\beta\pi_{1,t+1}-u_{1,t}\}+...$$
$$+2M_t^b\{\hat{b}_{t-1}^*-\tilde{b}_y y_t-\pi_t-(1-\beta)\bar{\tau}s_d^{-1}\sum_{k=1}^{K}m_k\check{\tau}_{k,t}-\beta E_t[\hat{b}_t^*-\tilde{\sigma}y_{t+1}-\pi_{t+1}]+\zeta_t\}+$$
$$+2M_t^y[y_t-\sum_{k=1}^{K}m_k y_{k,t}]+2M_t^{\pi}[\pi_t-\sum_{k=1}^{K}m_k\pi_{k,t}]\}\}$$
$$+2\sum_{k=1}^{K}m_kM_{k,-1}^{\pi}[-\pi_{k,0}]+2M_{-1}^b[\pi_0]+2M_{-1}^b[\tilde{\sigma}y_0]$$

where M_t^x denotes the multiplier of equation referred to variable x and where the last line correspond to the preconditions that allow the problem to be valid for all $t\geq 0$. FOCs with respect to $\pi_{t,k}$, π_t, $\check{\tau}_{k,t}$, y_t, $y_{k,t}$ and b_t^* are, respectively, given by:

$$\lambda_{\pi,k}\pi_{t,k}+M_{k,t}^{\pi}-M_{k,t-1}^{\pi}=M_t^{\pi}, \tag{4-80}$$

$$M_t^{\pi}=M_t^b-M_{t-1}^b, \tag{4-81}$$

$$M_{k,t}^{\pi}=-M_t^b\frac{(1-\bar{\tau})(1-\beta)}{\kappa_k}s_d^{-1}, \tag{4-82}$$

$$-\sum_{k=1}^{K}m_kM_{k,t}^{\pi}\kappa_k(\tilde{\sigma}-\eta^{-1})-M_t^b\tilde{b}_y+M_{t-1}^b\tilde{\sigma}+M_t^y=0, \tag{4-83}$$

$$\lambda_{y_k}y_{k,t}-M_{k,t}^{\pi}[\kappa_k(\nu+\eta^{-1})]-M_t^y=0, \tag{4-84}$$

$$M_t^b=E_tM_{t+1}^b, \tag{4-85}$$

plus the problem's constraints. Substituting (4-81) and (4-82) into (4-80) yields the law of motion to sectorial inflation in terms of debt Lagrange Multiplier M_t^b:

$$\pi_{k,t}=\psi_k^{\pi}(M_t^b-M_{t-1}^b), \tag{4-86}$$

where

$$\psi_k^{\pi}\equiv\lambda_{\pi,k}^{-1}\left[1+\frac{(1-\beta)(1-\bar{\tau})s_d^{-1}}{\kappa_k}\right].$$

From (4-83),

$$M_t^y=\tilde{\Phi}_1M_t^b-\tilde{\Phi}_2M_{t-1}^b, \tag{4-87}$$

where $\tilde{\Phi}_1=\tilde{b}_y-(1-\bar{\tau})(1-\beta)s_d^{-1}(\tilde{\sigma}-\eta^{-1})$ and $\tilde{\Phi}_2=\tilde{\sigma}$. Taking (4-84), replacing for $M_{k,t}^{\pi}$ from (4-82) and isolating for $y_{k,t}$ yields

$$y_{k,t} = \varphi_1 M_t^b - \varphi_2 M_{t-1}^b, \quad (4\text{-}88)$$

where

$$\varphi_1 \equiv \lambda_{y_k}^{-1}[\tilde{\Phi}_1 - (1-\bar{\tau})(1-\beta)s_d^{-1}(\nu + \eta^{-1})],$$

$$\varphi_2 \equiv \lambda_{y_k}^{-1}\tilde{\Phi}_2.$$

Summing up across sectors yields the aggregate output in terms of debt Lagrange Multiplier:

$$y_t = \Sigma_1 M_t^b - \Sigma_2 M_{t-1}^b, \quad (4\text{-}89)$$

where we defined coefficients Σ_1 and Σ_2, respectively as $\Sigma_1 \equiv \varphi_1$ and $\Sigma_2 \equiv \varphi_2$.

Finally, it is relevant to notice that under commitment, optimal solution imply that policy is conducted in such a way that:

$$E_t \pi_{k,t+1} = 0, \quad (4\text{-}90)$$

every k. In order to see this, we take leads in (4-86), apply expectation and use relation (4-85). In its turn, (4-90) for every k imply the same behavior for aggregate inflation, or:

$$E_t \pi_{t+1} = 0. \quad (4\text{-}91)$$

Also, for very k, (4-86) and (4-88) imply

$$\Delta y_{k,t} = \frac{\varphi_1}{\psi_k^\pi}\pi_{k,t} - \frac{\varphi_2}{\psi_k^\pi}\pi_{k,t-1} \quad (4\text{-}92)$$

and the aggregate relation

$$\Delta y_t = \frac{\Sigma_1}{\psi^\pi}\pi_t - \frac{\Sigma_2}{\psi^\pi}\pi_{t-1}, \quad (4\text{-}93)$$

where

$$\psi^\pi \equiv \sum_{k=1}^{K} m_k \psi_k^\pi.$$

4.8
Appendix H - Figures and Tables

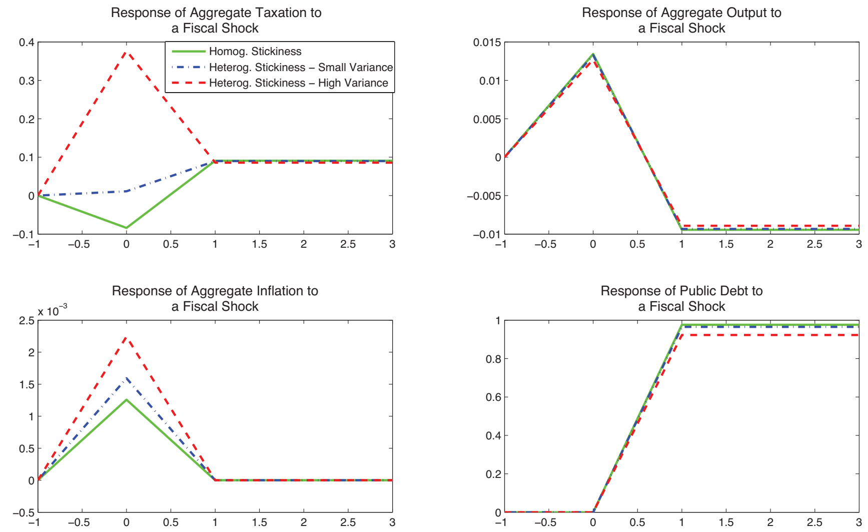

Figure 4.1: Effects of a Fiscal Shock on Aggregate Variables

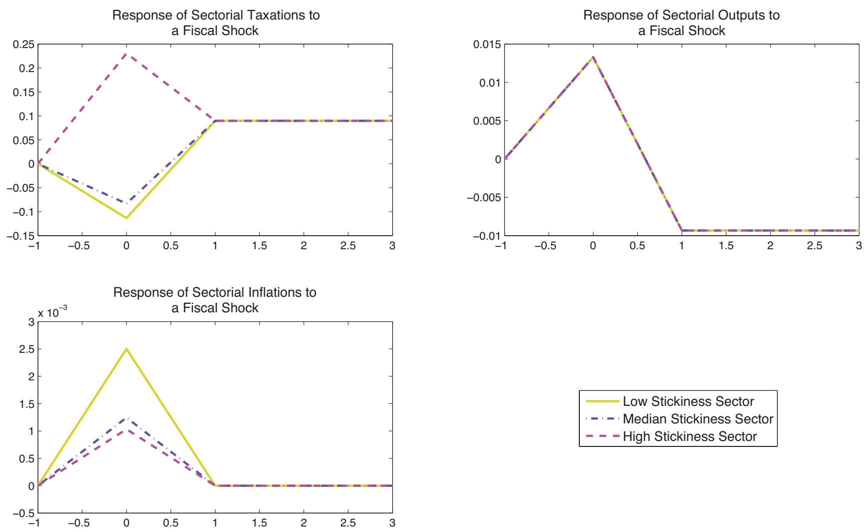

Figure 4.2: Effects of a Fiscal Shock on Sectorial Variables

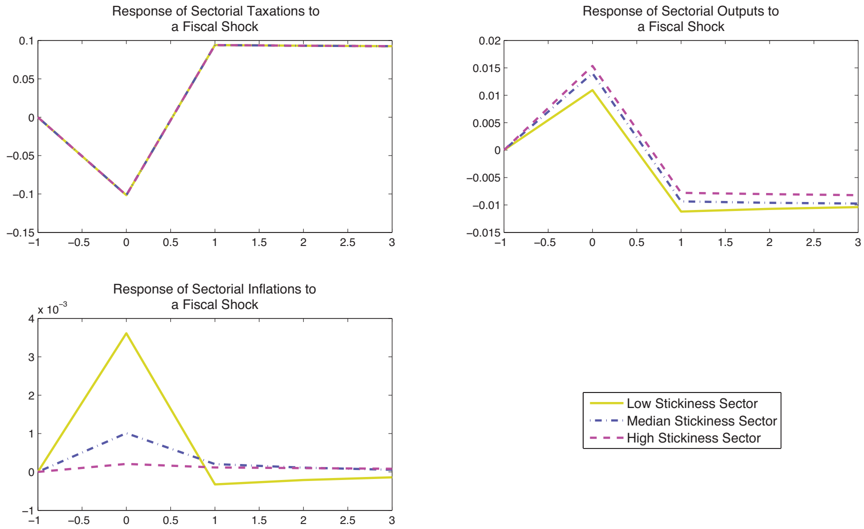

Figure 4.3: Effects of a Cost-Push Shock in Median Sector on Sectorial Variables

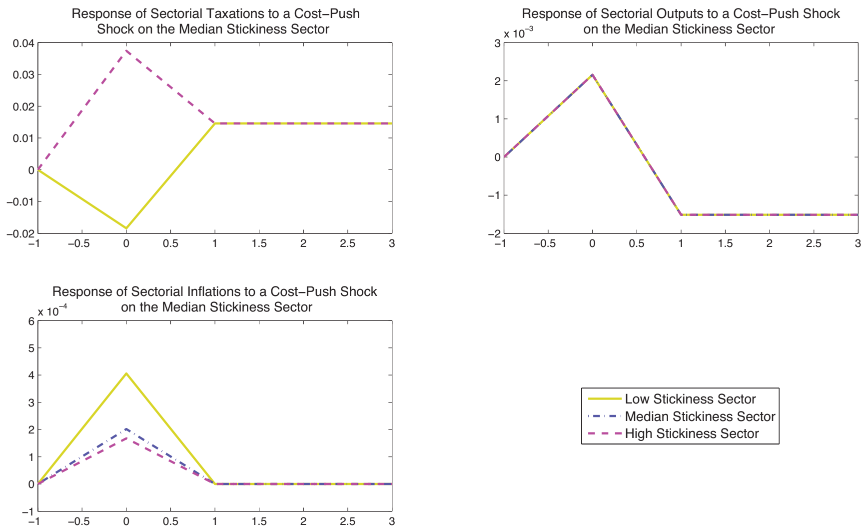

Figure 4.4: Effects of a Cost-Push Shock in the Median Sector on Aggregate Variables

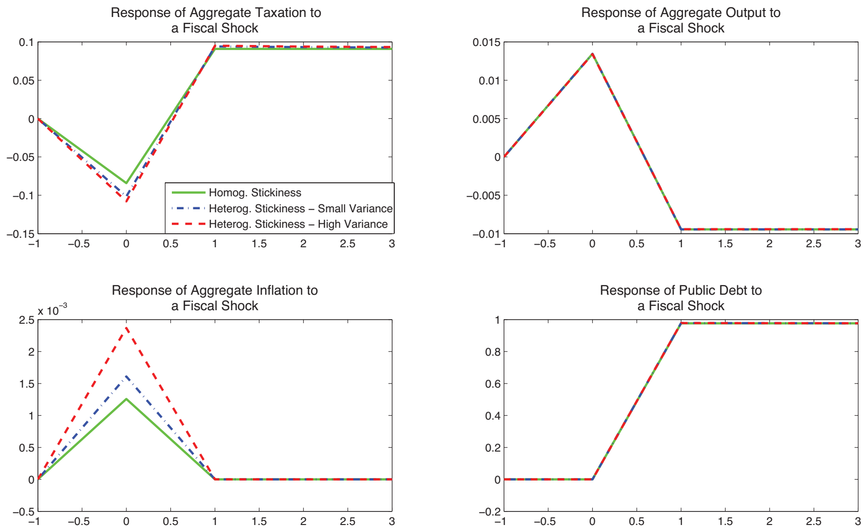

Figure 4.5: Effects of a Fiscal Shock on Aggregate Variables: Homogeneous Taxation Case

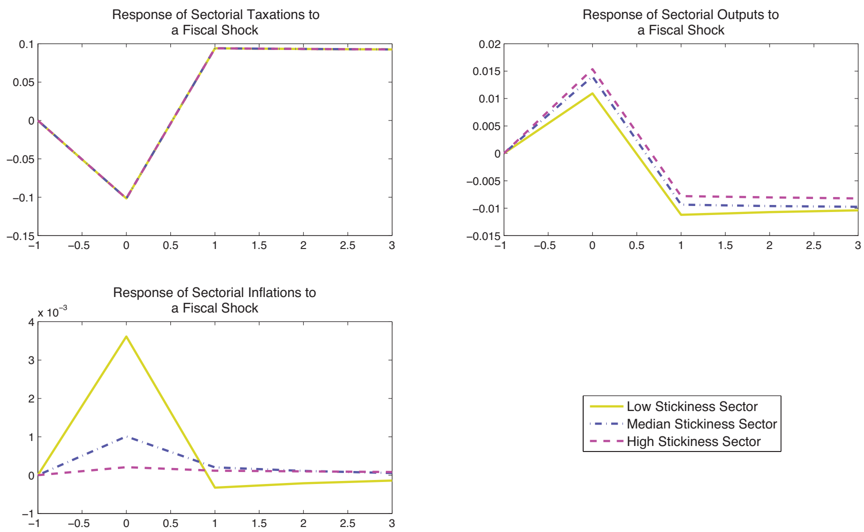

Figure 4.6: Effects of a Cost-Push Shock in the Median Stickiness Sector on Sectorial Variables: Homogeneous Taxation Case

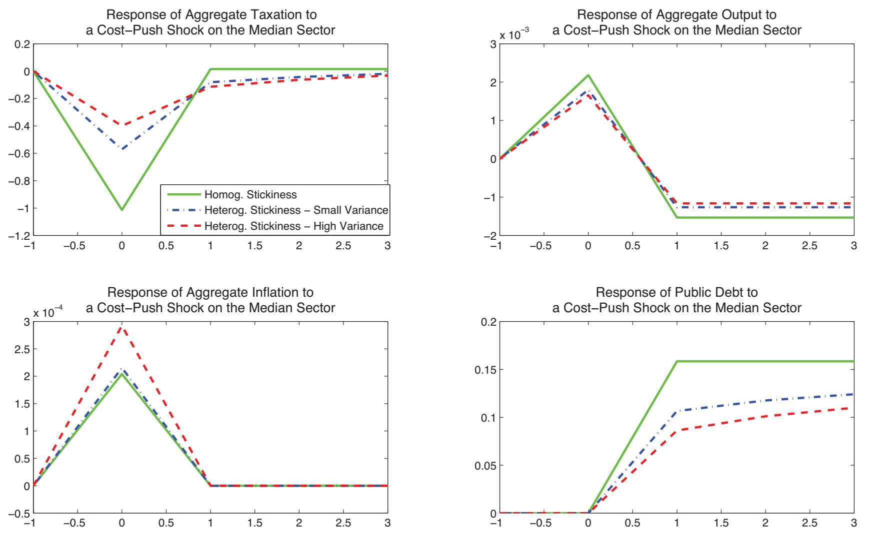

Figure 4.7: Effects of a Fiscal Shock on Sectorial Variables: Homogeneous Taxation Case

Table 4.1: Welfare losses under misperception of heterogeneity in price stickiness (% difference from 1st best in steady state equivalent consumption)

Reference for parameter estimation	Low Variance	High Variance
Smets and Wouters (2007)	0.0223%	0.2051%
US: 1966Q1 - 2004Q4		
Smets and Wouters (2005)[a]	0.0567%	0.4895%
US: 1974Q1 - 2002Q2		
Smets and Wouters (2005)[a]	0.0523%	0.4449%
Euro Area: 1974Q1 - 2002Q2		
Justiniano, Primiceri and Tambalotti (2008)[b]	0.0380%	0.3475%
US:1954Q3 - 2004Q4		

[a]Wage markups are estimated as random noises instead of AR(1)s.

[b]Only neutral technology shocks considered.

Table 4.2: Welfare losses under homogeneous taxation (% difference from 1st best in steady state equivalent consumption)

Reference for parameter estimation	Low Variance	High Variance
Smets and Wouters (2007)	0.0014%	0.0023%
US: 1966Q1 - 2004Q4		
Smets and Wouters (2005)[a]	0.0007%	0.0034%
US: 1974Q1 - 2002Q2		
Smets and Wouters (2005)[a]	0.0007%	0.0031%
Euro Area: 1974Q1 - 2002Q2		
Justiniano, Primiceri and Tambalotti (2008)[b]	0.0027%	0.0043%
US:1954Q3 - 2004Q4		

[a]Wage markups are estimated as random noises instead of AR(1)s.

[b]Only neutral technology shocks considered.

5
Appendix to Stabilizing Inflation under Heterogeneity: a welfare-based measure on what to target

5.1
Appendix A - The Firms' Problem

Noting that $\theta > 1$, FOC from firms' optimization problem is given by:

$$E_t \sum_{j=t}^{\infty} \alpha_k^{j-t} \Theta_{t,j} \frac{\partial \Psi_j \left(p_{k,t}\left(z\right),\ .\right)}{\partial p_{k,t}\left(z\right)} = 0; \tag{5-1}$$

taking derivatives and dividing resulting expression by $1-\theta$

$$E_t \sum_{j=t}^{\infty} \alpha_k^{j-t} \Theta_{t,j} \frac{p_{k,t}(z)}{P_{k,t+j}}^{-\theta} Y_{k,j} \{1 + \frac{\theta}{1-\theta} \frac{w_{k,j}(z)}{a_{k,j}} \frac{P_j}{P_{k,j}} \frac{P_{k,j}}{p_{k,t}(z)}\} = 0;$$

using expression in the main text for labor supply, production function and discount factor:

$$E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \frac{C_j^{-\sigma}}{P_j} \frac{p_{k,t}(z)}{P_{k,j}}^{-\theta} Y_{k,j} \{1+ \\ + \mu_{k,j} \frac{\theta\lambda}{1-\theta} \frac{y_{k,j}(z)^{\nu}}{C_j^{-\sigma}} \frac{1}{a_{k,j}^{\nu+1}} \frac{P_j}{P_{k,j}} \frac{P_{k,j}}{p_{k,t}(z)}\} = 0;$$

using expression for demand for good z in terms of sectorial aggregates and isolating terms $p_{k,t}(z)/P_{k,t}$.

$$\frac{p_{k,t}(z)}{P_{k,t}}^{1+\theta\nu} E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \frac{C_j^{-\sigma}}{P_j} \frac{P_{k,t}}{P_{k,j}}^{-\theta} Y_{k,j} = \\ \mu_{k,j} \frac{\theta\lambda}{\theta-1} m_k^{-\nu} E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \frac{P_{k,j}}{P_{k,t}}^{1+\theta(\nu+1)} \frac{Y_{k,j}}{a_{k,j}}^{\nu+1} \frac{1}{P_{k,j}}$$

$$\frac{p_{k,t}(z)}{P_{k,t}}^{1+\theta\nu} = \frac{\frac{\theta\lambda}{\theta-1} m_k^{-\nu} E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} \mu_{k,j} \frac{P_{k,j}}{P_{k,t}}^{\theta(\nu+1)} \frac{Y_{k,j}}{a_{k,j}}^{\nu+1}}{E_t \sum_{j=t}^{\infty} (\alpha_k \beta)^{j-t} C_j^{-\sigma} \frac{P_{k,j}}{P_{k,t}}^{\theta-1} p_{k,j} Y_{k,j}} \tag{5-2}$$

5.2 Appendix B - Steady State

There is a steady state characterized by zero inflation and constant values for all variables, where exogenous disturbances also assume constant values, that is: $\bar{\xi} = \{\bar{G}, \bar{a}_k, \bar{\mu}_k, \bar{e}_t\}$, where $\bar{a}_k = 1$ and $\bar{\mu}_k = \bar{\mu} > 1$, all k. We focus particular attention to a steady state with positive real debt, that is $\bar{b}_{-1} = \bar{b} > 0$, price dispersion equals one, $\Delta_{k,-1} = \bar{\Delta}_k = 1$ and relative price also equals one, $p_{k,-1} = \bar{p}_k = 1$, all k. Consider the government budget constraint, which in steady state is given by:

$$(1-\beta)\bar{b} = \bar{\tau} - \bar{G}. \tag{5-3}$$

Assuming government expenses are non-zero in steady state (i.e.: $\bar{G} > 0$), imply, according to the hypothesis of a Ricardian regime, that $\bar{\tau}$ is determined directly from (5-3) and proportional to both $\bar{G}$ and $\bar{b}$. From the firms' maximization problem, considering $\bar{\Pi}_k = 0$, all k:

$$\bar{K}_k = \bar{F}_k.$$

Using definitions for both terms:

$$\frac{\theta\lambda}{\theta-1}\bar{\mu} m_k^{-\nu} \bar{Y}_k^{\nu} = \bar{C}^{-\sigma}, \tag{5-4}$$

From (2-18) in the text, $\bar{Y}_k = m_k \bar{Y}$, which implies that

$$\bar{Y} = \left[\frac{\theta\lambda}{\theta-1}\bar{\mu} {s_c}^{\sigma}\right]^{-1/\nu+\sigma}, \tag{5-5}$$

while s_c is defined as

$$s_c = \bar{C}/\bar{Y},$$

which is determined through the market clearing and the fact that $\bar{G}$ is positive and exogenously given. Therefore, $\bar{Y}$ and $\bar{C}$ are defined by the equations above in terms of the parameters of the economy.

5.3 Appendix C - Approximation to Welfare Criterion

5.3.1
Second Order Approximation of Utility Function

I start with a second order Taylor expansion of the representative consumer's welfare function where ξ_t refers to the full vector of random disturbances, as in Benigno and Woodford (2003). Define hereafter, for any variable X_t,

$$\tilde{X}_t \equiv \frac{X_t - \bar{X}}{\bar{X}},$$

$$\hat{X}_t \equiv \log \frac{X_t}{\bar{X}}.$$

It is know that the following relation holds up to second order:

$$\tilde{X}_t \simeq \hat{X}_t + \frac{1}{2}\hat{X}_t^2. \tag{5-6}$$

Given the functional form assumed in the main text for the utility function, define

$$u\left(Y_t, \xi_t\right) \equiv \frac{(Y_t - G_t)^{1-\sigma}}{1-\sigma}.$$

A second order Taylor expansion yields:

$$u\left(Y_t, \xi_t\right) = \bar{C}^{-\sigma}\bar{Y}[\tilde{Y}_t - \frac{\sigma}{2}\frac{\bar{Y}}{\bar{C}}\tilde{Y}_t^2 + \sigma\frac{\bar{Y}}{\bar{C}}\tilde{Y}_t\tilde{G}_t] + tips + O_p^3, \tag{5-7}$$

where $\tilde{G}_t$ represents the absolute deviation *over GDP*. Defining $s_C = \bar{C}/\bar{Y}$, yields

$$u\left(Y_t, \xi_t\right) = \bar{C}^{-\sigma}\bar{Y}[\hat{Y}_t + \frac{1}{2}\hat{Y}_t^2(1 - \sigma s_C^{-1}) + \sigma s_C^{-1}\hat{Y}_t\hat{G}_t] + tips + O_p^3. \tag{5-8}$$

Define also:

$$v\left(Y_{k,t}, \xi_t\right)\Delta_{k.t} \equiv \frac{\lambda}{1+\nu}\left[\frac{Y_{k,t}}{m_k a_{k,t}}\right]^{1+\nu}\Delta_{k.j}$$

A second order Taylor expansion around steady state values yield

$$\begin{aligned} v\left(Y_{k,t}, \xi_t\right)\Delta_{k.t} &= v\left(\bar{Y}_k, \bar{\xi}\right)\tilde{\Delta}_{k.t} + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t} + \frac{1}{2}\hat{Y}_{k,t}^2) + \\ &+\frac{1}{2}v_{Y_kY_k}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k^2(\hat{Y}_{k,t}^2) + v_{Y_k}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t})\tilde{\Delta}_{k.t} + \\ &+v_{Y_k\xi}\left(\bar{Y}_k, \bar{\xi}\right)\bar{Y}_k(\hat{Y}_{k,t}\hat{a}_{k,t}) + v_{\xi}\left(\bar{Y}_k, \bar{\xi}\right)\tilde{\Delta}_{k.t}(\hat{a}_{k,t}) + \\ &+tips + O_p^3. \end{aligned} \tag{5-9}$$

Using the definition for $\Delta_{k,t}$ one can show that $\tilde{\Delta}_{k,t}$ is a term of second order. In this sense, interactions between $\tilde{\Delta}_{k,t}$ and $\hat{a}_{k,t}$ or $\tilde{\Delta}_{k,t}$ and $\hat{Y}_{k,t}$ can be ignored up to second order. Hence, expression (5-9) simplifies to

$$v\left(Y_{k,t},\xi_t\right)\Delta_{k.t}=\lambda\left[\frac{\bar{Y}_k}{m_k}\right]^{1+\nu}\{\frac{\hat{\Delta}_{k.t}}{1+\nu}+\hat{Y}_{k,t}+\frac{1+\nu}{2}\hat{Y}_{k,t}^2-(1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}\}+tips+O_p^3, \tag{5-10}$$

once one notice that $\hat{\Delta}_{k,t}^2$ is of higher order than O_p^2. Using a second order Taylor expansion over the law of motion for sectorial price dispersion given by (2-29) in the main text yields:

$$\hat{\Delta}_{k.t}=\alpha_k\hat{\Delta}_{k.t-1}+\frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\pi_{k,t}^2+O_p^3,$$

once one uses the relation $\hat{\Pi}_{k,t} = \pi_{k,t} + (1/2)\pi_{k,t}^2$, where $\pi_{k,t}$ is the percent variation of sectorial price level $\pi_{k,t} = \log P_{k,t}/P_{k,t-1}$. Iterating backwards yields

$$\hat{\Delta}_{k.t}=\alpha_k^{t-1}\hat{\Delta}_{k.-1}+\frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)}\theta(1+\nu)(1+\theta\nu)\sum_{j=0}^{t}\alpha_k^{t-j}\pi_{k,j}^2+O_p^3.$$

Here it is convenient to consider the sectorial price dispersion in the remote past as a "term independent of policy". Further considering that it is possible to change positions of sums over t and k on (5-10), and re-ordering the terms:

$$\sum_{t=0}^{\infty}\beta^t\hat{\Delta}_{k.t}=\frac{1}{2}\frac{\alpha_k}{(1-\alpha_k)(1-\alpha_k\beta)}\theta(1+\nu)(1+\theta\nu)\sum_{t=0}^{\infty}\beta^t\pi_{k,t}^2+tips+O_p^3. \tag{5-11}$$

Substituting (5-11) over (5-10) yields

$$\begin{aligned}v\left(Y_{k,t},\xi_t\right)\Delta_{k.t}=\lambda\left[\frac{\bar{Y}_{k,t}}{m_k}\right]^{1+\nu}&\{\frac{1}{2}\frac{\alpha_k\theta(1+\theta\nu)}{(1-\alpha_k)(1-\alpha_k\beta)}\pi_{k,t}^2+\hat{Y}_{k,t}+\\&+\frac{1+\nu}{2}\hat{Y}_{k,t}^2-(1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}\}+tips+O_p^3.\end{aligned}$$

Considering expressions for $u\left(Y_t,\xi_t\right)$ and $v\left(Y_{k,t},\xi_t\right)\Delta_{k.t}$, we can approximate the representative consumer utility up to second order by the following expression:

$$
\begin{aligned}
U_{t_0} &= \Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\hat{Y}_t+\frac{(1-\tilde{\sigma})}{2}\hat{Y}_t^2+\tilde{\sigma}\hat{Y}_t\hat{G}_t+ \\
&\quad -\sum_{k=1}^{K}m_k(1-\Phi)[\frac{\theta}{\kappa_k}\frac{\pi_{k,t}^2}{2}+\hat{Y}_{k,t}+\frac{1+\nu}{2}\hat{Y}_{k,t}^2+ \\
&\quad -(1+\nu)\hat{Y}_{k,t}\hat{a}_{k,t}]\}+tips+O_p^3,
\end{aligned} \tag{5-12}
$$

where

$$\Omega \equiv \bar{C}^{-\sigma}\bar{Y}, \tag{5-13}$$

$$\kappa_k \equiv \frac{(1-\alpha_k)(1-\alpha_k\beta)}{(1+\theta\nu)\alpha_k}, \tag{5-14}$$

$$\tilde{\sigma} \equiv \sigma s_C^{-1} \tag{5-15}$$

and

$$(1-\Phi) \equiv \frac{\theta-1}{\theta}\frac{1}{\bar{\mu}}, \tag{5-16}$$

where feasibility constraint in (5-4) was used to eliminate inconvenient terms in $v\left(Y_{k,t},\xi_t\right)\Delta_{k.t}$. Following Benigno and Woodford (2003), we seek to eliminate linear terms by obtaining second order approximations to all equations that describe the economy.

5.3.2
Second Order Approximation to AS Equation

The starting point is the expression for the sectorial non-linear Phillips Curve, given by:

$$\left(\frac{1-\alpha_k\Pi_{k,t}^{\theta-1}}{1-\alpha_k}\right)^{\frac{1+\theta\nu}{\theta-1}} = \frac{F_{k,t}}{K_{k,t}}. \tag{5-17}$$

We define $V_{k,t}$ as

$$V_{k,t} = \frac{1-\alpha_k\Pi_{k,t}^{\theta-1}}{(1-\alpha_k)}. \tag{5-18}$$

Using a second order Taylor expansion on $\hat{V}_{k,t}$:

$$\hat{V}_{k.t} = -\frac{\alpha_k(\theta-1)}{(1-\alpha_k)}\left[\pi_{k,t}+\frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2\right]+O_p^3. \tag{5-19}$$

Considering the expression for $K_{k,t}$ define $\Pi_{k,t,s}=P_{k,s}/P_{k,t}$, where $s\geq t$ is some date in the future and $P_{k,t}$ the aggregate price level in sector k in period t. We use a second order Taylor expansion:

$$\tilde{K}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{k}_{k,j}+\frac{1}{2}\hat{k}^2_{k,j}\}+O_p^3, \tag{5-20}$$

where the term $\hat{k}_{k,t}$ can be defined as

$$\hat{k}_{k,j} = \theta(1+\nu)\pi_{k,t,j}+(1+\nu)\hat{Y}_{k,j}-(1+\nu)\hat{a}_{k,j}.$$

Taking a second order Taylor expansion of (6-22) in the text:

$$\tilde{F}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{f}_{k,j}+\frac{1}{2}\hat{f}^2_{k,j}\}+O_p^3, \tag{5-21}$$

where we define

$$\hat{f}_{k,j} = -\sigma\hat{C}_j+\hat{Y}_{k,j}+\hat{p}_{k,j}+(\theta-1)\,\pi_{k,t,j}.$$

Using $\tilde{F}_{k,t}$, $\tilde{K}_{k,t}$, as well as $\hat{V}_{k.t}$, $\hat{F}_{k,t}$ and $\hat{K}_{k,t}$,after some algebra, we get:

$$\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t} = (1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}]+$$
$$+\frac{1}{2}[z_{k,j}-(1+\theta\nu)\pi_{k,t,j}][\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}]\}$$
$$-\frac{1}{2}\left[\frac{1+\theta\nu}{\theta-1}\right]\hat{V}_{k,t}(1-\beta\alpha_k)E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\}+O_p^3,$$

where

$$\hat{X}_{k,j} \equiv -\sigma\hat{C}_j+(2+\nu)\hat{Y}_{k,j}+\hat{p}_{k,j}-(1+\nu)\hat{a}_{k,j}+\hat{\mu}_{k,t}, \tag{5-22}$$

$$\hat{f}_{k,j}-\hat{k}_{k,j} = z_{k,j}-(1+\theta\nu)\pi_{k,t,j}$$

and

$$z_{k,j} = -\sigma\hat{C}_j-\nu\hat{Y}_{k,j}+\hat{p}_{k,j}+(1+\nu)\hat{a}_{k,j}-\hat{\mu}_{k,t}. \tag{5-23}$$

Define

$$Z_{k,t} \equiv E_t\sum_{j=t}^{\infty}(\alpha_k\beta)^{j-t}\{\left[\hat{X}_{k,j}+[(\theta-1)+\theta(1+\nu)]\pi_{k,t,j}\right]\} \tag{5-24}$$

We can replace in the expression above and after some algebra we get:

$$\frac{(1+\theta\nu)}{(\theta-1)(1-\beta\alpha_k)}\hat{V}_{k,t}(\pi_{k,t+1}) = (\pi_{k,t+1})E_t\sum_{j=t+1}^{\infty}(\alpha_k\beta)^{j-t-1}\{z_{k,j}-(1+\theta\nu)(\pi_{k,t,j})\}+O_p^3 \tag{5-25}$$

We can use the definition for $\hat{V}_{k,t}$ and replace above, also ignoring the terms O_p^3 or of higher order:

$$\begin{aligned}
-\kappa_k^{-1}\left[\pi_{k,t}+\frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2-\alpha_k\beta E_t\pi_{k,t+1}-\frac{1}{2}\frac{(\theta-1)}{(1-\alpha_k)}\alpha_k\beta E_t\pi_{k,t+1}^2\right] = \\
z_{k,t}+\frac{1}{2}z_{k,t}\hat{X}_{k,t}-(1+\theta\nu)\frac{\alpha_k\beta}{(1-\alpha_k\beta)}E_t\pi_{k,t+1}+ \\
-\frac{1}{2}[(\theta-1)+\theta(1+\nu)]\frac{\beta}{\kappa_k}E_t\pi_{k,t+1}^2+ \\
-\frac{1}{2}(1+\theta\nu)(\alpha_k\beta)E_t[\pi_{k,t+1}Z_{k,t+1}]+ \\
+\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}[\pi_{k,t}Z_{k,t}-\alpha_k\beta E_t[\pi_{k,t+1}Z_{k,t+1}]]+O_p^3,
\end{aligned}$$

where we have defined κ_k elsewhere.

Further simplification yields

$$\begin{aligned}
-\kappa_k^{-1}\pi_{k,t}-\frac{1}{2}\kappa_k^{-1}\frac{(\theta-1)}{(1-\alpha_k)}\pi_{k,t}^2-\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\pi_{k,t}Z_{k,t} \\
= z_{k,t}+\frac{1}{2}z_{k,t}\hat{X}_{k,t}-\kappa_k^{-1}\beta E_t\pi_{k,t+1} \\
-\frac{1}{2}\kappa_k^{-1}\{\frac{(\theta-1)}{(1-\alpha_k)}+\theta(1+\nu)\}\beta E_t\pi_{k,t+1}^2 \\
-\frac{1}{2}\frac{(1+\theta\nu)\alpha_k}{(1-\alpha_k)}\beta E_t[\pi_{k,t+1}Z_{k,t+1}]+O_p^3.
\end{aligned}$$

Multiplying both sides for $-\kappa_k$ allow us to write above expression as

$$\mathcal{V}_{k,t} = -\kappa_k\{z_{k,t}+\frac{1}{2}z_{k,t}\hat{X}_{k,t}\}+\frac{\theta(1+\nu)}{2}\pi_{k,t}^2+\beta E_t\mathcal{V}_{k,t+1}+O_p^3, \tag{5-26}$$

where:

$$\mathcal{V}_{k,t} = \pi_{k,t}+\frac{1}{2}\{\frac{(\theta-1)}{(1-\alpha_k)}+\theta(1+\nu)\}\pi_{k,t}^2+\frac{1}{2}\frac{\kappa_k\alpha_k}{(1-\alpha_k)}[\pi_{k,t}Z_{k,t}]. \tag{5-27}$$

Log-approximation on consumption as a function of aggregate output and government expenses yields:

$$\hat{C}_t = s_C^{-1}\hat{Y}_t - s_C^{-1}\hat{G}_t + \frac{1}{2}s_C^{-1}(1-s_C^{-1})\hat{Y}_t^2 - \frac{1}{2}s_C^{-1}(1+s_C^{-1})\hat{G}_t^2 + s_C^{-2}\hat{Y}_t\hat{G}_t + O_p^3. \quad (5\text{-}28)$$

Using this result, one can be generally express (5-26) as

$$\mathcal{V}_{k,t} = E_{t_0}\sum_{j=t}^{\infty}\beta^{j-t}\{-\kappa_k[z_{k,t} + \frac{1}{2}z_{k,t}\hat{X}_{k,t}] + \frac{\theta(1+\nu)}{2}\pi_{k,t}^2\} + tips + O_p^3. \quad (5\text{-}29)$$

One could finally note that a first order approximation to (5-29) yields the known Phillips Curve of the form:

$$\pi_{k,t} = \kappa_k\{(\tilde{\sigma} - \eta^{-1})\hat{Y}_t + (\nu + \eta^{-1})\hat{Y}_{k,t} - \tilde{\sigma}\hat{G}_t - (1+\nu)\hat{a}_{k,t} + \hat{\mu}_{k,t}\} + \beta E_t\pi_{k,t+1}.$$

5.3.3 Aggregate and Sectorial Output Relation

Sectorial demand expressed is (2-18) can be log-linearized as

$$\hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}), \quad (5\text{-}30)$$

which establishes an exact (inverse) relation between sector relative price and sector relative product. It is used to eliminate references to relative prices in all equations. Also, using (2-35) in the text and (5-30), one gets:

$$\eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}) = \pi_{k,t} - \pi_t + \eta^{-1}(\hat{Y}_{t-1} - \hat{Y}_{k,t-1}), \quad (5\text{-}31)$$

all k, which is also an exact relation. Also using (2-35) and (5-30) over (2-36) in the main text yields:

$$Y_t^{(\eta-1)/\eta} = \sum_{k=1}^{K} m_k^{1/\eta} Y_{k,t}^{(\eta-1)/\eta}, \quad (5\text{-}32)$$

which relates aggregate and sectorial outputs. Log linearization of (5-32) yields

$$\hat{Y}_t + \frac{1}{2}(1-\eta^{-1})\hat{Y}_t^2 = \sum_{k=1}^{K} m_k\hat{Y}_{k,t} + \frac{1}{2}(1-\eta^{-1})\sum_{k=1}^{K} m_k\hat{Y}_{k,t}^2 + O_p^3. \quad (5\text{-}33)$$

whose first order approximation in simply the definition of aggregate output in terms of sectorial outputs:

$$\hat{Y}_t = \sum_{k=1}^{K} m_k\hat{Y}_{k,t}. \quad (5\text{-}34)$$

5.3.4 Matrix Notation

We start by defining

$$x_t' = \begin{bmatrix} \hat{Y}_t & \hat{Y}_{1,t} & ... & \hat{Y}_{K,t} & \pi_{1,t} & ... & \pi_{K,t} \end{bmatrix} \tag{5-35}$$

and

$$\xi_t' = \begin{bmatrix} \hat{G}_t & \hat{a}_{1,t} & ... & \hat{a}_{K,t} & \hat{\mu}_{k,t} & ... & \hat{\mu}_{K,t} \end{bmatrix}. \tag{5-36}$$

For notational convenience, we also define the following terms:

$$\upsilon \equiv 1 + \nu, \tag{5-37}$$

$$\omega_\eta \equiv 1 - \eta^{-1}, \tag{5-38}$$

$$\chi \equiv \nu + \eta^{-1}, \tag{5-39}$$

$$\tilde{\sigma} \equiv \sigma s_C^{-1}, \tag{5-40}$$

$$\varsigma \equiv \tilde{\sigma} - \eta^{-1}, \tag{5-41}$$

and

$$\omega_C \equiv \frac{\bar{Y} - \bar{C}}{\bar{C}}, \tag{5-42}$$

in addition to:

$$s_C \equiv \bar{C}/\bar{Y}. \tag{5-43}$$

Using the definitions above, expression in (5-12) can be written in matrix notation as

$$U_{t_0} \equiv \Omega E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{A_x' x_t - \frac{1}{2} x_t' A_{xx} x_t - x_t' A_\xi \xi_t\} + tips + O_p^3, \tag{5-44}$$

where A_x, A_{xx}, and A_ξ are, respectively, $(2K+1) \times 1$, $(2K+1) \times (2K+1)$ and $(2K+1) \times (2K+1)$ matrices, such as:

$$A_x' = \begin{bmatrix} 1 & -m_1(1-\Phi) & ... & -m_K(1-\Phi) & 0 & ... & 0 \end{bmatrix}, \tag{5-45}$$

$$A_{xx} = \begin{bmatrix} A_{xx}^{11} & 0 & 0 \\ 0 & A_{xx}^{22} & 0 \\ 0 & 0 & A_{xx}^{33} \end{bmatrix}, \tag{5-46}$$

where A_{xx}^{11} is a 1×1 matrix such as

$$A_{xx}^{11} = -(1 - \tilde{\sigma}),$$

A_{xx}^{22} is a $K \times K$ diagonal matrix such as its typical k^{th} element is

$$\left(A_{xx}^{22}\right)_{kk} = m_k(1 - \Phi)\upsilon,$$

A_{xx}^{33} is a $K \times K$ diagonal matrix such as its typical k^{th} element is

$$\left(A_{xx}^{33}\right)_{kk} = \frac{m_k(1 - \Phi)}{\kappa_k}\theta,$$

and

$$A_{\xi} = \begin{bmatrix} A_{\xi}^{11} & 0 & 0 \\ 0 & A_{\xi}^{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5-47}$$

where

$$A_{\xi}^{11} = -\tilde{\sigma}$$

and A_{ξ}^{22} is a $K \times K$ diagonal matrix such as its typical k^{th} element is

$$\left(A_{\xi}^{22}\right)_{kk} = -m_k(1 - \Phi)\upsilon,$$

and where we have observed the definitions in (5-13)-(5-16).

The Sectorial Phillips Curve expressed in (5-29) can also be written in matrix notation. We start by substituting expressions for $\hat{p}_{k,t}$ into definitions for $z_{k,t}$ and $\hat{X}_{k,t}$, underlined in (5-23) and (5-22). Quadratic and linear terms of random disturbances are placed into *tips*. After some manipulation one obtains:

$$\mathcal{V}_{k,t_0} = E_{t_0} \sum_{j=t_0}^{\infty} \beta^{j-t_0} \{C_{x,k}' x_t + \frac{1}{2} x_t' C_{xx,k} x_t + x_t' C_{\xi,k} \xi_t\} + tips + O_p^3, \tag{5-48}$$

for a generic sector k. As in (5-44), matrices $C_{x,k}$, $C_{xx,k}$, and $C_{\xi,k}$ have, respectively, dimension $(2K+1)\times 1$, $(2K+1)\times(2K+1)$ and $(2K+1)\times(2K+1)$, such as:

$$C_{x,k}' = \begin{bmatrix} C_{x,k}^{11\prime} & C_{x,k}^{12\prime} & 0 \end{bmatrix}, \tag{5-49}$$

where $C_{x,k}^{11\prime}$ is 1×1 matrix such as

$$C_{x,k}^{11\prime} = \kappa_k \varsigma$$

every k, $C_{x,k}^{12\prime}$ is $1 \times K$ matrix such as

$$\left(C^{12\prime}_{x,k}\right)_{1k} = \kappa_k \chi$$

and zeros elsewhere; and

$$C_{xx.k} = \begin{bmatrix} C^{11}_{xx.k} & C^{12}_{xx.k} & 0 \\ C^{21}_{xx.k} & C^{22}_{xx.k} & 0 \\ 0 & 0 & C^{33}_{xx.k} \end{bmatrix} \tag{5-50}$$

such that $C^{11}_{xx,k}$ is 1×1 matrix

$$C^{11}_{xx,k} = -\kappa_k [\tilde{\sigma}\omega_C + \varsigma^2]$$

for every k, $C^{12}_{xx,k}$ is $1 \times K$ matrix such that

$$\left(C^{12}_{xx,k}\right)_{1k} = \kappa_k \varsigma \omega_\eta$$

and zeros elsewhere, all k, and $C^{12\prime}_{xx,k} = C^{21}_{xx,k}$; $C^{22}_{xx,k}$ is $K \times K$ diagonal matrix such that, all k,

$$\left(C^{22}_{xx,k}\right)_{kk} = \chi \kappa_k (\upsilon + \omega_\eta)$$

$C^{33}_{xx,k}$ is $K \times K$ diagonal matrix such that, for all k,

$$\left(C^{33}_{xx,k}\right)_{kk} = \theta \upsilon$$

Also, matrix $C_{\xi,k}$ can be defined as

$$C_{\xi,k} = \begin{bmatrix} C^{11}_{\xi,k} & 0 & 0 \\ C^{21}_{\xi,k} & C^{22}_{\xi,k} & C^{23}_{\xi,k} \\ 0 & 0 & 0 \end{bmatrix} \tag{5-51}$$

where $C^{11}_{\xi,k}$ is 1×1 matrix, such that

$$C^{11}_{\xi,k} = \kappa_k [\omega_C + \tilde{\sigma} + \omega_\eta] \tilde{\sigma}$$

for every k; $C^{21}_{\xi,k}$ is a $K \times 1$ matrix, such as

$$\left(C^{21}_{\xi,k}\right)_{1k} = -\kappa_k \omega_\eta \tilde{\sigma}$$

and zero elsewhere, $C^{22}_{\xi,k}$ is $K \times K$ diagonal matrix such that

$$\left(C^{22}_{\xi,k}\right)_{kk} = -\kappa_k \upsilon^2$$

and zero elsewhere, $C^{23}_{\xi,k}$ is $K \times K$ diagonal matrix such that

$$\left(C^{23}_{\xi,k}\right)_{kk} = \kappa_k \upsilon$$

and zero elsewhere.

Equation (5-33) can be expressed in matrix notation as

$$0 = \sum_{j=t}^{\infty} \beta^{j-t} \{ H_x' x_t + \frac{1}{2} x_t' H_{xx} x_t \} + O_p^3 \tag{5-52}$$

where we have used the fact that the definition for aggregate output in terms of its sectorial counterparts expressed in (5-33) is valid at all dates. Matrices H_x and H_{xx} have, respectively, dimension $(2K+1)\times 1$ and $(2K+1)\times(2K+1)$, such as:

$$H_x' = \begin{bmatrix} 1 & -m_1 & ... & -m_K & 0 & ... & 0 \end{bmatrix}, \tag{5-53}$$

$$H_{xx} = \omega_\eta \begin{bmatrix} 1 & 0 & 0 \\ 0 & H_{xx}^{22} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5-54}$$

where H_{xx}^{22} is a $K \times K$ diagonal matrix such as

$$\left(H_{xx}^{22}\right)_{kk} = -m_k,$$

for every k.

5.3.5 Elimination of Linear Terms

In order to eliminate linear terms in (5-44), we need to find a set a multipliers $\vartheta_C^1, ..., \vartheta_C^K, \vartheta_H$, such as

$$\vartheta_C^1 C_x^{1\prime} + ... + \vartheta_C^K C_x^{K\prime} + \vartheta_H H_x' = A_x' \tag{5-55}$$

By solving the linear system of equations, one gets the following set of solution:

$$\vartheta_H = \frac{\chi + \varsigma(1-\Phi)}{\varsigma + \chi} \tag{5-56}$$

and, for every k,

$$\vartheta_C^k = \frac{m_k}{\kappa_k} \frac{\Phi}{\varsigma + \chi} \tag{5-57}$$

where we have used the definitions in (5-14)-(5-16).

Hence, using relations (5-44), (5-48), (5-52) and (5-55) one can write:

$$E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}A'_x x_t = E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}[\sum_{k=1}^{K}\vartheta_C^k C_x^{k\prime} + \vartheta_H H'_x]x_t \tag{5-58}$$

$$= -E_{t_0}\sum_{j=t_0}^{\infty}\beta^{j-t_0}\{\frac{1}{2}x'_t D_{xx}x_t + x'_t D_\xi \xi_t\} + \sum_{k=1}^{K}\vartheta_C^k \mathcal{V}_{k,t_0},$$

where

$$D_{xx} = \sum_{k=1}^{K}\vartheta_C^k C_{xx,k} + \vartheta_H H_{xx}$$

and

$$D_\xi = \sum_{k=1}^{K}\vartheta_C^k C_\xi^k.$$

We use this last relations in order to rewrite (5-44) as

$$U_{t_0} \equiv -\Omega E_{t_0}\sum_{t=t_0}^{\infty}\beta^{t-t_0}\{\frac{1}{2}x'_t Q_{xx}x_t + x'_t Q_\xi \xi_t\} + T_{t_0} + tips + O_p^3 \tag{5-59}$$

where

$$T_{t_0} = \Omega\{\sum_{k=1}^{K}\vartheta_C^k \mathcal{V}_{k,t_0}\} \tag{5-60}$$

is a vector of predetermined variables and where Q_{xx} and Q_ξ can be defined, respectively, as

$$Q_{xx} = \begin{bmatrix} Q_{xx}^{11} & Q_{xx}^{12} & 0 \\ Q_{xx}^{21} & Q_{xx}^{22} & 0 \\ 0 & 0 & Q_{xx}^{33} \end{bmatrix}, \tag{5-61}$$

where Q_{xx}^{11} is a 1×1 matrix such as

$$Q_{xx}^{11} = -(1-\tilde{\sigma}) - [\tilde{\sigma}\omega_C + \varsigma^2]\frac{\Phi}{\varsigma+\chi} + \omega_\eta\frac{\chi+\varsigma(1-\Phi)}{\varsigma+\chi},$$

Q_{xx}^{22} is a $K\times K$ diagonal matrix such as, for a generic k diagonal element,

$$\left(Q_{xx}^{22}\right)_{kk} = m_k\{(1-\Phi)\upsilon + \frac{\omega_\eta}{\varsigma+\chi}[2\varsigma\Phi - \chi - \varsigma]\},$$

Q_{xx}^{33} is a $K\times K$ diagonal matrix such as, for a generic k diagonal element,

$$\left(Q_{xx}^{33}\right)_{kk} = \frac{m_k}{\kappa_k}\theta[1-\Phi + \frac{\Phi}{\varsigma+\chi}\upsilon],$$

Q_{xx}^{12} a $1 \times K$ such as its typical k^{th}-column element is

$$\left(Q_{xx}^{12}\right)_{1k} = m_k \varsigma \omega_\eta \frac{\Phi}{\varsigma + \chi},$$

and $Q_{xx}^{21} = Q_{xx}^{12\prime}$. In the same fashion, we define the matrix Q_ξ as

$$Q_\xi = \begin{bmatrix} Q_\xi^{11} & 0 & 0 \\ Q_\xi^{21} & Q_\xi^{22} & Q_\xi^{23} \\ 0 & 0 & 0 \end{bmatrix}, \tag{5-62}$$

where Q_ξ^{11} is a 1×1 matrix such as

$$Q_\xi^{11} = -\tilde{\sigma} + [\omega_C + \tilde{\sigma} + \omega_\eta]\tilde{\sigma}\frac{\Phi}{\varsigma + \chi},$$

Q_ξ^{22} is a $K \times K$ diagonal matrix such as, for a generic k diagonal element,

$$\left(Q_\xi^{22}\right)_{kk} = -m_k \upsilon [1 - \Phi + \frac{\Phi}{\varsigma + \chi}\upsilon],$$

Q_ξ^{21} a $K \times 1$ dimension matrix such as its typical k^{th}-line element is

$$\left(Q_\xi^{21}\right)_{k1} = -m_k \omega_\eta \tilde{\sigma} \frac{\Phi}{\varsigma + \chi},$$

Q_ξ^{23} a $K \times K$ diagonal matrix such as its typical k^{th}-line element is

$$\left(Q_\xi^{23}\right)_{k1} = m_k \upsilon \frac{\Phi}{\varsigma + \chi}.$$

Simplifying (5-59) further by getting rid-off tax rates references and by separating terms referring to sectorial and overall outputs from references to sectorial inflation. Proceeding in such fashion yields

$$U_{t_0} = -\frac{\Omega}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{x_{y,t}' \tilde{Q}_y x_{y,t} + 2x_{y,t}' \tilde{Q}_\xi \xi_t + x_{\pi,t}' \tilde{Q}_\pi x_{\pi,t}\} + T_{t_0} + tips + O_p^3, \tag{5-63}$$

where $x_{y,t}$ is a $K + 1 \times 1$ vector containing only references to aggregate and sectorial outputs measures, or

$$x_{y,t}' = \begin{bmatrix} \hat{Y}_t & \hat{Y}_{1,t} & ... & \hat{Y}_{K,t} \end{bmatrix},$$

$x_{\pi,t}$ is a $K \times 1$ vector containing only sectorial inflation measures, or

$$x_{\pi,t}' = \begin{bmatrix} \pi_{1,t} & ... & \pi_{K,t} \end{bmatrix},$$

and $\tilde{Q}_y$, $\tilde{Q}_\xi$ and $\tilde{Q}_\pi$ are given, respectively, by:

$$\tilde{Q}_y = \begin{bmatrix} Q_{xx}^{11} & Q_{xx}^{12} \\ Q_{xx}^{21} & Q_{xx}^{22} \end{bmatrix},$$

$$\tilde{Q}_\pi = \begin{bmatrix} Q_{xx}^{33} \end{bmatrix},$$

$$\tilde{Q}_\xi = \begin{bmatrix} Q_\xi^{11} & 0 & 0 \\ Q_\xi^{21} & Q_\xi^{22} & Q_\xi^{23} \end{bmatrix},$$

where accurate specifications for submatrices Q_{xx}^{ij} and Q_ξ^{ij} are given in (5-61) and (5-62). From (5-63), we now focus on the term

$$x'_{y,t}\tilde{Q}_y x_{y,t} = q_y Y_t^2 + \sum_{k=1}^{K} m_k q_{y_k} Y_{k,t}^2 + 2\sum_{k=1}^{K} m_k q_{y,y_k} Y_t Y_{k,t}, \tag{5-64}$$

where q terms are defined according to

$$q_y = -(1-\tilde{\sigma}) - [\tilde{\sigma}\omega_C + \varsigma^2]\frac{\Phi}{\varsigma+\chi} + \omega_\eta \frac{\chi + \varsigma(1-\Phi)}{\varsigma+\chi}, \tag{5-65}$$

$$q_{y_k} = (1-\Phi)\upsilon + \frac{\omega_\eta}{\varsigma+\chi}[2\varsigma\Phi - \chi - \varsigma], \tag{5-66}$$

$$q_{y,y_k} = \varsigma\omega_\eta \frac{\Phi}{\varsigma+\chi}. \tag{5-67}$$

Under the assumption that wage markups is steady state as well as markups over marginal costs are the same across sectors ($\bar{\mu}_k = \bar{\mu}$ and $\theta_k = \theta$) , q coefficients are all independent of k. We use the following proposition in order to simplify (5-64) further:

Proposition 21 *The following expression relating sum of sectorial output variances and covariances of sectorial outputs and aggregate output is of third order:*

$$\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3.$$

Proof: On one hand, from (5-33)

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = \frac{(1-\eta^{-1})}{2}\left(\sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 - \hat{Y}_t^2\right) + O_p^3. \tag{5-68}$$

On the other hand, from the definition of sectorial demand it is possible to establish the following exact relation:

$$\hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \hat{Y}_{k,t}). \tag{5-69}$$

Summing across sectors yields:

$$\sum_{k=1}^{K} m_k \hat{p}_{k,t} = \eta^{-1}(\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}). \tag{5-70}$$

From the definition of aggregate price level in terms of sectorial prices:

$$1 = \sum_{k=1}^{K} m_k p_{k,t}^{1-\eta}. \tag{5-71}$$

Log-approximation on (5-71) yields:

$$\sum_{k=1}^{K} m_k \hat{p}_{k,t} = \frac{1}{2}(1-\eta)\sum_{k=1}^{K} m_k \hat{p}_{k,t}^2 + O_p^3.$$

One can use (5-69) and (5-70) in order to replace for $\hat{p}_{k,t}$, which yields:

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = -\frac{(1-\eta^{-1})}{2}(\hat{Y}_t^2 - 2\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} + \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2) + O_p^3. \tag{5-72}$$

Comparing (6-1) and (6-2) yields the result.

Given proposition above, (5-64) is equivalent to:

$$x'_{y,t}\tilde{Q}_y x_{y,t} = q_y Y_t^2 + q'_{y_k} \sum_{k=1}^{K} m_k Y_{k,t}^2 + O_p^3, \tag{5-73}$$

where:

$$q'_{y_k} = q_{y_k} + 2q_{y,y_k}.$$

We now focus on the second term of (5-63), containing the interactions between endogenous variables and exogenous processes:

$$x'_{y,t}\tilde{Q}_\xi \xi_t = q_{yG}\hat{Y}_t\hat{G}_t + q_{y_kG}\sum_{k=1}^{K} m_k Y_{k,t}\hat{G}_t + \sum_{k=1}^{K} m_k \hat{Y}_{k,t}[q_{y_k a_k}\hat{a}_{k,t} + q_{y_k \mu_k}\hat{\mu}_{k,t}]. \tag{5-74}$$

where coefficients defined as

$$q_{yG} = -\tilde{\sigma} + \tilde{\sigma}[\omega_C + \tilde{\sigma} + \omega_\eta]\frac{\Phi}{\varsigma + \chi}, \tag{5-75}$$

$$q_{y_k a_k} = -\upsilon[1 - \Phi + \frac{\Phi}{\varsigma + \chi}\upsilon], \tag{5-76}$$

$$q_{y_k G} = -\omega_\eta \tilde{\sigma}\frac{\Phi}{\varsigma + \chi}, \tag{5-77}$$

$$q_{y_k\mu_k} = \frac{\Phi}{\varsigma + \chi}\upsilon \tag{5-78}$$

are all independent of sector-specific characteristics.

Proposition 22 *The following expression is, at least, of second order:*

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = O_p^2.$$

Proof: Follows directly from (5-33).

From above, the following holds:

Proposition 23 *The following expression holds:*

$$[\hat{Y}_t - \sum_{k=1}^{K} m_k Y_{k,t}]\hat{G}_t = O_p^3.$$

Proof: From proposition above plus the fact that all exogenous processes are O_p^1.

From (5-74), one can use above to get:

$$x'_{y,t}\tilde{Q}_\xi \xi_t = \sum_{k=1}^{K} m_k Y_{k,t}[q'_{y_k G}\hat{G}_t + q_{y_k a_k}\hat{a}_{k,t} + q_{y_k\mu_k}\hat{\mu}_{k,t}] + O_p^3, \tag{5-79}$$

where

$$q'_{y_k G} = q_{yG} + q_{y_k G}.$$

We now focus our attention on (5-73). The following lemma can help us simplify the expression even further.

Proposition 24 *The following expression is of third order:*

$$\hat{Y}_t^2 - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3.$$

Proof: From the first proposition:

$$\hat{Y}_t \sum_{k=1}^{K} m_k \hat{Y}_{k,t} - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3. \tag{5-80}$$

From the second proposition:

$$\hat{Y}_t - \sum_{k=1}^{K} m_k \hat{Y}_{k,t} = O_p^2. \tag{5-81}$$

Replacing (5-81) over (5-80) yields:

$$\hat{Y}_t^2 - \sum_{k=1}^{K} m_k \hat{Y}_{k,t}^2 = O_p^3,$$

once we notice that $\hat{Y}_t O_p^2$ is O_p^3.

From (5-73):

$$x'_{y,t} \tilde{Q}_y x_{y,t} = q_y [Y_t^2 - \sum_{k=1}^{K} m_k Y_{k,t}^2] + [q'_{y_k} + q_y] \sum_{k=1}^{K} m_k Y_{k,t}^2 \tag{5-82}$$

Applying the last Proposition above:

$$x'_{y,t} \tilde{Q}_y x_{y,t} = q''_{y_k} \sum_{k=1}^{K} m_k Y_{k,t}^2 + O_p^3, \tag{5-83}$$

where

$$q''_{y_k} = q'_{y_k} + q_y.$$

Replacing (5-79) and (5-83) over (5-63) yields the expression for the second order approximation for the utility function:

$$U_{t_0} = -\frac{\Omega}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{\lambda_{y_k} \sum_{k=1}^{K} m_k y_{k,t}^2 + \sum_{k=1}^{K} m_k \lambda_{k,\pi} \pi_{k,t}^2\} + T_{t_0} + tips + O_p^3,$$

where

$$y_{k,t} = \hat{Y}_{k,t} - \hat{Y}_{k,t}^*$$

and

$$-\hat{Y}_{k,t}^* = \lambda_{y_k}^{-1} [(q_{yG} + q_{y_k G}) \hat{G}_t + q_{y_k a_k} \hat{a}_{k,t} + q_{y_k \mu_k} \hat{\mu}_{k,t}], \tag{5-84}$$

all k, and, most importantly,

$$\lambda_{y_k} \equiv q_{y_k} + 2q_{y,y_k} + q_y, \tag{5-85}$$

$$\lambda_{k,\pi} \equiv \frac{\theta}{\kappa_k}[1 - \Phi + \frac{\Phi}{\varsigma + \chi}\upsilon], \tag{5-86}$$

while terms such as q_{y_k}, q_y , and q_{y,y_k} are defined from (5-65) to (5-67) and terms such as q_{yG}, q_{y_kG}, $q_{y_ka_k}$ and $q_{y_k\mu_k}$ are defined from (5-75) to (5-78).

5.4 Appendix D - Log-linear Model

5.4.1 Definition of Target Variables

Explicitly using the assumption that sector specific tax rates as well as wage markups in steady state are the same across sectors, we can define the target level of aggregate output using (5-84):

$$-\hat{Y}_t^* = \lambda_{y_k}^{-1}[(q_{yG} + q_{y_kG})\hat{G}_t + q_{y_ka_k}\hat{a}_t + q_{y_k\mu_k}\hat{\mu}_t], \tag{5-87}$$

where coefficients q are defined elsewhere and $\hat{a}_t$ and $\hat{\mu}_t$ are respectively defined as:

$$\hat{a}_t = \sum_{k=1}^{K} m_k \hat{a}_{k,t} \tag{5-88}$$

and

$$\hat{\mu}_t = \sum_{k=1}^{K} m_k \bar{\mu}_{k,t}. \tag{5-89}$$

5.4.2 Aggregate supply and cost-push disturbance term

We take the first order terms of AS equation in (5-29), valid for all k. Adding and subtracting, respectively, the terms referring to overall and sectorial output targets with the appropriate coefficients yield

$$\pi_{k,t} = \kappa_k\{\varsigma y_t + \chi y_{k,t}\} + \beta E_t \pi_{k,t+1} + u_{k,t}, \tag{5-90}$$

for all k, where the definition for the cost-push $u_{k,t}$ is given in terms of primitive shocks as

$$\begin{aligned} u_{k,t} = -\kappa_k\{&[(\varsigma + \chi)\lambda_{y_k}^{-1}(q_{yG} + q_{y_kG}) + \tilde{\sigma}]\hat{G}_t + \varsigma\lambda_{y_k}^{-1} q_{y_ka_k}\hat{a}_t + \\ &+ \varsigma\lambda_{y_k}^{-1} q_{y_k\mu_k}\hat{\mu}_t + [\chi\lambda_{y_k}^{-1} q_{y_ka_k} + \upsilon]\hat{a}_{k,t} + [\chi\lambda_{y_k}^{-1} q_{y_k\mu_k} - 1]\hat{\mu}_{k,t}\}. \end{aligned}$$

5.4.3
Aggregate and Sectorial Output Relations

First order approximation to (5-33) can be redefined in terms of deviation from aggregate and sectorial output targets, yielding

$$y_t = \sum_{k=1}^{K} m_k y_{k,t}. \tag{5-91}$$

First order approximation to aggregate inflation measured by consumer prices is:

$$\pi_t = \sum_{k=1}^{K} m_k \pi_{k,t}. \tag{5-92}$$

In the same way, targeting inflation measure is given by

$$\breve{\pi}_t = \sum_{k=1}^{K} \omega_k \pi_{k,t}. \tag{5-93}$$

Finally, from (5-31)

$$y_t - y_{k,t} = \eta[\pi_{k,t} - \pi_t] + y_{t-1} - y_{k,t-1} + \Delta\zeta_{k,t}, \tag{5-94}$$

where

$$\zeta_{k,t} = \lambda_{y_k}^{-1} q_{y_k a_k}[\hat{a}_t - \hat{a}_{k,t}] + \lambda_{y_k}^{-1} q_{y_k \mu_k}[\hat{\mu}_t - \hat{\mu}_{k,t}]. \tag{5-95}$$

5.4.4
Euler Equation

Taking the first order approximation of the Euler equation in the main text yields

$$\hat{R}_t = \tilde{\sigma} E_t \Delta\hat{Y}_{t+1} - \tilde{\sigma} E_t \Delta\hat{G}_{t+1} + E_t \pi_{t+1} + O_p^2,$$

where we have used the relation in (5-28) to substitute for $\hat{C}_t$ in terms of $\hat{Y}_t$ and $\hat{G}_t$. Expressing equilibrium interest rates in terms of aggregate output gap by using definition in (5-84), which yields

$$\hat{R}_t = \tilde{\sigma} E_t \Delta y_{t+1} + E_t \pi_{t+1} - E_t \Delta r_{t+1}, \tag{5-96}$$

where

$$r_t = \tilde{\sigma}[\lambda_{y_k}^{-1}(q_{yG} + q_{y_k G}) + 1]\hat{G}_t + \tilde{\sigma}\lambda_{y_k}^{-1} q_{y_k a_k}\hat{a}_t + \tilde{\sigma}\lambda_{y_k}^{-1} q_{y_k \mu_k}\hat{\mu}_t. \tag{5-97}$$

5.4.5
Taylor Rule

Taking the first order approximation of the Taylor rule yields

$$\hat{R}_t = \rho_R \hat{R}_{t-1} + (1 - \rho_R)[\phi_\pi \breve{\pi}_t + \phi_y y_t] + e_t. \tag{5-98}$$

5.5
Appendix E - Benchmark Calibration

The following table presents the parameter values for the benchmark calibration, along with its definitions.

Table 5.1: Benchmark Calibration

Symbol	Parameter Definition	Assigned Value
K	Number of Sectors	2
σ	Coeff. of risk aversion	1.1
ν	Inv. of the Frisch elasticity of labor supply	.47
β	Discount parameter	.99
α_k	Calvo prob. of price stickiness	.5
m_k	Sector size	$1/K$
η	Cross-sector elasticity of substitution	1.5
θ	Within-sector elasticity of substitution	11
λ	Desutility of sectorial labor	.98
ρ_g	AR(1) coeff. of fiscal shock	.5
ρ_e	AR(1) coeff. of monetary shock	.5
ρ_{μ_k}	AR(1) coeff. of wage markup shock	.5
ρ_{a_k}	AR(1) coeff. of productivity shock	.5
σ_g	Standard deviation of fiscal shock	.2
σ_e	Standard deviation of monetary shock	.2
σ_{μ_k}	Standard deviation of wage markup shock	.2
σ_{a_k}	Standard deviation of productivity shock	.2
s_C	Steady state consumption over GDP	78%
$\overline{\tau}$	Steady state lump sum tax level over GDP	22%
$\overline{G}$	Steady state gov. expenses over GDP	19.5%
$\overline{\mu}$	Steady state wage markup	5%
$\overline{b}$	Steady state public debt level over GDP	50% (annual)
$\overline{R}$	Steady state interest rate level	4.05% (annual)
ϕ_π	Taylor rule reaction parameter to inflation	1.5
ϕ_y	Taylor rule reaction parameter to output gap	.25
ρ_R	Taylor rule interest rate smooth parameter	.85

For simplicity, only two sectors are considered. Shocks follow an AR(1) defined for any variable x as:

$$x_{t+1} = \rho_x x_t + \varepsilon_{t+1},$$

where ε_t follows a Normal Distribution, with mean zero and variance σ_x^2. Parameters ρ_x are calibrated at .5 for reasons of symmetry. σ_x parameters are calibrated at .2. All other parameters have approximated values of those used in the literature.

5.6
Appendix F - Bayesian Estimation

5.6.1
Sector Weights and Prior Distributions

Table below present the PCE sectors with respective weights. These are averages on the sample period of 1954, last quarter, to the first quarter of 2008. The following table presents the prior distributions of the estimated parameters.

Table 5.2: Sectors of PCE and respective weights

k	Categories	Weight (m_k), in %.
1	Motor vehicles and parts	4.91
2	Furniture and household equipment	2.52
3	Other durable goods	1.71
4	Food	18.94
5	Clothing and shoes	3.69
6	Gasoline, fuel oil, and other energy goods	4.21
7	Other nondurable goods	7.96
8	Housing	16.18
9	Household operation	5.63
10	Transportation	4.19
11	Medical care	14.37
12	Recreation	2.91
13	Other services	12.77

Table 5.3: Prior Distributions

Parameter	Distribution	Prior Mean	Prior Std.
α_k	Uniform $(0, 1)$	.5	.28
ρ_{a_k}	Beta	.5	.2
σ_{a_k}	Inverse Gamma	.01	1
σ_{μ_k}	Inverse Gamma	.01	1
ρ_g	Beta	.8	.1
σ_g	Inverse Gamma	.01	1
σ_e	Inverse Gamma	.01	1
ρ_R	Beta	.85	.1
ϕ_π	Gamma, truncated at 1	1.5	.1
ϕ_y	Gamma, truncated at 0	.25	.05

5.6.2 Estimation Results

This section presents the posterior distributions for the estimated parameters. Other aggregate parameters not displayed are calibrated according to the benchmark values, presented in Appendix E.

Degrees of Nominal Rigidity

Table 5.4: Posterior Distribution - Degrees of Price Stickiness

Categories	Symbol	Posterior Mean	95% Confidence Interval
1	α_1	0.7190	$[0.6966, 0.7412]$
2	α_2	0.6129	$[0.4921, 0.7158]$
3	α_3	0.5648	$[0.5201, 0.6071]$
4	α_4	0.3626	$[0.3140, 0.4127]$
5	α_5	0.3606	$[0.2992, 0.4173]$
6	α_6	0.0291	$[0.0033, 0.0525]$
7	α_7	0.4628	$[0.4097, 0.5146]$
8	α_8	0.5306	$[0.4829, 0.5793]$
9	α_9	0.3680	$[0.3089, 0.4266]$
10	α_{10}	0.1547	$[0.1230, 0.1846]$
11	α_{11}	0.4404	$[0.3754, 0.5041]$
12	α_{12}	0.5528	$[0.5075, 0.5986]$
13	α_{13}	0.2304	$[0.1592, 0.3006]$

Productivity Shock Parameters

Table 5.5: Posterior Distribution - Productivity Shocks: AR(1) Coeffs.

Categories	Symbol	Posterior Mean	95% Confidence Interval
1	ρ_{a_1}	0.0261	$[0.0031, 0.0494]$
2	ρ_{a_2}	0.6394	$[0.4717, 0.7777]$
3	ρ_{a_3}	0.0672	$[0.0066, 0.1236]$
4	ρ_{a_4}	0.4116	$[0.3364, 0.4854]$
5	ρ_{a_5}	0.1735	$[0.0648, 0.2754]$
6	ρ_{a_6}	0.0206	$[0.0024, 0.0384]$
7	ρ_{a_7}	0.3067	$[0.1931, 0.4242]$
8	ρ_{a_8}	0.5690	$[0.5175, 0.6217]$
9	ρ_{a_9}	0.0433	$[0.0066, 0.0795]$
10	$\rho_{a_{10}}$	0.0202	$[0.0024, 0.0376]$
11	$\rho_{a_{11}}$	0.5170	$[0.4520, 0.5823]$
12	$\rho_{a_{12}}$	0.0692	$[0.0093, 0.1266]$
13	$\rho_{a_{13}}$	0.1602	$[0.0402, 0.2788]$

Table 5.6: Posterior Distribution - Productivity Shocks: Std. Deviations

Categories	Symbol	Posterior Mean	95% Confidence Interval
1	σ_{a_1}	0.1074	$[0.0979, 0.1167]$
2	σ_{a_2}	0.0878	$[0.0514, 0.1173]$
3	σ_{a_3}	0.0395	$[0.0348, 0.0439]$
4	σ_{a_4}	0.0257	$[0.0222, 0.0294]$
5	σ_{a_5}	0.0291	$[0.0244, 0.0337]$
6	σ_{a_6}	0.0906	$[0.0833, 0.0980]$
7	σ_{a_7}	0.0241	$[0.0194, 0.0288]$
8	σ_{a_8}	0.0250	$[0.0218, 0.0282]$
9	σ_{a_9}	0.0298	$[0.0269, 0.0326]$
10	$\sigma_{a_{10}}$	0.0433	$[0.0396, 0.0471]$
11	$\sigma_{a_{11}}$	0.0345	$[0.0293, 0.0393]$
12	$\sigma_{a_{12}}$	0.0315	$[0.0278, 0.0350]$
13	$\sigma_{a_{13}}$	0.0290	$[0.0245, 0.0334]$

Wage Markup Shock Parameters

Table 5.7: Posterior Distribution - Wage Markup Shocks: Std. Deviations

Categories	Symbol	Posterior Mean	95% Confidence Interval
1	σ_{μ_1}	0.7232	$[0.6071, 0.8460]$
2	σ_{μ_2}	0.3604	$[0.1711, 0.5155]$
3	σ_{μ_3}	0.2020	$[0.1565, 0.2445]$
4	σ_{μ_4}	0.0588	$[0.0469, 0.0713]$
5	σ_{μ_5}	0.0846	$[0.0636, 0.1047]$
6	σ_{μ_6}	0.0713	$[0.0592, 0.0829]$
7	σ_{μ_7}	0.0680	$[0.0527, 0.0830]$
8	σ_{μ_8}	0.0535	$[0.0415, 0.0656]$
9	σ_{μ_9}	0.0667	$[0.0490, 0.0833]$
10	$\sigma_{\mu_{10}}$	0.0410	$[0.0347, 0.0471]$
11	$\sigma_{\mu_{11}}$	0.0553	$[0.0394, 0.0711]$
12	$\sigma_{\mu_{12}}$	0.1441	$[0.1113, 0.1768]$
13	$\sigma_{\mu_{13}}$	0.0123	$[0.0026, 0.0238]$

Other Estimated Parameters

Table 5.8: Prior Distributions - Other Parameters

Parameter Definition	Posterior Mean	95% Confidence Interval
ρ_g	0.9874	$[0.9828, 0.9921]$
σ_g	0.0814	$[0.0609, 0.1013]$
σ_e	0.0026	$[0.0024, 0.0029]$
ρ_R	0.7329	$[0.7000, 0.7683]$
ϕ_π	1.5197	$[1.3990, 1.6422]$
ϕ_y	0.5372	$[0.4300, 0.6439]$

6
Appendix to Real Business Cycle Dynamics under Rational Inattention

6.1
Appendix A - Definition of Steady State

From the main text, we take the suggested transformation to make the problem stationary. The representative consumer maximizes the transformed utility function:

$$U_{t_0} \equiv E_{t_0} \sum_{t=t_0}^{\infty} \beta^t [\ln \hat{C}_t + \theta \ln(1 - H_t)],$$

subject to the following restriction set:

$$\begin{aligned} & \hat{K}_{e,t}^{\alpha_e} \hat{K}_{s,t}^{\alpha_s} [\hat{A}_t H_t]^{1-a_e-\alpha_s} \\ = \quad & \hat{C}_t + g_{e,t+1} \frac{\hat{K}_{e,t+1}}{\hat{Q}_t} - (1-\delta^e) \frac{\hat{K}_{e,t}}{\hat{Q}_t} + g_{s,t+1} \hat{K}_{s,t+1} - (1-\delta^s) \hat{K}_{s,t}, \end{aligned}$$

$$\text{given } \hat{K}_{e,t} \text{ and } \hat{K}_{s,t}.$$

The objective is to show that there is a deterministic steady state for the detrended system above, where all endogenous variables assume constant values. FOCs are given by:

– with respect to $\hat{C}_t$:

$$1/\hat{C}_t = \hat{\lambda}_t; \tag{6-1}$$

– with respect to H_t:

$$\theta \frac{H_t}{1 - H_t} = \hat{\lambda}_t (1 - a_e - \alpha_s) \hat{K}_{e,t}^{\alpha_e} \hat{K}_{s,t}^{\alpha_s} [\hat{A}_t H_t]^{1-a_e-\alpha_s}; \tag{6-2}$$

– with respect to $\hat{K}_{e,t+1}$:

$$\frac{\hat{\lambda}_t}{\hat{Q}_t}E_t(g_{e,t+1}) = E_t\hat{\lambda}_{t+1}\beta\{\alpha_e\hat{K}_{e,t+1}^{\alpha_e-1}\hat{K}_{s,t+1}^{\alpha_s}[\hat{A}_{t+1}H_{t+1}]^{1-a_e-\alpha_s} + \frac{(1-\delta^e)}{\hat{Q}_{t+1}}\}; \quad (6\text{-}3)$$

– with respect to $\hat{K}_{s,t+1}$:

$$\hat{\lambda}_t E_t(g_{s,t+1}) = E_t\hat{\lambda}_{t+1}\beta\{\alpha_s\hat{K}_{e,t+1}^{\alpha_e}\hat{K}_{s,t+1}^{\alpha_s-1}[\hat{A}_{t+1}H_{t+1}]^{1-a_e-\alpha_s} + (1-\delta^s)\}. \quad (6\text{-}4)$$

These equations, along with restrictions (3-3), (3-5), (3-6), (3-4) in the main text, can be use to determine the steady state values for endogenous variables. In order to do that, we need to show that FOCs are satisfied for time-invariant Lagrange multipliers. We start by noticing that, once stochastic terms are dropped out, there are no deviations from productivity factors from their (constant) growth trends. Also, given no population growth, $\hat{H} = H$. Other endogenous variables assume constant values. Therefore, we can drop the subscripts and expectation operators. From (6-1), $\hat{\lambda}$ is constant:

$$1/\hat{C} = \hat{\lambda};$$

We can use this fact over expressions (6-3) (6-4), yielding, respectively:

$$\frac{\hat{K}_e}{\hat{Y}} = \alpha_e[g_e\beta^{-1} - (1-\delta^e)]^{-1}, \quad (6\text{-}5)$$

$$\frac{\hat{K}_s}{\hat{Y}} = \alpha_s[g_s\beta^{-1} - (1-\delta^e)]^{-1}, \quad (6\text{-}6)$$

where $g_s = \gamma_a\gamma_q^{\alpha_e/(1-\alpha_e-\alpha_s)}$ and $g_e = \gamma_a\gamma_q^{1+\alpha_e/(1-\alpha_e-\alpha_s)}$.These establish the steady state level of capital stocks over GDP in terms of exogenous parameters, where we have used the relation in (3-3). Investment over GDP can be established from (3-5) and (3-6):

$$\frac{\hat{I}_e}{\hat{Y}} = [g_e - (1-\delta^e)]\frac{\hat{K}_e}{\hat{Y}}, \quad (6\text{-}7)$$

$$\frac{\hat{I}_s}{\hat{Y}} = [g_s - (1-\delta^s)]\frac{\hat{K}_s}{\hat{Y}}. \quad (6\text{-}8)$$

Using the previous results, we can use the demand equation (3-4) in order to determine consumption over GDP in terms of exogenous parameters:

$$1 = \frac{\hat{C}}{\hat{Y}} + \frac{\hat{I}_e}{\hat{Y}} + \frac{\hat{I}_s}{\hat{Y}}. \quad (6\text{-}9)$$

From (6-2), it is then possible to establish the steady state level of labor hours using the previous result:

$$\theta \frac{H}{1-H} \frac{\hat{C}}{\hat{Y}} = (1 - a_e - \alpha_s). \tag{6-10}$$

Finally, it is possible to recover the level of output in steady state using (3-3) and the previous results:

$$\hat{Y}^{1-a_e-\alpha_s} = \left(\frac{\hat{K}_e}{\hat{Y}}\right)^{\alpha_e} \left(\frac{\hat{K}_e}{\hat{Y}}\right)^{\alpha_s} [H]^{1-a_e-\alpha_s}. \tag{6-11}$$

6.2
Appendix B - The Quadratic Policy Problem

6.2.1
Second Order Approximation to Objective Function and Restrictions

From the previous section, we use the deterministic steady state for the detrended problem in order to establish an approximation point, hereby characterized by hat-variables *without* the subscript t. We follow Benigno and Woodford (2006, 2008) by applying a second order Taylor expansion for the objective function and restrictions. The objective is to define a purely quadratic approximation to the objective function and a set o linear restrictions that result on policy functions for the policy problem, equivalent to the ones produced by a second order approximation for both objective function and restrictions.

Second order approximation on objective function yields:

$$u(\hat{C}_t, H_t) = \tilde{C}_t - \theta\varphi\tilde{H}_t - \frac{1}{2}\tilde{C}^2 + \frac{\theta}{2}\varphi^2\tilde{H}_t^2 + tips + O_p^3,$$

where "*tips*" stands for "*terms independent of policy*" and φ is defined by $\varphi \equiv H/(1-H)$. Also, for any original variable X_t, denote:

$$\tilde{X}_t = \frac{(\hat{X}_t - \hat{X})}{\hat{X}},$$

where $\hat{X}$ (without t-subscript) denotes the steady state level for the detrended problem described in the previous section and $\hat{X}_t$ the detrended variable itself. The following relation applies up to second order:

$$\tilde{X}_t = \hat{x}_t + \frac{1}{2}\hat{x}_t^2,$$

where

$$\hat{x}_t = \ln(\hat{X}_t/\hat{X}).$$

Substitution on the original expression results:

$$u(\hat{C}_t, H_t) = c_t - \theta\varphi h_t - \frac{\theta\varphi}{2}(1+\varphi)h_t^2 + tips + O_p^3, \tag{6-12}$$

which give variables in terms of log-deviations from their steady state levels. We proceed by log-linearizing the restrictions to the policy problem.

– Technology:

$$\begin{aligned} \hat{y}_t + \frac{1}{2}\hat{y}_t^2 &= \alpha_e \hat{k}_{e,t} + \alpha_s \hat{k}_{s,t} + (1-\alpha_e-\alpha_s)(\hat{a}_t + \hat{h}_t) + \\ &+ \frac{1}{2}\left[\alpha_e \hat{k}_{e,t} + \alpha_s \hat{k}_{s,t} + (1-\alpha_e-\alpha_s)(\hat{a}_t + \hat{h}_t)\right]^2 + O_p^3. \end{aligned} \tag{6-13}$$

– Law of motion for stocks on equipment and structures:

$$\begin{aligned} \frac{\hat{I}_s}{\hat{K}_s}[\hat{\imath}_{s,t} + \frac{1}{2}\hat{\imath}_{s,t}^2] &= g_s(\hat{k}_{s,t+1} + \varepsilon_{a,t+1}^P + \frac{\alpha_e}{\alpha_h}\varepsilon_{q,t+1}^P) - (1-\delta^s)\hat{k}_{s,t} \\ &+ \frac{1}{2}[g_s(\hat{k}_{s,t+1} + \varepsilon_{a,t+1}^P + \frac{\alpha_e}{\alpha_h}\varepsilon_{q,t+1}^P)^2 + \\ &-(1-\delta^s)\hat{k}_{s,t}^2] + O_p^3 \end{aligned} \tag{6-14}$$

and

$$\begin{aligned} \frac{\hat{I}_e}{\hat{K}_e}[\hat{\imath}_{e,t} + \frac{1}{2}\hat{\imath}_{e,t}^2] &= g_e[\hat{k}_{e,t+1} - \hat{q}_t + \varepsilon_{a,t+1}^P + (1+\frac{\alpha_e}{\alpha_h})\varepsilon_{q,t+1}^P] + \\ &-(1-\delta^e)\hat{k}_{e,t} \\ &+ \frac{1}{2}\{g_e[\hat{k}_{e,t+1} - \hat{q}_t + \varepsilon_{a,t+1}^P + (1+\frac{\alpha_e}{\alpha_h})\varepsilon_{q,t+1}^P]^2 + \\ &-(1-\delta^e)(\hat{k}_{e,t} - \hat{q}_t)^2\} + O_p^3, \end{aligned} \tag{6-15}$$

where we have defined for notational convenience:

$$\alpha_h \equiv 1 - \alpha_e - \alpha_s.$$

– Finally, second-order approximation on demand equation yields:

$$\hat{y}_t + \frac{1}{2}\hat{y}_t^2 = s_c\hat{c}_t + s_{i_s}\hat{\imath}_{s,t} + s_{i_e}\hat{\imath}_{e,t} + \frac{1}{2}\left[s_c\hat{c}_t^2 + s_{i_s}\hat{\imath}_{s,t}^2 + s_{i_e}\hat{\imath}_{e,t}^2\right] + O_p^3. \tag{6-16}$$

where:

$$s_c \equiv \frac{\hat{C}}{\hat{Y}},$$

$$s_{i_e} \equiv \frac{\hat{I}_e}{\hat{Y}},$$

$$s_{i_s} \equiv \frac{\hat{I}_s}{\hat{Y}}.$$

We can combine expressions in (6-13) and (6-16) with (6-14), and use (6-15) in order to obtain a set of two restrictions, respectively:

$$\begin{aligned} 0 = & -\alpha_e \hat{k}_{e,t} - \alpha_s \hat{k}_{s,t} - \alpha_h(\hat{a}_t + \hat{h}_t) + s_c \hat{c}_t + s_{i_e} \hat{\imath}_{e,t} + \\ & + s_{k_s} g_s(\hat{k}_{s,t+1} + \varepsilon^P_{a,t+1} + \frac{\alpha_e}{\alpha_h}\varepsilon^P_{q,t+1}) - s_{k_s}(1-\delta^s)\hat{k}_{s,t} \\ & -\frac{1}{2}\left[\alpha_e \hat{k}_{e,t} + \alpha_s \hat{k}_{s,t} + \alpha_h(\hat{a}_t + \hat{h}_t)\right]^2 + \frac{s_c}{2}\hat{c}_t^2 + \frac{s_{i_e}}{2}\hat{\imath}_{e,t}^2 \\ & + \frac{s_{k_s}}{2}[g_s(\hat{k}_{s,t+1} + \varepsilon^P_{a,t+1} + \frac{\alpha_e}{\alpha_h}\varepsilon^P_{q,t+1})^2 - (1-\delta^s)\hat{k}_{s,t}^2] + O_p^3, \end{aligned} \tag{6-17}$$

and

$$\begin{aligned} 0 \quad = \quad & -s_{i_e}(\hat{\imath}_{e,t} + \hat{q}_t) - \frac{s_{i_e}}{2}(\hat{\imath}_{e,t} + \hat{q}_t)^2 - s_{k_e}(1-\delta^e)\hat{k}_{e,t} + \\ & + s_{k_e} g_e(\hat{k}_{e,t+1} + \varepsilon^P_{a,t+1} + (1 + \frac{\alpha_e}{\alpha_h})\varepsilon^P_{q,t+1}) + \\ & + \frac{s_{k_e}}{2}[g_e(\hat{k}_{e,t+1} + \varepsilon^P_{a,t+1} + (1 + \frac{\alpha_e}{\alpha_h})\varepsilon^P_{q,t+1})^2 - (1-\delta^e)\hat{k}_{e,t}^2] \\ & + O_p^3. \end{aligned} \tag{6-18}$$

where:

$$s_{k_s} \equiv \frac{\hat{K}_s}{\hat{Y}},$$

$$s_{k_e} \equiv \frac{\hat{K}_e}{\hat{Y}}.$$

By adding and subtracting the proper terms and using the definition for steady state variables, (6-17) can be written recursively, such that:

$$V_{s,t} = F(\hat{c}_t, \hat{h}_t, \hat{\imath}_{e,t}, \hat{k}_{e,t}, \hat{k}_{s,t}, \boldsymbol{\xi}_t) + \beta V_{s,t+1}, \tag{6-19}$$

where $F(.)$ is a linear-quadratic function of log-deviation of endogenous variables and the vector of exogenous shocks $\boldsymbol{\xi}_t$ at t, defined as:

$$
\begin{aligned}
F(.) &= \bar{f}\{-\alpha_e\hat{k}_{e,t} - \alpha_h\hat{h}_t + s_c[\hat{c}_t + \frac{1}{2}\hat{c}_t^2] + s_{i_e}[\hat{\imath}_{e,t} + \frac{1}{2}\hat{\imath}_{e,t}^2] + \\
&+ \frac{\alpha_s}{2}\hat{k}_{s,t}^2 - \frac{1}{2}[\alpha_e\hat{k}_{e,t} + \alpha_s\hat{k}_{s,t} + \alpha_h(\hat{a}_t + \hat{h}_t)]^2\} + \\
&+ tips + O_p^3
\end{aligned}
$$

where:

$$\bar{f} \equiv [\alpha_s + s_{k_s}(1-\delta^s)]^{-1},$$

and the pre-determined term $V_{s,t}$ is defined as

$$V_{s,t} \equiv \hat{k}_{s,t} + \frac{1}{2}\hat{k}_{s,t}^2.$$

One could notice that interactions between current capital stock and i.i.d innovations of permanent shocks have been included at "*tips*".

Proceeding in an analogous way for (6-18), we have

$$V_{e,t} = G(\hat{c}_t, \hat{h}_t, \hat{\imath}_{e,t}, \hat{k}_{e,t,}\hat{k}_{s,t}, \boldsymbol{\xi}_t) + \beta V_{e,t+1}. \tag{6-20}$$

where:

$$
\begin{aligned}
G(.) &= \bar{g}\{-s_{i_e}[(\hat{\imath}_{e,t} + q_t) + \frac{1}{2}(\hat{\imath}_{e,t} + q_t)^2] + \\
&+ \alpha_e[\hat{k}_{e,t} + \frac{1}{2}\hat{k}_{e,t}^2]\} + tips + O_p^3,
\end{aligned}
$$

$$\bar{g} \equiv [\alpha_e + s_{k_e}(1-\delta^e)]^{-1},$$

$$V_{e,t} \equiv \hat{k}_{e,t} + \frac{1}{2}\hat{k}_{e,t}^2.$$

6.2.2 Elimination of Linear terms

By follow Benigno and Woodford (2008) we can use matrix notation in order to rewrite expressions (6-12), (6-19) and (6-20). We start by noticing the corresponding log-approximation of the (detrended) policy problem can be stated in the following way:

$$\max_{\boldsymbol{c}_t} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} u(c_t) \tag{6-21}$$

subject to intertemporal restrictions

$$F_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} F(c_t, k_t, \boldsymbol{\xi}_t) \tag{6-22}$$

and

$$G_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} G(c_t, k_t, \boldsymbol{\xi}_t), \tag{6-23}$$

where F_{t_0} and G_{t_0} are predetermined terms at t_0 and therefore independent of policy from that date on. Notice that (6-22) and (6-23) are obtained by iterating forward expressions (6-19) and (6-20). Vector definitions are:

$$c_t = \begin{bmatrix} \hat{c}_t \\ \hat{h}_t \\ \hat{\imath}_{e,t} \end{bmatrix};\ k_t = \begin{bmatrix} \hat{k}_{e,t} \\ \hat{k}_{s,t} \end{bmatrix};\ \boldsymbol{\xi}_t = \begin{bmatrix} \hat{a}_t \\ \hat{q}_t \\ \varepsilon^P_{a,t} \\ \varepsilon^P_{q,t} \end{bmatrix}.$$

Equation (6-21) can be expressed in matrix notation as:

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [U_c.c_t + \frac{1}{2} c_t' U_{cc}.c_t] + tips + O_p^3. \tag{6-24}$$

The following definitions for the underlined terms apply:

$$U_c = \begin{bmatrix} 1 & -\theta\varphi & 0 \end{bmatrix};$$

$$U_{cc} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\theta\varphi[1-\varphi] & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In the same fashion, restriction (6-22) is expressed as:

$$\begin{aligned} 0 \;=\; & \sum_{t=t_0}^{\infty} \beta^{t-t_0} \{ \bar{\lambda}(F_c.c_t + F_k.k_t) + \frac{1}{2}\bar{\lambda}[c_t' F_{cc}.c_t + \\ & +2c_t' F_{c\xi}.\boldsymbol{\xi}_t + 2k_t' F_{k\xi}.\boldsymbol{\xi}_t + k_t' F_{kk}.k_t + 2c_t' F_{ck}.k_t]\} \\ & -\bar{\lambda} F_{t_0} + tips + O_p^3 \end{aligned} \tag{6-25}$$

where $\bar{\lambda}$ is the associated Lagrange multiplier to be determined. Recall the definition of $\bar{f}$ as

$$\bar{f} \equiv [\alpha_s + s_{k_s}(1-\delta^s)]^{-1}.$$

Matrices can then be expressed as:

$$F_c = \bar{f}.\begin{bmatrix} s_c & -\alpha_h & s_{i_e} \end{bmatrix};$$

$$F_k = \bar{f}.\begin{bmatrix} -\alpha_e & 0 \end{bmatrix};$$

$$F_{cc} = \bar{f}.\begin{bmatrix} s_c & 0 & 0 \\ 0 & -\alpha_h^2 & 0 \\ 0 & 0 & s_{i_e} \end{bmatrix};$$

$$F_{kk} = \bar{f}.\begin{bmatrix} -\alpha_e^2 & -\alpha_e\alpha_s \\ -\alpha_e\alpha_s & -\alpha_s(\alpha_s - 1) \end{bmatrix};$$

$$F_{c\xi} = \bar{f}.\begin{bmatrix} 0 & 0 & 0 & 0 \\ -\alpha_h^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix};$$

$$F_{k\xi} = \bar{f}.\begin{bmatrix} -\alpha_e\alpha_h & 0 & 0 & 0 \\ -\alpha_s\alpha_h & 0 & 0 & 0 \end{bmatrix};$$

$$F_{ck} = \bar{f}.\begin{bmatrix} 0 & 0 \\ -\alpha_e\alpha_h & -\alpha_s\alpha_h \\ 0 & 0 \end{bmatrix}.$$

Proceeding in the same way, restriction (6-23) can be expressed:

$$\begin{aligned} 0 \quad = \quad & \sum_{t=t_0}^{\infty} \beta^{t-t_0}\{\bar{\vartheta}(G_c.c_t + G_k.k_t) + \frac{1}{2}\bar{\vartheta}[c_t' G_{cc}.c_t + \\ & +2c_t' G_{c\xi}.\boldsymbol{\xi}_t + 2k_t' G_{k\xi}.\boldsymbol{\xi}_t + k_t' G_{kk}.k_t + 2c_t' G_{ck}.k_t]\} \\ & -\bar{\vartheta} G_{t_0} + tips + O_p^3 \end{aligned} \tag{6-26}$$

where notational choices are analogous as above and $\bar{\vartheta}$ is the associate Lagrange multiplier to be determined. Noticing that we have defined:

$$\bar{g} = [\alpha_e + s_{k_e}(1-\delta^e)]^{-1}.$$

Matrices can then be defined:

$$G_c = \bar{g}.\begin{bmatrix} 0 & 0 & -s_{i_e} \end{bmatrix};$$

$$G_k = \bar{g}.\begin{bmatrix} \alpha_e & 0 \end{bmatrix};$$

$$G_{cc} = \bar{g}. \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -s_{i_e} \end{bmatrix};$$

$$G_{kk} = \bar{g}. \begin{bmatrix} \alpha_e & 0 \\ 0 & 0 \end{bmatrix};$$

$$G_{c\xi} = \bar{g}. \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -s_{i_e} & 0 & 0 \end{bmatrix};$$

$$G_{k\xi} = \vec{0};$$

$$G_{ck} = \vec{0}.$$

Constants $\bar{\lambda}$ and $\bar{\vartheta}$ are then defined in such a way that the following holds:

$$\bar{\lambda}(F_c + F_k) + \bar{\vartheta}(G_c + G_k) = -U_c.$$

The solution is, therefore:

$$\bar{\lambda} = -\frac{1}{\bar{f}.s_c} \tag{6-27}$$

and

$$\bar{\vartheta} = -\frac{1}{\bar{g}.s_c} \tag{6-28}$$

By using the definitions for $\bar{\lambda}$ and $\bar{\vartheta}$, it is possible to show that the following relation holds up to second order:

$$\begin{aligned} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0}[U_c.c_t] &= \frac{1}{2} \sum_{t=t_0}^{\infty} \beta^{t-t_0}[c_t' H_{cc}.c_t + k_t' H_{kk}.k_t + \\ &\quad +2c_t' R.k_t + 2c_t' Z_{c\xi}.\boldsymbol{\xi}_t + 2k_t' Z_{k\xi}.\boldsymbol{\xi}_t] + T_0 + tips, \end{aligned}$$

while we have defined the new terms as:

$$H_{cc} \equiv \bar{\lambda} F_{cc} + \bar{\vartheta} G_{cc},$$

$$H_{kk} \equiv \bar{\lambda} F_{kk} + \bar{\vartheta} G_{kk},$$

$$R \equiv \bar{\lambda} F_{ck} + \bar{\vartheta} G_{ck},$$

$$Z_{c\xi} \equiv \bar{\lambda} F_{c\xi} + \bar{\vartheta} G_{c\xi},$$

$$Z_{k\xi} \equiv \bar{\lambda} F_{k\xi} + \bar{\vartheta} G_{k\xi},$$

and

$$T_0 = -(\bar{\lambda} F_{t_0} + \bar{\vartheta} G_{t_0}).$$

Plugging this last expression into (6-24), yields:

$$U_{t_0} = \frac{1}{2} E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} [c_t' Q.c_t + k_t' H_{kk}.k_t + 2c_t' R.k_t + 2c_t' Z_{c\xi}.\boldsymbol{\xi}_t + 2k_t' Z_{k\xi}.\boldsymbol{\xi}_t], \quad (6\text{-}29)$$

where:

$$Q \equiv U_{cc} + H_{cc},$$

and Z_{xy} and R are defined elsewhere.

6.2.3
Additional Simplifications

Consider now the following definition for the vector of state variables S_t and control variables:

$$S_t \equiv \begin{bmatrix} \hat{k}_{e,t} \\ \hat{k}_{s,t} \\ \hat{a}_t \\ \hat{q}_t \end{bmatrix}; \qquad c_t \equiv \begin{bmatrix} \hat{c}_t \\ \hat{h}_t \\ \hat{\imath}_{e,t} \end{bmatrix}.$$

Expression (6-29) can finally be expresses as a quadratic objective function in terms of control and state variables as

$$U_{t_0} = E_{t_0} \sum_{t=t_0}^{\infty} \beta^{t-t_0} \left\{ c_t' B c_t + 2c_t' D S_t + S_t' A S_t \right\}, \quad (6\text{-}30)$$

where:

$$B \equiv Q$$

$$A \equiv \begin{bmatrix} H_{kk} & Z_{k\xi} \\ Z'_{k\xi} & tips \end{bmatrix}$$

and

$$D \equiv \begin{bmatrix} R & Z_{c\xi} \end{bmatrix}.$$

In addition to the objective function given in (6-30), the decisionmaker is subject to the following set of linear constraints,

$$\begin{aligned} \hat{k}_{e,t+1} &= \frac{(1-\delta^e)}{g_e}\hat{k}_{e,t} + \frac{s_{i_e}}{s_{k_e}g_e}\hat{\imath}_{e,t} + \\ &+[1-\frac{(1-\delta^e)}{g_e}]\hat{q}_t - \varepsilon^P_{a,t+1} - (1+\frac{\alpha_e}{\alpha_h})\varepsilon^P_{q,t+1}, \end{aligned}$$

$$\begin{aligned} \hat{k}_{s,t+1} &= \frac{(1-\delta^s)}{g_s}\hat{k}_{s,t} + \frac{1}{g_s s_{k_s}}[\alpha_e\hat{k}_{e,t} + \alpha_s\hat{k}_{s,t} + \alpha_h(\hat{a}_t + \hat{h}_t) + \\ &-s_c\hat{c}_t - s_{i_e}\hat{\imath}_{e,t}] - \varepsilon^P_{a,t+1} - \frac{\alpha_e}{\alpha_h}\varepsilon^P_{q,t+1}. \end{aligned}$$

and the following AR(1) processes for stationary component of exogenous shocks:

$$\hat{a}_{t+1} = \rho_a\hat{a}_t + \varepsilon^T_{a,t+1},$$

$$\hat{q}_{t+1} = \rho_q\hat{q}_t + \varepsilon^T_{q,t+1}.$$

Restrictions can then be written in matrix notation according to the following:

$$S_{t+1} = G_1S_t + G_2c_t + G_3\varepsilon_{t+1}. \tag{6-31}$$

where ε_t stands for a vector of i.i.d. innovations to shocks, or $\varepsilon_t = [\varepsilon^T_{a,t}, \varepsilon^T_{q,t}, \varepsilon^P_{a,t}, \varepsilon^P_{q,t}]'$. Expression above represents the law of motion for state variables in terms of its past values, present control variables and exogenous shocks. Matrices are defined by:

$$G_1 \equiv \begin{bmatrix} G^1_{11} & G^1_{12} \\ 0_{(2x2)} & G^1_{22} \end{bmatrix};$$

$$G_2 \equiv \begin{bmatrix} G^2_{11} & G^2_{12} \\ 0_{(2x2)} & 0_{(1x2)} \end{bmatrix};$$

$$G_3 \equiv \begin{bmatrix} 0_{(2x2)} & G_{12}^3 \\ I_{(2x2)} & 0_{(2x2)} \end{bmatrix};$$

where I is an identity matrix. In particular, we have:

$$G_{11}^1 = \begin{bmatrix} (1-\delta^e)/g_e & 0 \\ \alpha_e/(s_{k_s} g_s) & \alpha_s/(s_{k_s} g_s) + (1-\delta^s)/g_s \end{bmatrix};$$

$$G_{12}^1 = \begin{bmatrix} 0 & 1-(1-\delta^e)/g_e \\ \alpha_h/(s_{k_s} g_s) & 0 \end{bmatrix};$$

$$G_{22}^1 = \begin{bmatrix} \rho_a & 0 \\ 0 & \rho_q \end{bmatrix};$$

$$G_{11}^2 = \begin{bmatrix} 0 & 0 \\ -s_c/(s_{k_s} g_s) & \alpha_h/(s_{k_s} g_s) \end{bmatrix};$$

$$G_{12}^2 = \begin{bmatrix} s_{i_e}/(s_{k_e} g_e) \\ -s_{i_e}/(s_{k_s} g_s) \end{bmatrix};$$

$$G_{12}^3 = \begin{bmatrix} -1 & -(1+\alpha_e/\alpha_h) \\ -1 & -\alpha_e/\alpha_h \end{bmatrix}.$$

The final problem reduces to maximize(6-30) subject to (6-31).

6.3
Appendix C - Derivation of Optimal Signal

6.3.1
Value Function

The problem with full information is given by:

$$\max_{c_t} E_{t_0} \left[\sum_{t=0}^{\infty} \beta^{t-t_0} (S_t' A S_t + 2 S_t' D c_t + c_t' B c_t) \right]$$

s.t.:

$$S_{t+1} = G_1 S_t + G_2 c_t + G_3 \varepsilon_{t+1}$$

where c_t is the vector of control variables and S_t the full vector of state variables. We assume further that ε_t has a multidimensional Normal distribution, such that $E(\varepsilon_t) = 0$ and $E(\varepsilon_t' \varepsilon_t) = \Omega$, all t. The LQ problem has some desired features: policy function is linear on state variables and, hence, certainty equivalence applies. Information frictions does not change the problem (Sims, 2003a). In addition, the value function is quadratic. This property is desirable to show that the optimal signal has a Gaussian distribution.

In order to do that, the first stage is to solve the deterministic problem. Writing the objective equation recursively and replacing the restriction in the objective function, we have:

$$V(S_t) = \max_{c_t}\{S_t'AS_t + 2S_t'Dc_t + c_t'Bc_t + \beta E_t V(G_1S_t + G_2c_t + G_3\varepsilon_{t+1})\}.$$

Conjecture: The value function in quadratic in the state vector, or $V(S_t) = S_t'P_1S_t + P_2S_t + d$, P_1 following the properties described by Benigno and Woodford (2008) and d an unknown constant.

Then, we can write the expression above as:

$$\begin{aligned} S_t'P_1S_t + P_2S_t + d \;\; = \;\; & \max_{c_t}\{S_t'AS_t + 2S_t'Dc_t + c_t'Bc_t + \beta d + \\ & +\beta S_t'G_1'P_1G_1S_t + \beta[S_t'G_1'P_1G_2c_t + c_t'G_2'P_1G_1S_t + \\ & +\beta c_t'G_2'P_1G_2c_t + \beta tr(P_1G_3\Omega G_3') + \\ & +\beta P_2G_1S_t + \beta P_2G_2c_t\}, \end{aligned}$$

after evaluating conditional expectations and exploring the fact that ε_t is i.i.d. and $E(\varepsilon_t) = 0$. We next take FOC with respect to the control variable vector c_t. It is clear that the resulting policy function is indeed linear:

$$c_t = H_0 + H_1S_t,$$

where

$$\begin{aligned} H_0 &= -2[B + \beta G_2'P_1G_2]^{-1}[\beta G_2'P_2'], \\ H_1 &= -[B + \beta G_2'P_1G_2]^{-1}[\beta G_2'P_1G_1 + D'], \end{aligned}$$

a function both of P_1 and P_2. Replacing the policy function back to value function, it is possible to determine the values for P_1, P_2 and d. Define:

$$\hat{\Omega} \equiv G_3\Omega G_3'. \tag{6-32}$$

For, d:

$$\begin{aligned}(1-\beta)d &= H_0'BH_0 + \beta tr(P_1\hat{\Omega}) + \\ &\quad +\beta H_0'G_2'P_1G_2H_0 + \beta P_2G_2H_0,\end{aligned}$$

For P_2:

$$P_2 = \vec{0}.$$

For P_1:

$$\begin{aligned}P_1 = A + 2DH_1 + H_1'BH_1 + \beta G_1'P_1G_1 + \\ +2\beta H_1'G_2'P_1G_1 + \beta H_1'G_2'P_1G_2H_1.\end{aligned} \tag{6-33}$$

Equation (6-33) describes P_1 recursively, a matrix Riccati equation:

$$\begin{aligned}P_1(s+1) &= A + \beta G_1'P_1(s)G_1 + \\ &\quad -(D' + \beta G_2'P_1(s)G_1)'(B + \beta G_2'P_1(s)G_2)^{-1}(D' + \beta G_2'P_1(s)G_1)\end{aligned} \tag{6-34}$$

It can be solved by iterating the matrix difference equation starting from some initial value and converging to a fixed point or using a method based on eigenvalue-eigenvector decomposition (such as Blanchard-Quah). Finally, $d = \beta tr(P_1\hat{\Omega})/(1-\beta)$. Equations defining the value function are independent of S_t, which means that the value function given by the problem with information friction is analogous: $\hat{V}(\hat{S}_t) = \hat{S}_t'P_1\hat{S}_t + d$.

6.3.2
Gaussianity of Optimal Signal

Define the welfare loss in t due to imperfect information as $\Delta V_t = V(S_t) - \hat{V}(\hat{S}_t)$. The expected welfare loss is given by[1]:

$$E_t[\Delta V_t] = E_t[V(S_t) - \hat{V}(\hat{S}_t)].$$

Substituting for $V(S_t)$ and $\hat{V}(\hat{S}_t)$, and noticing that $E_t(S_t) = E_t(\hat{S}_t)$, yields:

$$E_t[\Delta V_t] = -E_t[(S_t - \hat{S}_t)'P_1(S_t - \hat{S}_t)] + 2E_t[S_t'P_1(S_t - \hat{S}_t)],$$

[1]For simplicity, we use $E_t[.]$ as short for $E_t[.|\mathcal{I}_t]$.

Considering that

$$E_t[(S_t - \hat{S}_t)] = 0,$$

by hypothesis, then it is clear that

$$E_t[S_t' P_1(S_t - \hat{S}_t)] = P_1 Cov[S_t, (S_t - \hat{S}_t)] = 0.$$

Then:

$$E_t[\Delta V_t] = -E_t[(S_t - \hat{S}_t)' P_1 (S_t - \hat{S}_t)].$$

The problem then becomes to choose a joint distribution of state variables and signals that minimize the loss function

$$\min_{q(S_t, \hat{S}_t)} -E_t[(S_t - \hat{S}_t)' P_1 (S_t - \hat{S}_t)],$$

subject to:

$$-\mathcal{H}(S_t, \hat{S}_t) + \mathcal{H}(\hat{S}_t) + \mathcal{H}(S_t) \leq 2\kappa$$

plus the conditions on $q(S_t, \hat{S}_t)$ being a pdf. $\mathcal{H}$ corresponds to the definition of entropy and κ is the channel capacity on the mutual information between S_t and $\hat{S}_t$. More explicitly, we can apply the definition of entropy to the problem above, yielding:

$$\min_{q(S_t, \hat{S}_t)} - \int\int (S_t - \hat{S}_t)' P_1 (S_t - \hat{S}_t) q(S_t, \hat{S}_t) dS_t d\hat{S}_t,$$

subject to:

$$\begin{aligned} & \int\int \log[q(S_t, \hat{S}_t)] q(S_t, \hat{S}_t) dS_t d\hat{S}_t - \int \log[q(S_t)] q(S_t) dS_t + \\ & - \int \left[\log \left[\int q(S, \hat{S}_t) dS_t \right] \int q(S, \hat{S}_t) dS_t \right] d\hat{S}_t \\ \leq \; & 2\kappa, \end{aligned}$$

$$\int q(S_t, \hat{S}_t) d\hat{S}_t = q(S_t),$$

and

$$q(S_t, \hat{S}_t) \geq 0.$$

We disregard this last restriction, assuming that it always holds. Following Sims (2003b) and Luo (2006), the maximization can be carried out pointwise by taking derivatives with respect to $q(S_t, \hat{S}_t)$. FOC yields:

$$-(S_t - \hat{S}_t)'P_1(S_t - \hat{S}_t) - \lambda\{1 + \log[q(S_t, \hat{S}_t)] - 1 - \log\left[\int q(S, \hat{S}_t)dS_t\right]\} - \mu = 0.$$

Define:

$$q(S_t|\hat{S}_t) \equiv \frac{q(S_t, \hat{S}_t)}{\int q(S_t, \hat{S}_t)dS_t},$$

then:

$$q(S_t|\hat{S}_t) = \theta_0 e^{\theta_1 (S_t - \hat{S}_t)' P_1 (S_t - \hat{S}_t)}.$$

where θ_0 and can θ_1 be conveniently chosen so as the right-hand side is a multivariate normal distribution. The result implies that it is optimal to choose the joint distribution of S_t and $\hat{S}_t$ in such a way that the conditional distribution of the state variable given the signal is a multivariate normal distribution:

$$q(S_t|\hat{S}_t) \sim N(\hat{S}_t, \Sigma).$$

The infinite dimensional problem simplifies to one in which it is only necessary to establish the variance-covariance matrix of the posterior distribution of state variables given the signal, Σ.

6.3.3 Determination of Σ

Following Sims(2003a), the loss function can be written as:

$$\begin{aligned} E_t[V(S_t) - \hat{V}(\hat{S}_t)] &= -tr([A + DH_1 + H_1'BH_1]\Sigma) + \\ &+\beta E_t[V(S^*_{t+1}) - V(S_{t+1}) + (S_{t+1}) - \hat{V}(\hat{S}_{t+1})] \end{aligned} \quad (6\text{-}35)$$

where $S^*_{t+1} = (G_1 + G_2H_1)S_t + G_3\varepsilon_t$ is the value of state variables that would emerge in the case where control variables are chosen optimally and without uncertainty upon the true state at t: S_t. Note that S_{t+1} is the true value of the state vector at $t+1$ when control variables are chosen under information capacity constraint, that is, the state at t is merely perceived: $\hat{S}_t$. Define

$$S^*_{t+1} - S_{t+1} = G_2 H_1 (S_t - \hat{S}_t)].$$

Because of the LQ structure, the left-hand side is constant. Expression simplifies to:

$$(1-\beta)M = -tr([A + DH_1 + H_1' B H_1]\Sigma) + \\ -\beta E_t[(S^*_{t+1} - S_{t+1})' P_1 (S^*_{t+1} - S_{t+1}) + 2(S^*_{t+1} - S_{t+1})' P_1 S_{t+1}] \quad (6\text{-}36)$$

By replacing the definition above, one gets:

$$(1-\beta)M = -tr(W\Sigma),$$

where W is given by:

$$W = A + DH_1 + H_1' B H_1 + \beta(H_1' G_2' P_1 G_2 H_1 + H_1' G_2' P_1 G_1 + G_1' P_1 G_2 H_1).$$

The optimization problem takes the following form:

$$\min_{\Sigma}\{tr(W\Sigma)\}$$

subject to the information capacity constraint:

$$-\log_2 |\Sigma| + \log_2 \left| G_1 \Sigma G_1' + \hat{\Omega} \right| \leq 2\kappa$$

and an additional condition to ensure that $G_1 \Sigma G_1' + \hat{\Omega} - \Sigma$ is a positive definite matrix:

$$\Sigma \preceq G_1 \Sigma G_1' + \hat{\Omega}.$$

As shown by Sims (2003a), the problem is the one of maximizing a linear objective function subject to a convex restriction set. In order to establish Σ numerically, we reparameterize the problem in terms of the upper triangular matrix ϕ^*, such that $\phi^{*\prime}\phi^* = \Lambda^*$ and $\Lambda^* = \Psi - \Sigma$. For an initial value of ϕ^* it is then possible to establish Σ by solving the Lyanpunov equation $\Lambda^* + \Sigma = \hat{\Omega} + G_1 \Sigma G_1'$. For a given value of the Lagrangian multiplier, it is then possible to compute the value for the objective function subject to the information capacity constraint. Once the optimal ϕ^* has been found, it is possible to recover Σ by solving the same equation and then recovering the covariance matrix of the noise variables $\Lambda = var(\zeta_t)$ using the following expression:

$$\Lambda^{-1} = \Sigma^{-1} - \Psi^{-1},$$

which derives from the usual formula for the variance of a stationary Gaussian distribution updated based on a linear observation, according to Sims (2003a).

6.4
Appendix D - Parameter Choices

The following table presents the parameter values for the benchmark calibration, along with its definitions.

Table 6.1: Benchmark Calibration (Quarterly Data)

Symbol	Parameter Definition	Assigned Value
β	Intertemporal discount factor	$.96^{1/4}$
θ	Preference parameter on labor supply	2.74
α_e	Equipment share of income	.19
α_s	Structures share of income	.16
H	Steady state labor hours	.24
φ	Steady state labor-leisure hours ratio	.31
$1-\delta^e$	Gross depreciation rate of equipment	$(1-.035)^{1/4}$
$1-\delta^s$	Gross depreciation rate of structures	$(1-.075)^{1/4}$
ρ_a	AR(1) coeff. of neutral tech. shock	.75
ρ_q	AR(1) coeff. of investment tech. shock	.75
σ_a^T	Sd. of transitory neutral tech. shock	.0050
σ_q^T	Sd. of transitory relative investment shock	.0050
σ_a^P	Sd. of permanent neutral tech. shock	.0050
σ_q^P	Sd. of permanent relative investment shock	.0050
$\sigma_{a,q}^{T,P}$	Correlations among innovations	*zero*
γ_a	Gross growth trend on investment prod.	$(1+.004)^{1/4}$
γ_q	Gross growth trend on neutral prod.	$(1+.025)^{1/4}$
s_c	Steady state consumption over GDP	81%
s_{ie}	Steady state investment in equipment over GDP	12%
s_{is}	Steady state investment in structures over GDP	7%
s_{ke}	Steady state capital stock in equipment over GDP	480%
s_{ks}	Steady state capital stock in structures over GDP	770%

6.5
Appendix E - Model Dynamics

In this section, we present the model general dynamics in response to shocks. The perfect information case is contrasted with responses of endogenous variables under limited information. Capacity constraint is calibrated to 0.7 bits.

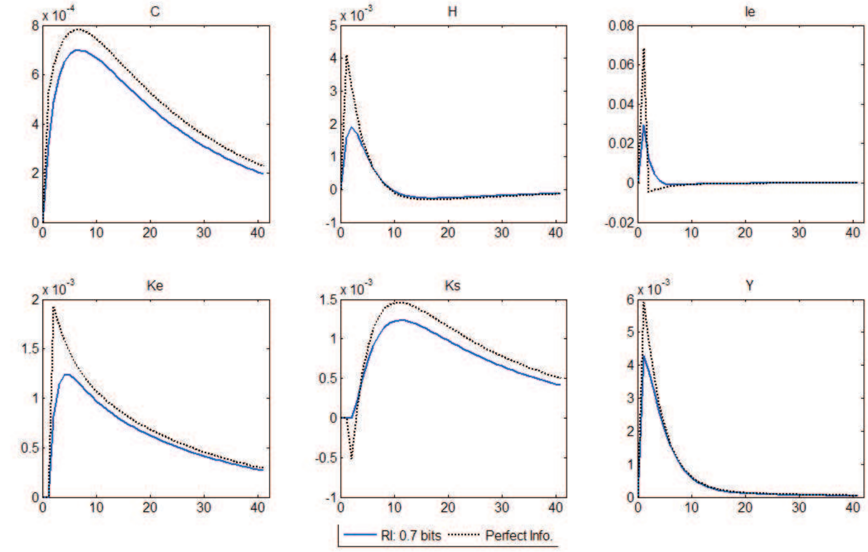

Figure 6.1: Impulse Response Functions to a one s.d. stationary shock on Labor Productivity (neutral technology shock)

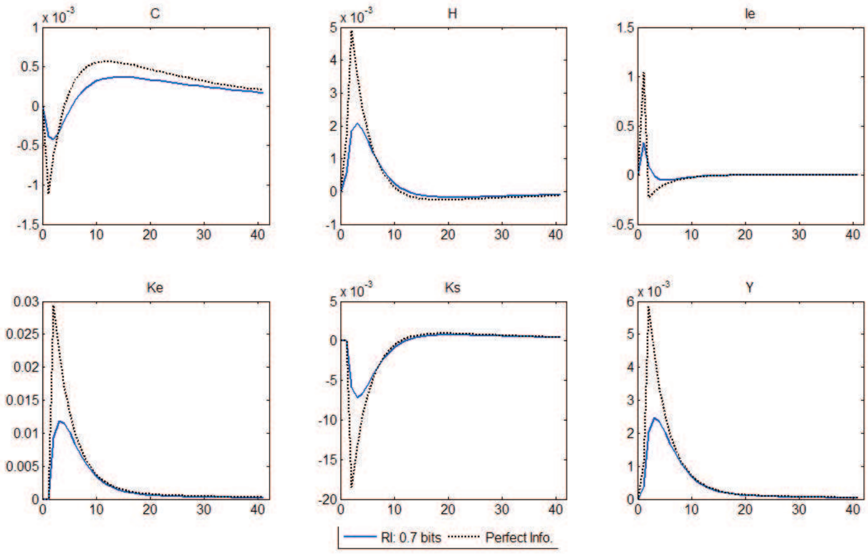

Figure 6.2: Impulse Response Functions to a one s.d. stationary shock on Investment Relative Productivity

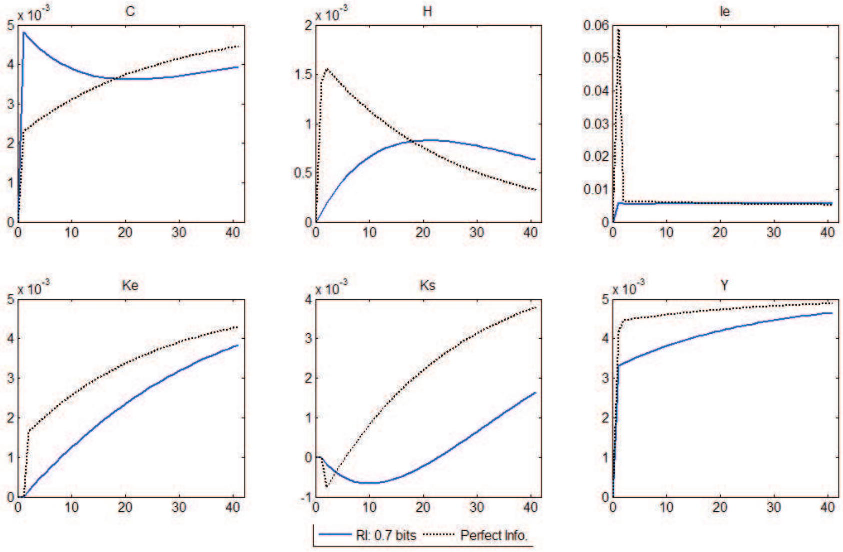

Figure 6.3: Impulse Response Functions to a one s.d. permanent shock on Labor Productivity (neutral technology shock)

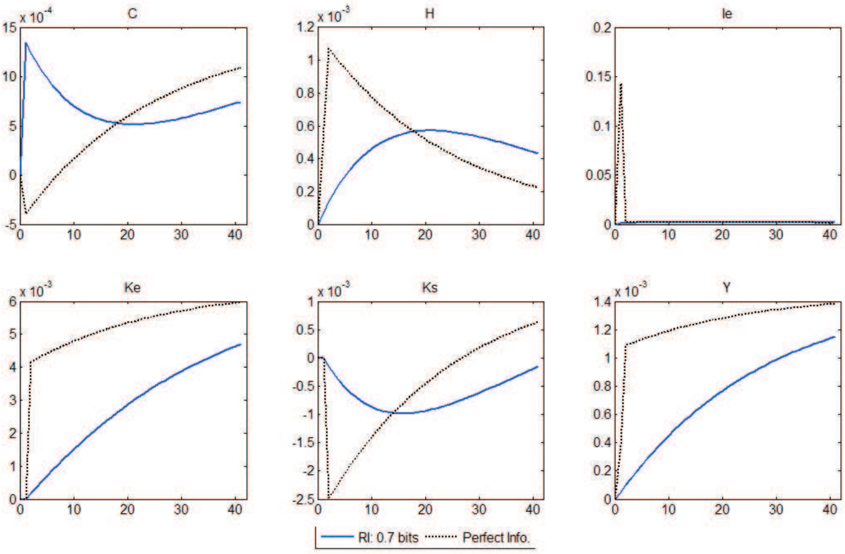

Figure 6.4: Impulse Response Functions to a one s.d. permanent shock on Investment Relative Productivity

6.6
Appendix F - S.D.s for RBC Statistics

The following table presents the standard errors for simulations of RBC statistics, presented at Section 3.

Table 6.2: RBC Basic Statistics - Standard Errors

	Full Inf.	$\kappa = 4.3$	$\kappa = 1.4$	$\kappa = .80$	$\kappa = .24$
Relative S.D.					
σ_y^a	0.08	0.09	0.08	0.07	0.06
σ_c/σ_y	0.01	0.06	0.10	0.09	0.10
σ_h/σ_y	0.01	0.03	0.04	0.04	0.03
σ_i/σ_y	0.05	0.20	0.32	0.32	0.35
Cross-Correlation					
$\rho(c, y)$	0.08	0.10	0.11	0.10	0.07
$\rho(h, y)$	0.00	0.02	0.03	0.04	0.08
$\rho(i, y)$	0.00	0.02	0.05	0.05	0.08
Autocorrelation					
$\rho(\Delta y)$	0.05	0.05	0.04	0.05	0.05
$\rho(\Delta c)$	0.05	0.06	0.05	0.05	0.05
$\rho(\Delta h)$	0.04	0.04	0.03	0.03	0.03
$\rho(\Delta i)$	0.04	0.04	0.04	0.04	0.05

[a]In percentage.