III
The Sequent Calculus for $\mathcal{ALC}$

III.1 A Sequent Calculus for logicALC

The Sequent Calculus for $\mathcal{ALC}$ that it is shown in Figures III.1 and III.2 considers the extension of the language $\mathcal{ALC}$ presented in Section II.1 for labeled concepts. The labels are a list of existencial or universal quantified roles names. Its syntax is as following:

$$L \rightarrow \forall R, L \mid \exists R, L \mid \emptyset$$
$$\phi_{lc} \rightarrow {}^{L}\phi_c$$

where R stands for atomic role names, L for list of labels and ϕ_c for $\mathcal{ALC}$ concept descriptions defined in Section II.1.

Each labeled $\mathcal{ALC}$ concept has a straightforward $\mathcal{ALC}$ concept equivalent. For example, the $\mathcal{ALC}$ concept $\exists R_2.\forall Q_2.\exists R_1.\forall Q_1.\alpha$ has the same semantics of the labeled concept ${}^{\exists R_2, \forall Q_2, \exists R_1, \forall Q_1}\alpha$.

In other words, the list of labels are just the roles prefix of a concept. Labels are syntactic artifacts of our system, which means that labeled concepts and its equivalent $\mathcal{ALC}$ have the same semantics. The system was designed to be extended to description logics with role constructors and subsumptions of roles. This is one of the main reasons to use roles-as-labels in its formulation. Besides that, whenever roles are promoted to labels the rules of the calculus can compose or decompose concept description preserving role prefix stored as labels. In that way, labels are a kind of "context" where concept manipulation occurs.

Given that any labeled concept has an equivalent $\mathcal{ALC}$ concept, the semantics of a labeled concept can be given with the support of a formal transformation of labeled concepts into $\mathcal{ALC}$ concepts. We defined the function $\sigma : \phi_{lc} \rightarrow \phi_c$ that takes a labeled $\mathcal{ALC}$ concept an returns a $\mathcal{ALC}$ concept. Considering α an $\mathcal{ALC}$ concept description, the function σ is recursively defined as:

$$\sigma\left({}^{\emptyset}\alpha\right) = \alpha$$
$$\sigma\left({}^{\forall R,L}\alpha\right) = \forall R.\sigma\left({}^{L}\alpha\right)$$
$$\sigma\left({}^{\exists R,L}\alpha\right) = \exists R.\sigma\left({}^{L}\alpha\right)$$

Given σ, the semantics of a labeled concept γ is given by $\sigma(\gamma)^{\mathcal{I}}$.

We define $\Delta \Rightarrow \Gamma$ as a *sequent* where Δ and Γ are finite sequences of labeled concepts. The natural interpretation of the sequent $\Delta \Rightarrow \Gamma$ is the $\mathcal{ALC}$ formula:

$$\prod_{\delta \in \Delta} \sigma(\delta) \sqsubseteq \bigsqcup_{\gamma \in \Gamma} \sigma(\gamma)$$

The $SC_{\mathcal{ALC}}$ system is presented in Figures III.1 and III.2. In all rules of the figures, the greek letters α and β stands for $\mathcal{ALC}$ concepts (formulas without labels), γ_i and δ_i stands for labeled concepts, Γ_i and Δ_i for lists of labeled concepts. For a clean presentation, the lists of labels are omitted whenever they are not used in the rule, this is the case of all structural rules in Figure III.1. The notation ${}^{L}\Gamma$ has to be taken as a list of labeled formulas of the form ${}^{L}\gamma_1, \ldots, {}^{L}\gamma_k$ for all $\gamma \in \Gamma$. The notation ${}^{+\forall R}\gamma$ (resp. ${}^{+\exists R}\gamma$) which can also be used with list of labeled concepts, ${}^{+\forall R}\Gamma$ (resp. ${}^{+\exists R}\Gamma$), means the addition of a label, $\forall R$ or $\exists R$ of a given role R, in front of the list of labels of γ, respectively in all $\gamma \in \Gamma$. Finally, we write ${}^{\exists L}\alpha$ to denote that all labels of L are existential quantificated and ${}^{\forall L}\alpha$ whenever all labels are universal quantificated (value restricted).

Considering the labeled formula ${}^{L}\alpha$, the notation ${}^{\neg L}\beta$ denotes exchanging the universal roles occurring in L for existential roles and vice-versa in a consistent way. Thus, if $\beta \equiv \neg\alpha$ them the formulas will be a negation each other. For example, ${}^{\neg\forall R,\exists Q}\beta$ is ${}^{\exists R,\forall Q}\neg\alpha$.

The system ought to be used by applying propositional rules, then the introduction of labels and then the quantification rules. This procedure will derive a normal derivation. Example 1 was taken from [7] and is useful to give an idea of how the rules of the $SC_{\mathcal{ALC}}$ system can be used.

Example 1 *Given the $\mathcal{ALC}$ subsumption axiom:*

$$\exists child.\top \sqcap \forall child.\neg(\exists child.\neg Doctor) \sqsubseteq \exists child.\forall child.Doctor \tag{1}$$

In $SC_{\mathcal{ALC}}$, we can prove that it is a theorem with the proof:

$$\frac{}{\alpha \Rightarrow \alpha} \qquad \frac{}{\bot \Rightarrow \alpha}$$

$$\frac{\Delta \Rightarrow \Gamma}{\Delta, \delta \Rightarrow \Gamma}\text{weak-l} \qquad \frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, \gamma}\text{weak-r}$$

$$\frac{\Delta, \delta, \delta \Rightarrow \Gamma}{\Delta, \delta \Rightarrow \Gamma}\text{contraction-l} \qquad \frac{\Delta \Rightarrow \Gamma, \gamma, \gamma}{\Delta \Rightarrow \Gamma, \gamma}\text{contraction-r}$$

$$\frac{\Delta_1, \delta_1, \delta_2, \Delta_2 \Rightarrow \Gamma}{\Delta_1, \delta_2, \delta_1, \Delta_2 \Rightarrow \Gamma}\text{perm-l} \qquad \frac{\Delta \Rightarrow \Gamma_1, \gamma_1, \gamma_2, \Gamma_2}{\Delta \Rightarrow \Gamma_1, \gamma_2, \gamma_1, \Gamma_2}\text{perm-r}$$

$$\frac{\Delta_1 \Rightarrow \Gamma_1, {}^{L}\alpha \qquad {}^{L}\alpha, \Delta_2 \Rightarrow \Gamma_2}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\text{cut}$$

Figure III.1: The System $\mathrm{SC}_{\mathcal{ALC}}$: structural rules

$$\frac{\Delta, {}^{L,\forall R}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}(\forall R.\alpha)L_2 \Rightarrow \Gamma}\forall\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\forall R}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}(\forall R.\alpha)}\forall\text{-r}$$

$$\frac{\Delta, {}^{L,\exists R}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}(\exists R.\alpha) \Rightarrow \Gamma}\exists\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\exists R}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}(\exists R.\alpha)}\exists\text{-r}$$

$$\frac{\Delta, {}^{\forall L}\alpha, {}^{\forall L}\beta \Rightarrow \Gamma}{\Delta, {}^{\forall L}(\alpha \sqcap \beta) \Rightarrow \Gamma}\sqcap\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{\forall L}\alpha \qquad \Delta \Rightarrow \Gamma, {}^{\forall L}\beta}{\Delta \Rightarrow \Gamma, {}^{\forall L}(\alpha \sqcap \beta)}\sqcap\text{-r}$$

$$\frac{\Delta, {}^{\exists L}\alpha \Rightarrow \Gamma \qquad \Delta, {}^{\exists L}\beta \Rightarrow \Gamma}{\Delta, {}^{\exists L}(\alpha \sqcup \beta) \Rightarrow \Gamma}\sqcup\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{\exists L}\alpha, {}^{\exists L}\beta}{\Delta \Rightarrow \Gamma, {}^{\exists L}(\alpha \sqcup \beta)}\sqcup\text{-r}$$

$$\frac{\Delta \Rightarrow \Gamma, {}^{\neg L}\alpha}{\Delta, {}^{L}\neg\alpha \Rightarrow \Gamma}\neg\text{-l} \qquad \frac{\Delta, {}^{\neg L}\alpha \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, {}^{L}\neg\alpha}\neg\text{-r}$$

$$\frac{\delta \Rightarrow \Gamma}{{}^{+\exists R}\delta \Rightarrow {}^{+\exists R}\Gamma}\text{prom-}\exists \qquad \frac{\Delta \Rightarrow \gamma}{{}^{+\forall R}\Delta \Rightarrow {}^{+\forall R}\gamma}\text{prom-}\forall$$

Figure III.2: The System $\mathrm{SC}_{\mathcal{ALC}}$: logical rules

$$
\begin{array}{ll}
Doctor \Rightarrow Doctor & \\
\hline
{}^{\forall child}Doctor \Rightarrow {}^{\forall child}Doctor & \textit{prom-}\forall \\
\hline
\top, {}^{\forall child}Doctor \Rightarrow {}^{\forall child}Doctor & \textit{weak-l} \\
\hline
\top \Rightarrow {}^{\exists child}\neg Doctor, {}^{\forall child}Doctor & \neg\textit{-r} \\
\hline
\top \Rightarrow \exists child.\neg Doctor, {}^{\forall child}Doctor & \exists\textit{-r} \\
\hline
{}^{\exists child}\top \Rightarrow {}^{\exists child}(\exists child.\neg Doctor), {}^{\exists child,\forall child}Doctor & \textit{prom-}\exists \\
\hline
{}^{\exists child}\top, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow {}^{\exists child,\forall child}Doctor & \neg\textit{-l} \\
\hline
{}^{\exists child}\top, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow {}^{\exists child}\forall child.Doctor & \forall\textit{-l} \\
\hline
{}^{\exists child}\top, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \exists\textit{-r} \\
\hline
{}^{\exists child}\top, \forall child.\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \forall\textit{-l} \\
\hline
\exists child.\top, \forall child.\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \exists\textit{-l} \\
\hline
\exists child.\top \sqcap \forall child.\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \sqcap\textit{-l}
\end{array}
$$

III.2 $\mathrm{SC}_{\mathcal{ALC}}$ Soundness

The soundness of $\mathrm{SC}_{\mathcal{ALC}}$ is proved by taking into account the intuitive meaning of each sequent and establishing that the truth preservation holds. From Section III.1, a sequent $\Delta \Rightarrow \Gamma$ is equivalent in meaning to the $\mathcal{ALC}$ formula:

$$\bigsqcap_{\delta\in\Delta} \sigma(\delta) \sqsubseteq \bigsqcup_{\gamma\in\Gamma} \sigma(\gamma)$$

A sequent is defined to be *valid* or a *tautology* if and only if its corresponding $\mathcal{ALC}$ formula is.

When using the calculus, the usual axioms of a particular DL theory (TBox or an ontology) of the form $C \sqsubseteq D$ should be taken as sequents $C \Rightarrow D$. Labeled formulas occur only during the proof procedure, since they are in practical terms taken as intermediate data.

Theorem 11 ($\mathrm{SC}_{\mathcal{ALC}}$ is sound) *Considering Ω a set of sequents, a theory or a TBox, let a Ω-proof be any $\mathrm{SC}_{\mathcal{ALC}}$ proof in which sequents from Ω are permitted as initial sequents (in addition to the logical axioms). The soundness of $\mathrm{SC}_{\mathcal{ALC}}$ states that if a sequent $\Delta \Rightarrow \Gamma$ has a Ω-proof, then $\Delta \Rightarrow \Gamma$ is satisfied by every interpretation which satisfies Ω. That is,*

$$\textit{if} \quad \Omega \vdash_{\mathrm{SC}_{\mathcal{ALC}}} \Delta \Rightarrow \Gamma \quad \textit{then} \quad \Omega \models \bigsqcap_{\delta\in\Delta} \sigma(\delta) \sqsubseteq \bigsqcup_{\gamma\in\Gamma} \sigma(\gamma)$$

In the proof of Theorem 11 we will write $\Delta^{\mathcal{I}}$ as an abbreviation for the set interpretation of the conjunction of concepts in Δ, that is, $\bigcap_{\delta\in\Delta} \sigma(\delta)^{\mathcal{I}}$, and $\Gamma^{\mathcal{I}}$ as an abbreviation for the set interpretation of the disjunction of the concepts in Γ, $\bigcup_{\gamma\in\Gamma} \sigma(\gamma)^{\mathcal{I}}$.

During the proof below, we will use many times the axioms and facts from Section II.6.

Proof: We proof Theorem 11 by induction on the length of the Ω-proofs. The length of a Ω-proof is the number of applications for any derivation rule of the calculus in a top-down approach.

base case Proofs with length zero are proofs $\Omega \vdash \Delta \Rightarrow \Gamma$ where $\Delta \Rightarrow \Gamma$ occurs in Ω. In that case, it is easy to see that the theorem holds.

For the initial sequents, logical axioms like $C \Rightarrow C$, it is easy to see that $\sigma(C)^{\mathcal{I}} \subseteq \sigma(C)^{\mathcal{I}}$ for every interpretation $\mathcal{I}$ since every set is a subset of itself.

Induction hypothesis As inductive hypothesis, we will consider that for proofs of length n the theorem holds. It is now sufficient to show that each of the derivation rules preserves the truth. That is, if the premises holds, the conclusion must also hold.

Cut rule Given the sequents $\Delta_1 \Rightarrow \Gamma_1, {}^{L}C$ and ${}^{L}C, \Delta_2 \Rightarrow \Gamma_2$ then, by hypothesis, we know that they are valid and so

$$\bigcap_{\delta \in \Delta_1} \sigma(\delta)^{\mathcal{I}} \subseteq \bigcup_{\gamma \in \Gamma_1} \sigma(\gamma)^{\mathcal{I}} \cup \sigma({}^{L}C)^{\mathcal{I}}$$

and

$$\sigma({}^{L}C)^{\mathcal{I}} \cap \bigcap_{\delta \in \Delta_2} \sigma(\delta)^{\mathcal{I}} \subseteq \bigcup_{\gamma \in \Gamma_2} \sigma(\gamma)^{\mathcal{I}}$$

Let $\Delta_1^{\mathcal{I}} = \bigcap_{\delta \in \Delta_1} \sigma(\delta)^{\mathcal{I}}$, $\Gamma_1^{\mathcal{I}} = \bigcup_{\gamma \in \Gamma_1} \sigma(\gamma)^{\mathcal{I}}$, $\Delta_2^{\mathcal{I}} = \bigcap_{\delta \in \Delta_2} \sigma(\delta)^{\mathcal{I}}$, $\Gamma_2^{\mathcal{I}} = \bigcup_{\gamma \in \Gamma_2} \sigma(\gamma)^{\mathcal{I}}$ and $X = \sigma({}^{L}C)^{\mathcal{I}}$. Now me must show that the application of the *cut* rule preserves the set inclusion. In other words, given $\Delta_1^{\mathcal{I}} \subseteq (\Gamma_1^{\mathcal{I}} \cup X)$ and $(X \cap \Delta_2^{\mathcal{I}}) \subseteq \Gamma_2^{\mathcal{I}}$, we must have $(\Delta_1^{\mathcal{I}} \cap \Gamma_2^{\mathcal{I}}) \subseteq (\Gamma_1^{\mathcal{I}} \cup \Gamma_2^{\mathcal{I}})$. What is easy to show using the standard set theory.

***Rules* weak-l *and* weak-r** Given the sequent $\Delta \Rightarrow \Gamma$, by the inductive hypothesis we know that

$$\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}}$$

By set theory, $\Delta^{\mathcal{I}} \cap X \subseteq \Gamma^{\mathcal{I}}$ and $\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup X$ for any set X interpretation of a labeled concept α. In the first case, we have the interpretation of $\Delta, \alpha \Rightarrow \Gamma$. In the second case, we have the interpretation of $\Delta \Rightarrow \Gamma, \alpha$. This is sufficient to show the soundness of both rules.

***Rules* perm-l *and* perm-r** By the definition of the meaning of a sequent and its semantics, it is easy to see that both rules are sound. Note that the order of the formulas in both sides of a sequent do not change the sequent semantics.

***Rules* prom-∀ *and* prom-∃** The soundness of rule prom-∃ if easily proved using the Fact 1 and Axiom 3. The soundness of rule prom-∀ is proved using Fact 2 and the Axiom 1.

Rules* ∀*-r,* ∀*-l,* ∃*-r and* ∃*-l From the definition of σ function, we know that in all those four rules, both the premises and the conclusions have, given a interpretation function, the same semantics.

Rules* ⊓*-l and* ⊓*-r In order to prove the soundness of those rules we need the $\mathcal{ALC}$ Axiom 1 that states the distributivity of the universal quantified constructor over the conjunction. Moreover, we must observe that both rules have an important proviso. That is, they are restricted to details only with labeled concepts were all labels are universal quantified. This restriction permit us to apply the Axiom 1 inductively.

Taking the sequent $\Delta, {}^{L}\alpha, {}^{L}\beta \Rightarrow \Gamma$ valid as hypothesis, we have:

$$\left(\Delta^{\mathcal{I}} \cap \sigma({}^{L}\alpha)^{\mathcal{I}} \cap \sigma({}^{L}\beta)^{\mathcal{I}}\right) \subseteq \Gamma^{\mathcal{I}}$$

To show that the rule (⊓-l) is sound, We must prove that $\Delta, {}^{L}(\alpha \sqcap \beta) \Rightarrow \Gamma$ is also valid. In other words, that $\Delta^{\mathcal{I}} \cap \sigma({}^{L}(\alpha \sqcap \beta))^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}}$ holds. What is true by the definition of σ, the Axiom 1 and the rule proviso which allows us to apply the Axiom 1 over the list of labels L.

Now consider the rule (⊓-r). By induction hypothesis, the sequents $\Delta \Rightarrow \Gamma, {}^{L}\alpha$ and $\Delta \Rightarrow \Gamma, {}^{L}\beta$ are valid, and so,

$$\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup \sigma({}^{L}\alpha)^{\mathcal{I}} \qquad \textit{and} \qquad \Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup \sigma({}^{L}\beta)^{\mathcal{I}}$$

holds for all interpretations $\cdot^{\mathcal{I}}$. Now, suppose the application of the rule (⊓-r) over the two sequent above. We must show that $\Delta \Rightarrow \Gamma, {}^{L}(\alpha \sqcap \beta)$ is also valid, that is,

$$\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup \sigma({}^{L}(\alpha \sqcap \beta))^{\mathcal{I}}$$

holds. But by basic set theory we have

$$\Delta^{\mathcal{I}} \subseteq ((\Gamma^{\mathcal{I}} \cup \sigma({}^{L}\alpha)^{\mathcal{I}}) \cap (\Gamma^{\mathcal{I}} \cup \sigma({}^{L}\beta)^{\mathcal{I}}))$$

And by distributive law

$$\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup (\sigma({}^{L}\alpha)^{\mathcal{I}} \cap \sigma({}^{L}\beta)^{\mathcal{I}})$$

Finally, by the definition of σ, the Axiom (1) and the rule proviso, we can conclude that the rule conclusion if valid.

Rules $\sqcup$***-l and*** $\sqcup$***-r*** In both rules the proviso is that the labels list L must contain only existential quantified roles. The soundness of both rules are proved with the support of this proviso and the Axiom 3, applied inductively over the labels lists.

As inductive hypothesis the sequents $\Delta, {}^{L}\alpha \Rightarrow \Gamma$ and $\Delta, {}^{L}\beta \Rightarrow \Gamma$ are valid. That is, given $\Delta^{\mathcal{I}} = \bigcap_{\delta\in\Delta} \sigma(\delta)^{\mathcal{I}}$ and $\Gamma^{\mathcal{I}} = \bigcup_{\gamma\in\Gamma} \sigma(\gamma)^{\mathcal{I}}$, we know that

$$\Delta^{\mathcal{I}} \cap \sigma({}^{L}\alpha)^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \qquad \text{and} \qquad \Delta^{\mathcal{I}} \cap \sigma({}^{L}\beta)^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}}$$

holds. Now considering the application of the rule ($\sqcup$-l) over the two sequents above we must prove that the resulting sequent $\Delta, {}^{L}(\alpha \sqcup \beta) \Rightarrow \Gamma$ is also valid:

$$\Delta^{\mathcal{I}} \cap \sigma({}^{\emptyset}(\alpha \sqcup \beta)^{L})^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}}$$

Following from the hypothesis and basic set theory we know that if $\Delta^{\mathcal{I}} \cap X_1 \subseteq \Gamma^{\mathcal{I}}$ an $\Delta^{\mathcal{I}} \cap X_2 \subseteq \Gamma^{\mathcal{I}}$ than $(\Delta^{\mathcal{I}} \cap X_1) \cup (\Delta^{\mathcal{I}} \cap X_2) \subseteq \Gamma^{\mathcal{I}}$ what gives us

$$\Delta^{\mathcal{I}} \cap (\sigma({}^{L}\alpha)^{\mathcal{I}} \cup \sigma({}^{L}\beta)^{\mathcal{I}}) \subseteq \Gamma^{\mathcal{I}}$$

and by the Axiom 3 applied inductively over the list L we have the desired semantics of the resulting sequent:

$$\Delta^{\mathcal{I}} \cap (\sigma({}^{L}(\alpha \sqcup \beta))^{\mathcal{I}}) \subseteq \Gamma^{\mathcal{I}}$$

For rule ($\sqcup$r) the inductive hypothesis is that $\Delta \Rightarrow \Gamma, {}^{L}\alpha, {}^{L}\beta$ is valid. And so, the following statement must holds:

$$\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup \sigma({}^{L}\alpha)^{\mathcal{I}} \cup \sigma({}^{L}\beta)^{\mathcal{I}}$$

Now by the Axiom 3 we can rewrite to

$$\Delta^{\mathcal{I}} \subseteq \Gamma^{\mathcal{I}} \cup \sigma({}^{L}(\alpha \sqcup \beta))^{\mathcal{I}}$$

What is the semantics of the rule conclusion.

Rules* $\neg$*-l and* $\neg$*-r Given a concept ${}^{L}\alpha$ and a interpretation $\cdot^{\mathcal{I}}$ we define the set $X = \sigma({}^{L}\alpha)^{\mathcal{I}}$ and the interpretation of its negation, $\sigma({}^{\neg L}\neg\alpha)^{\mathcal{I}}$, will be the set $\overline{X} = \Delta^{\mathcal{I}} \setminus X$.

For rule ($\neg$-l), the inductive hypothesis is that the premise $\Delta \Rightarrow \Gamma, {}^{L}\alpha$ is valid. Which means that $\Delta^{\mathcal{I}} \subseteq (\Gamma^{\mathcal{I}} \cup X)$. From the basic set theory this implies that $(\Delta^{\mathcal{I}} \cap \overline{X}) \subseteq \Gamma^{\mathcal{I}}$, witch is the interpretation of the conclusion.

For rule ($\neg$-r), the inductive hypothesis is that the premise $\Delta, {}^{L}\alpha \Rightarrow \Gamma$ is valid. Which means that $(\Delta^{\mathcal{I}} \cap X) \subseteq \Gamma^{\mathcal{I}}$. From the basic set theory this implies that $\Delta^{\mathcal{I}} \subseteq (\Gamma^{\mathcal{I}} \cup \overline{X})$, the interpretation of the conclusion as desired. ■

III.3 The Completeness of $\mathrm{SC}_{\mathcal{ALC}}$

We show the relative completeness of $\mathrm{SC}_{\mathcal{ALC}}$ regarding the axiomatic presentation of $\mathcal{ALC}$ from Section II.6. Since $\mathcal{ALC}$ formulas are not labeled, the completeness must take into account only formulas with empty list of labels. Proceeding in this way, the $\mathcal{ALC}$ sequent calculus deduction rules without labels behave exactly as the sequent calculus rules for classical propositional logic. Thus, in order to prove that $\mathrm{SC}_{\mathcal{ALC}}$ is complete, we have only to derive the axioms above.

The derivation of the rule of necessitation is accomplished by

$$\dfrac{\dfrac{\Rightarrow \alpha}{\Rightarrow {}^{\forall R}\alpha}\ \text{prom-}\forall}{\Rightarrow \forall R.\alpha}\ \forall\text{-r}$$

The derivation of the Axiom 1 is obtained from the following derivations. First we consider the case:

$$\forall R.(\alpha \sqcap \beta) \sqsubseteq \forall R.\alpha \sqcap \forall R.\beta$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{\forall R}\alpha \Rightarrow {}^{\forall R}\alpha}{{}^{\forall R}\alpha, {}^{\forall R}\beta \Rightarrow {}^{\forall R}\alpha}\ \text{weak-l}}{{}^{\forall R}\alpha, {}^{\forall R}\beta \Rightarrow \forall R.\alpha}\ \forall\text{-r} \quad \dfrac{\dfrac{{}^{\forall R}\beta \Rightarrow {}^{\forall R}\beta}{{}^{\forall R}\alpha, {}^{\forall R}\beta \Rightarrow {}^{\forall R}\beta}\ \text{weak-l}}{{}^{\forall R}\alpha, {}^{\forall R}\beta \Rightarrow \forall R.\beta}\ \forall\text{-r}}{{}^{\forall R}\alpha, {}^{\forall R}\beta \Rightarrow \forall R.\alpha \sqcap \forall R.\beta}\ \sqcap\text{-r}}{{}^{\forall R}(\alpha \sqcap \beta) \Rightarrow \forall R.\alpha \sqcap \forall R.\beta}\ \sqcap\text{-l}}{\forall R.(\alpha \sqcap \beta) \Rightarrow \forall R.\alpha \sqcap \forall R.\beta}\ \forall\text{-l}$$

Finally, we prove the subsumption from right to left:

$$\forall R.\alpha \sqcap \forall R.\beta \sqsubseteq \forall R.(\alpha \sqcap \beta)$$

by

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{{}^{\forall R}\alpha \Rightarrow {}^{\forall R}\alpha}{\forall R.\beta, {}^{\forall R}\alpha \Rightarrow {}^{\forall R}\alpha}\ \text{weak-l}}{\forall R.\beta, \forall R.\alpha \Rightarrow {}^{\forall R}\alpha}\ \forall\text{-l} \qquad \dfrac{\dfrac{{}^{\forall R}\beta \Rightarrow {}^{\forall R}\beta}{\forall R.\alpha, {}^{\forall R}\beta \Rightarrow {}^{\forall R}\beta}\ \text{weak-l}}{\forall R.\alpha, \forall R.\beta \Rightarrow {}^{\forall R}\beta}\ \forall\text{-l}}{\forall R.\alpha, \forall R.\beta \Rightarrow {}^{\forall R}(\alpha \sqcap \beta)}\ \sqcap\text{-r}}{\forall R.\alpha \sqcap \forall R.\beta \Rightarrow {}^{\forall R}(\alpha \sqcap \beta)}\ \sqcap\text{-l}}{\forall R.\alpha \sqcap \forall R.\beta \Rightarrow \forall R.(\alpha \sqcap \beta)}\ \forall\text{-r}$$

III.4 The cut-elimination theorem

In this section we adopt the usual terminology of proof theory for sequent calculus presented in [13, 66]. We follow Gentzen's original proof for cut elimination with the introduction of the *mix* rule.

Let δ be a labeled formula. An inference of the following form is called *mix* with respect to ψ, a labeled concept:

$$\dfrac{\Delta_1 \Rightarrow\ \Gamma_1 \quad \Delta_2 \Rightarrow\ \Gamma_2}{\Delta_1, \Delta_2^* \Rightarrow\ \Gamma_1^*, \Gamma_2}\ (\psi)$$

where both Γ_1 and Δ_2 contain the formula δ, and Γ_1^* and Δ_2^* are obtained from Γ_1 and Δ_2 respectively by deleting all the occurrences of δ in them.

But in order to obtain an easier presentation of our cut elimination we introduce four additional rules of inference called *quasi-mix* rules.

$$\dfrac{{}^{L}\delta \Rightarrow \Gamma_1 \quad \Delta_2 \Rightarrow \Gamma_2}{{}^{\exists R,L}\delta, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1^*, \Gamma_2}\ ({}^{L}\alpha, {}^{\exists R,L}\alpha) \qquad \dfrac{\Delta_1 \Rightarrow \Gamma_1 \quad {}^{L}\alpha \Rightarrow \Gamma_2}{\Delta_1 \Rightarrow \Gamma_1^*, {}^{+\exists R}\Gamma_2}\ ({}^{\exists R,L}\alpha, {}^{L}\alpha)$$

$$\dfrac{\Delta_1 \Rightarrow {}^{L}\alpha \quad \Delta_2 \Rightarrow \Gamma_2}{{}^{+\forall R}\Delta_1, \Delta_2^* \Rightarrow \Gamma_2}\ ({}^{L}\alpha, {}^{\forall R,L}\alpha) \qquad \dfrac{\Delta_1 \Rightarrow \Gamma_1 \quad \Delta_2 \Rightarrow {}^{L}\gamma}{\Delta_1, {}^{+\forall R}\Delta_2^* \Rightarrow \Gamma_1^*, {}^{\forall R,L}\gamma}\ ({}^{\forall R,L}\alpha, {}^{L}\alpha)$$

where in each rule, the tuple of concepts on the right indicates the two mix formulas of this inference rule. Γ_1, the list of formulas on the right from the left-side premisse, contains the first projection of the tuple, Δ_2, the list of formulas on the left from the right-side premisse, contains the second projection. Γ_1^* and Δ_2^* are obtained from Γ_1 and Δ_2 by deleting all occurrences of the first and second tuple's projection, respectively. The notation ${}^{+\exists R}\Delta$ (resp. ${}^{+\forall R}\Delta$) means the addition of $\exists R$ (resp. $\forall R$) on the list of labels of all $\delta \in \Delta$.

By the definitions of *mix* and *quasi-mix* rules, the *mix* rule is a special case of *quasi-mix* rules in which both mix formulas in the tuple are equal. Therefore, we can also consider a *quasi-mix* the *mix* rule.

Definition 12 (The $\mathrm{SC^*}_{\mathcal{ALC}}$ system) *We call $\mathrm{SC^*}_{\mathcal{ALC}}$ the new system obtained from $\mathrm{SC}_{\mathcal{ALC}}$ by replacing the cut rule by the* quasi-mix *(and mix) rules.*

Lemma 13 *The systems $\mathrm{SC}_{\mathcal{ALC}}$ and $\mathrm{SC^*}_{\mathcal{ALC}}$ are equivalent, that is, a sequent is $\mathrm{SC}_{\mathcal{ALC}}$-provable if and only if that sequent is also $\mathrm{SC^*}_{\mathcal{ALC}}$-provable.*

Proof: The four *quasi-mix* rules are derived from inferences where the promotional rules (prom-∀ and prom-∃) are applied just before a *mix* rule. In that way, one can transformed all the applications of *quasi-mix* rule into a sequence of prom-∀ or prom-∃ followed by *mix* rules applications. All applications of *mix* rule can than be replaced by applications of *cut* rule provide that all the repetitions of the cut formula in the upper sequents being first transformed into just one occurrence on each sequent. This is easily done by one or more application of the contraction and permutation rules.

To illustrate the process, let us consider an application of a *quasi-mix* rule in the $\mathrm{SC^*}_{\mathcal{ALC}}$-proof fragment bellow where Π_n are proof fragments. The double-line labeled with "perm*; contract*" means the application of rule permutation one or more times followed by one or more applications of contraction rule.

$$\dfrac{\begin{array}{c}\Pi_1\\ \Delta_1 \Rightarrow \Gamma_1\end{array} \qquad \begin{array}{c}\Pi_2\\ {}^{L}\alpha \Rightarrow \Gamma_2\end{array}}{\begin{array}{c}\Delta_1 \Rightarrow \Gamma_1^*, {}^{+\exists R}\Gamma_2\\ \Pi_3\end{array}}\ ({}^{\exists R,L}\alpha, {}^{L}\alpha)$$

And its corresponding $\mathrm{SC}_{\mathcal{ALC}}$-proof:

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ \Delta_1 \Rightarrow \Gamma_1\end{array}}{\Delta_1 \Rightarrow \Gamma_1^*, {}^{\exists R,L}\alpha}\ \text{perm*; contract*} \qquad \dfrac{\begin{array}{c}\Pi_2\\ {}^{L}\alpha \Rightarrow \Gamma_2\end{array}}{{}^{\exists R,L}\alpha \Rightarrow {}^{+\exists R}\Gamma_2}\ \text{prom-}\exists}{\begin{array}{c}\Delta_1 \Rightarrow \Gamma_1^*, {}^{+\exists R}\Gamma_2\\ \Pi_3\end{array}}\ \text{cut}$$

■

By the proof of Lemma 13, a derivation Π of $\Delta \Rightarrow \Gamma$ in $\mathrm{SC}_{\mathcal{ALC}}$ with cuts can be transformed in a derivation Π' of $\Delta \Rightarrow \Gamma$ in $\mathrm{SC^*}_{\mathcal{ALC}}$ with *quasi-mixes* (and *mixes*). So that, it is sufficient to show that the *quasi-mix* (and *mix*) rules are redundant in $\mathrm{SC^*}_{\mathcal{ALC}}$, since a proof in $\mathrm{SC^*}_{\mathcal{ALC}}$ without *quasi-mix* (and *mix*) is at the same time a proof in $\mathrm{SC}_{\mathcal{ALC}}$ without cut.

Definition 14 ($\mathrm{SC^{\mathcal{T}}}_{\mathcal{ALC}}$ system) $\mathrm{SC}_{\mathcal{ALC}}$ *was defined with initial sequents of the form $\alpha \Rightarrow \alpha$ with α a $\mathcal{ALC}$ concept definition (logical axiom). However, it is often convenient to allow for other initial sequents. So if $\mathcal{T}$ is a set of* *sequents of the form $\Delta \Rightarrow \Gamma$, where Δ and Γ are sequences of $\mathcal{ALC}$ concept descriptions (non-logical axioms), we define* $\mathrm{SC}^{\mathcal{T}}{}_{\mathcal{ALC}}$ *to be the proof system defined like* $\mathrm{SC}_{\mathcal{ALC}}$ *but allowing initial sequents to be from $\mathcal{T}$ too.*

The Definition 14 can be extend to the system $\mathrm{SC}^{*}{}_{\mathcal{ALC}}$ in the same way, obtaining the systems $\mathrm{SC}^{*\mathcal{T}}{}_{\mathcal{ALC}}$.

Definition 15 (Free-quasi-mix free proof) *Let P be an $\mathrm{SC}^{*\mathcal{T}}{}_{\mathcal{ALC}}$-proof. A formula occurring in P is anchored (by an $\mathcal{T}$-sequent) if it is a direct descendent of a formula from $\mathcal{T}$ occurring in an initial sequent. A* quasi-mix *inference in P is* anchored *if either:*

(i) the mix formulas are not atomic and at least one of the occurrences of the mix formulas in the upper sequents is anchored, or

(ii) the mix formulas are atomic and both of the occurrences of the mix formulas in the upper sequents are anchored.

A quasi-mix *inference which is not anchored is said to be* free. *A proof P is* free-quasi-mix free *if it contains no free* quasi-mixes.

Given that a *mix* is a special quase of *quasi-mix*, the Definition 15 can also be used to define *free* mixes. If a proof P is *free-quasi-mix free* it is also *free-mix free.*

Theorem 16 (Free-quasi-mix Elimination) *Let $\mathcal{T}$ be a set of sequents. If* $\mathrm{SC}^{*\mathcal{T}}{}_{\mathcal{ALC}} \vdash \Delta \Rightarrow \Gamma$ *then there is a* free-quasi-mix *free* $\mathrm{SC}^{*\mathcal{T}}{}_{\mathcal{ALC}}$*-proof of* $\Delta \Rightarrow \Gamma$.

Theorem 16 is a consequence of the following lemma.

Lemma 17 *If P is a proof of S (in $\mathrm{SC}^{*\mathcal{T}}{}_{\mathcal{ALC}}$) which contains only one* free-quasi-mix, *occurring as the last inference, then S is provable without any* free-quasi-mix.

Theorem 16 is obtained from Lemma 17 by simple induction over the number of *quasi-free-mix* occurring in a proof P.

We can now concentrate our attention on Lemma 17. First we define three scalars as a measure of the complexity of the proof. The *grade* of a formula ${}^{L}\alpha$ is defined as the number of logical symbols of α (denoted by $g({}^{L}\alpha)$). The *label-degree* of a formula ${}^{L}\alpha$ is defined as $ld({}^{L}\alpha) = |L|$ where $|L|$ means the length of the list L.

Let P be a proof containing only one *quasi-mix* as its last inference:

$$J \, \frac{\Delta_1 \Rightarrow \Gamma_1 \qquad \Delta_2 \Rightarrow \Gamma_2}{\Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2} \, (\gamma, \gamma')$$

The grade of a *quasi-mix* is

$$g(\gamma, \gamma') = g(\gamma) + g(\gamma')$$

Given that, the grade of a *mix* (a special quase of *quasi-mix*) is the double of the grade of the *mix* formula.

In a similar way, the label-degree of a *quasi-mix* is

$$ld(\gamma, \gamma') = ld(\gamma) + ld(\gamma')$$

and the label-degree of a *mix* is again the double of the lable-degree of the *mix* formula.

We say that the grade of P (denote by $g(P)$) and the label-degree of P (denoted by $ld(P)$) is the grade and label-degree of that *quasi-mix*.

We refer to the left and right sequents as S_1 and S_2 respectively, and to the lower sequent as S. We call a thread in P a left (or right) thread if it contains the left (or right) upper sequent of the *quasi-mix* J. The *rank* of the thread $\mathcal{F}$ in P is defined as the number of consecutive sequents, counting upward from the left (right) upper sequent of J, that contains γ (γ') in its succedent (antecedent). Since the left (right) upper sequent always contains the mix formulas, the rank of a thread in P is at least 1. The rank of a thread $\mathcal{F}$ in P is denoted by $rank(\mathcal{F}; P)$ and is defined as follows:

$$rank_l(P) = \max_{\mathcal{F}}(rank(\mathcal{F}; P)),$$

where $\mathcal{F}$ ranges over all the left threads in P, and

$$rank_r(P) = \max_{\mathcal{F}}(rank(\mathcal{F}; P)),$$

where $\mathcal{F}$ ranges over all the right threads in P. The rank of P is defined as

$$rank(P) = rank_l(P) + rank_r(P),$$

where $rank(P) \geq 2$.

Proof: We prove Lemma 17 by lexicographically induction on the ordered triple (grade,label-degree,rank) of the proof P. We divide the proof into two main cases, namely $rank = 2$ and $rank > 2$ (regardless of the grade and label-degree).

Case 1: $rank = 2$ We shall consider several cases according to the form of the proofs of the upper sequents of the *quasi-mix*.

1.1) The left upper sequent S_1 is an logical initial sequent. There are several cases to be examined.

a) P has the form:

$$J\;\dfrac{\alpha \Rightarrow \alpha \qquad \begin{array}{c} P_1 \\ \Delta_2 \Rightarrow \Gamma_2 \end{array}}{{}^{\exists R}\alpha, \Delta_2^* \Rightarrow \Gamma_2}\;(\alpha, {}^{\exists R}\alpha)$$

We can easily obtain the same end-sequent without using the *quasi-mix* as follows: [1]

$$\begin{array}{c} P_1 \\ \Delta_2 \Rightarrow \Gamma_2 \\ \hline\hline {}^{\exists R}\alpha, \ldots, {}^{\exists R}\alpha, \Delta_2^* \Rightarrow \Gamma_2 \\ \hline\hline {}^{\exists R}\alpha, \Delta_2^* \Rightarrow \Gamma_2 \end{array} \begin{array}{l} \\ \text{perm}^* \\ \text{contract}^* \end{array}$$

All other cases, that it, other *quasi-mix* occurrences in a similar proof format, are treated in a similar way. Note also that a logical initial sequent can only have $\mathcal{ALC}$ formulas on both sides of the sequent.

1.2) The right upper sequent S_2 is an logical initial sequent. Similar as Case 1 above.

1.3) S_1 or S_2 (or both) are non-logical initial sequents. In this case, it is obvious that the *quasi-mix* is not a *free* and it will be not eliminated.

1.4) Neither S_1 nor S_2 are initial sequents, and S_1 is the lower sequent of a structural inference J_1. Since $rank_l(P) = 1$, the mix formula ψ cannot appear in the succedent of the premisse of J_1, that is, J_1 must be the *weak-r* that introduced ψ. Again there are several cases to be examined for each possible *quasi-mix* rule used.

a) Let us consider the *quasi-mix* case $({}^{L}\alpha, {}^{\exists R,L}\alpha)$:

$$J\;\dfrac{\begin{array}{c} P_1 \\ \dfrac{\delta \Rightarrow \Gamma_1}{\delta \Rightarrow \Gamma_1, {}^{L}\alpha}\;J_1 \end{array} \qquad \begin{array}{c} P_2 \\ \Delta_2 \Rightarrow \Gamma_2 \end{array}}{{}^{+\exists R}\delta, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1, \Gamma_2}\;({}^{L}\alpha, {}^{\exists R,L}\alpha)$$

where Γ_1 does not contain ${}^{L}\alpha$. We can eliminate the *quasi-mix* as follows:

[1] The notation *contract∗* (*perm∗*) means zero or more applications of contraction (permutation) rule.

$$\cfrac{\cfrac{\cfrac{\begin{matrix}P_1\\ \delta \Rightarrow \Gamma_1\end{matrix}}{^{+\exists R}\delta \Rightarrow {}^{+\exists R}\Gamma_1}\text{ prom-1}}{\Delta_2^*, {}^{+\exists R}\delta \Rightarrow {}^{+\exists R}\Gamma_1, \Gamma_2}\text{ weak*}}{^{+\exists R}\delta, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1, \Gamma_2}\text{ perm*}$$

All other cases are treated in a similar way.

1.5) The same conditions that hold for Case 4 but with S_2 as the lower sequent of structural inference instead of S_1. As in Case 4.

1.6) Neither S_1 nor S_2 are an initial sequents and S_1 is the lower sequent of a prom-∃ rule application and J is a *mix* rule application.

$$\text{J}\ \cfrac{\cfrac{\begin{matrix}P_1\\ \delta \Rightarrow \Gamma_1\end{matrix}}{^{+\exists R}\delta \Rightarrow {}^{+\exists R}\Gamma_1}\text{ prom-}\exists \qquad \begin{matrix}P_2\\ \Delta_2 \Rightarrow \Gamma_2\end{matrix}}{^{+\exists R}\delta, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1^*, \Gamma_2}\ ({}^{+\exists R}\gamma)$$

where by assumption none of the proofs P_n for $n \in \{1, 2\}$ contain a mix or *quasi-mix*. Moreover, Γ_1 does not contain ${}^{+\exists R}\gamma$ since $rank_l(P) = 1$. That is, the prom-∃ rule introduced the mix formula of J. We can replace the application of the *mix* rule by an application of *quasi-mix* rule as follows:

$$\cfrac{\begin{matrix}P_1\\ \delta \Rightarrow \Gamma_1\end{matrix} \qquad \begin{matrix}P_2\\ \Delta_2 \Rightarrow \Gamma_2\end{matrix}}{^{+\exists R}\delta, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1^*, \Gamma_2}\ (\gamma, {}^{+\exists R}\gamma)$$

The new *quasi-mix* rule has label-degree less than the label-degree of the original *mix* rule, $ld({}^{+\exists R}\gamma, {}^{+\exists R}\gamma)$. So by the induction hypothesis, we can obtain a proof which contains no mixes.

1.7) Similar case as above with S_1 being lower sequent of a prom-∀ or S_2 being lower sequent of prom-∃ or prom-∀. We apply similar transformation of *mix* application into *quasi-mix* rules applications. Always "moving" the *mix* upward into the direction of the prom-∀ or prom-∃ inference.

1.8) Both S_1 and S_2 are lower sequents of logical inferences and $rank_l(P) = rank_r(P) = 1$, J being a *mix* with the mix formula γ of each side being the principal formula of the logical inference. We use induction on the grade, distinguishing several cases according to the outermost logical symbol of γ:

   i) The outermost logical symbol is $\sqcap$. P has the form:

$$\dfrac{\dfrac{\dfrac{P_1}{\Delta_1 \Rightarrow \Gamma_1, {}^L\alpha} \quad \dfrac{P_2}{\Delta_1 \Rightarrow \Gamma_1, {}^L\beta}}{\Delta_1 \Rightarrow \Gamma_1, {}^L(\alpha \sqcap \beta)}\ \sqcap\text{-r} \qquad \dfrac{\dfrac{P_3}{\Delta_2, {}^L\alpha, {}^L\beta \Rightarrow \Gamma_2}}{\Delta_2, {}^L(\alpha \sqcap \beta) \Rightarrow \Gamma_2}\ \sqcap\text{-l}}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ ({}^L(\alpha \sqcap \beta))$$

where by assumption none of the proofs P_n for $n \in \{1, 2, 3\}$ contain a *quasi-mix*. We transform P into:

$$\dfrac{\dfrac{\dfrac{P_2}{\Delta_1 \Rightarrow \Gamma_1, {}^L\beta} \qquad \dfrac{\dfrac{P_1}{\Delta_1 \Rightarrow \Gamma_1, {}^L\alpha} \quad \dfrac{P_3}{\Delta_2, {}^L\alpha, {}^L\beta \Rightarrow \Gamma_2}}{\Delta_1, \Delta_2, {}^L\beta \Rightarrow \Gamma_1, \Gamma_2}\ ({}^L\alpha)}{\Delta_1, \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_1, \Gamma_2}\ ({}^L\beta)}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ \text{perm*; contract*}$$

which contains two mix but both with grade less than $g({}^L(\alpha \sqcap \beta))$. So by induction hypothesis, we can obtain a proof which contains no mixes. Note that the mix $({}^L\alpha)$ is now the last inference rule of a proof which contains no mix. Given that, this mix can be omitted using the transformations defined above.

ii) The outermost logical symbol is $\sqcup$. In this case S_1 and S_2 must be lower sequents of $\sqcup$-r and $\sqcup$-l rule, respectively:

$$\dfrac{\dfrac{\dfrac{P_1}{\Delta_1 \Rightarrow \Gamma_1, {}^L\alpha, {}^L\beta}}{\Delta_1 \Rightarrow \Gamma_1, {}^L(\alpha \sqcup \beta)}\ \sqcup\text{-r} \qquad \dfrac{\dfrac{P_2}{\Delta_2, {}^L\alpha \Rightarrow \Gamma_2} \quad \dfrac{P_3}{\Delta_2, {}^L\beta \Rightarrow \Gamma_2}}{\Delta_2, {}^L(\alpha \sqcup \beta) \Rightarrow \Gamma_2}\ \sqcup\text{-l}}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ ({}^L(\alpha \sqcup \beta))$$

where, by hypothesis, none of the proofs P_n for $n \in \{1, 2, 3\}$ contain a *quasi-mix*. This proof can be transformed into:

$$\dfrac{\dfrac{\dfrac{\dfrac{P_1}{\Delta_1 \Rightarrow \Gamma_1, {}^L\alpha, {}^L\beta} \quad \dfrac{P_2}{\Delta_2, {}^L\alpha \Rightarrow \Gamma_2}}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2, {}^L\beta}\ ({}^L\alpha) \qquad \dfrac{P_3}{\Delta_2, {}^L\beta \Rightarrow \Gamma_2}}{\Delta_1, \Delta_2, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2, \Gamma_2}\ ({}^L\beta)}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ \text{perm*; contract*}$$

This proof contains two mix, but both with grade less than $g({}^L(\alpha \sqcup \beta))$. So by the induction hypothesis, we can obtain a proof which contains no mixes. As mentioned above, the new created mixes are now the last inference rule of proofs which contains no mix.

iii) The outermost logical symbol is $\forall$. In this case S_1 and S_2 must be lower sequents of $\forall$-r and $\forall$-l rule, respectively. P is:

$$\dfrac{\dfrac{\dfrac{P_1}{\Delta_1 \Rightarrow \Gamma_1, {}^{L,R}\alpha}}{\Delta_1 \Rightarrow \Gamma_1, {}^L\forall R.\alpha}\ \forall\text{-r} \qquad \dfrac{\dfrac{P_2}{\Delta_2, {}^{L,R}\alpha \Rightarrow \Gamma_2}}{\Delta_2, {}^L\forall R.\alpha \Rightarrow \Gamma_2}\ \forall\text{-l}}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ ({}^L\forall R.\alpha)$$

which again by hypothesis, none of the proofs P_n for $n \in \{1,2\}$ contain a mix. These proof can be transformed into:

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1, {}^{L,R}\alpha \end{matrix} \qquad \begin{matrix} P_2 \\ \Delta_2, {}^{L,R}\alpha \Rightarrow \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ ({}^{L,R}\alpha)$$

which contains one mix with grade less than $g({}^{L}\forall R.\alpha)$. So by induction hypothesis, we can obtain a proof which contains no mixes.

iv) The outermost logical symbol is $\exists$. The treatment is similar to the case above.

v) The outermost logical symbol is $\neg$ and P is:

$$\dfrac{\dfrac{\begin{matrix} P_1 \\ \Delta_1, {}^{L}\alpha \Rightarrow \Gamma_1 \end{matrix}}{\Delta_1 \Rightarrow \Gamma_1, {}^{\neg L}\neg\alpha}\ \textit{neg}\text{-r} \qquad \dfrac{\begin{matrix} P_2 \\ \Delta_2 \Rightarrow \Gamma_2, {}^{L}\alpha \end{matrix}}{\Delta_2, {}^{\neg L}\neg\alpha \Rightarrow \Gamma_2}\ \textit{neg}\text{-l}}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ ({}^{\neg L}\neg\alpha)$$

This proof can be transformed into:

$$\dfrac{\dfrac{\begin{matrix} P_2 \\ \Delta_2 \Rightarrow \Gamma_2, {}^{L}\alpha \end{matrix} \qquad \begin{matrix} P_1 \\ \Delta_1, {}^{L}\alpha \Rightarrow \Gamma_1 \end{matrix}}{\Delta_2, \Delta_1 \Rightarrow \Gamma_2, \Gamma_1}\ ({}^{L}\alpha)}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\ \text{perm*}$$

which contains one mix with grade less than $g({}^{\neg L}\neg\alpha)$. So by the induction hypothesis, we can obtain a proof which contains no mixes.

1.9) Both S_1 and S_2 are lower sequents of logical inferences, $rank_l(P) = rank_r(P) = 1$ and J being a *quasi-mix* $(\gamma, {}^{+\exists R}\gamma)$ where the mix formulas on each side is the principal formula of the logical inferences. Let us here present just the case $\sqcup$. In this case S_1 and S_2 must be lower sequents of $\sqcup$-r and $\sqcup$-l rule, respectively:

$$\dfrac{\dfrac{\begin{matrix} P_1 \\ \delta \Rightarrow \Gamma_1, {}^{L}\alpha, {}^{L}\beta \end{matrix}}{\delta \Rightarrow \Gamma_1, {}^{L}(\alpha \sqcup \beta)}\ \sqcup\text{-r} \qquad \dfrac{\begin{matrix} P_2 \\ \Delta_2, {}^{\exists R,L}\alpha \Rightarrow \Gamma_2 \end{matrix} \qquad \begin{matrix} P_3 \\ \Delta_2, {}^{\exists R,L}\beta \Rightarrow \Gamma_2 \end{matrix}}{\Delta_2, {}^{\exists R,L}(\alpha \sqcup \beta) \Rightarrow \Gamma_2}\ \sqcup\text{-l}}{{}^{+\exists R}\delta, \Delta_2 \Rightarrow {}^{+\exists R}\Gamma_1, \Gamma_2}\ ({}^{L}(\alpha \sqcup \beta), {}^{\exists R,L}(\alpha \sqcup \beta))$$

This proof can be transformed into:

$$\dfrac{\dfrac{\dfrac{\begin{matrix} P_1 \\ \delta \Rightarrow \Gamma_1, {}^{L}\alpha, {}^{L}\beta \end{matrix} \qquad \begin{matrix} P_2 \\ \Delta_2, {}^{\exists R,L}\alpha \Rightarrow \Gamma_2 \end{matrix}}{{}^{+\exists R}\delta, \Delta_2 \Rightarrow {}^{+\exists R}\Gamma_1, {}^{\exists R,L}\beta, \Gamma_2}\ ({}^{L}\alpha, {}^{\exists R,L}\alpha) \qquad \begin{matrix} P_3 \\ \Delta_2, {}^{\exists R,L}\beta \Rightarrow \Gamma_2 \end{matrix}}{{}^{+\exists R}\delta, \Delta_2, \Delta_2 \Rightarrow {}^{+\exists R}\Gamma_1, \Gamma_2, \Gamma_2}\ ({}^{\exists R,L}\beta)}{{}^{+\exists R}\delta, \Delta_2 \Rightarrow {}^{+\exists R}\Gamma_1, \Gamma_2}\ \text{perm*; contract*}$$

which again contains one *mix* and one *quasi-mix*, but both with grade less than the grade of *quasi-mix* on P. So by the induction hypothesis, we can obtain a proof which contains no *quasi-mixes* at all. All other cases of outermost logical symbol in *quasi-mix* inferences can be obtained in a similar way.

Case 2: *rank* > 2***, i.e.,*** $rank_l(P) > 1$ ***and/or*** $rank_r(P) > 1$ The induction hypothesis is that from every proof Q which contains a *quasi-mix* only as the last inference, and which satisfies either $g(Q) < g(P)$, or $g(Q) = g(P)$ and $rank(Q) < rank(P)$, we can eliminate the application of the *quasi-mix.*

2.1) $rank_r(P) > 1$

2.1.1) Let us consider a *quasi-mix* of the form $\left({}^{L}\alpha, {}^{\exists R,L}\alpha\right)$ in which Γ_2 contains ${}^{\exists R,L}\alpha$ or ${}^{L}\delta$ is ${}^{L}\alpha$. In this case, we construct a new proof as follows.

$$\dfrac{\dfrac{\dfrac{{}^{L}\delta \Rightarrow \Gamma_1}{{}^{\exists R,L}\delta \Rightarrow {}^{+\exists R}\Gamma_1}\ \text{prom-}\exists}{{}^{\exists R,L}\delta \Rightarrow {}^{+\exists R}\Gamma_1^*, {}^{\exists R,L}\alpha}\ \text{perm*; contract*}}{{}^{\exists R,L}\delta, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1^*, \Gamma_2}\ \text{weak*; perm*}$$

where the assumption Γ_2 contains ${}^{\exists R,L}\alpha$ were used in the last inference to construct Γ_2. When Δ_1 is ${}^{L}\alpha$, we construct a new proof as follows:

$$\dfrac{\dfrac{\Delta_2 \Rightarrow \Gamma_2}{{}^{\exists R,L}\alpha, \Delta_2^* \Rightarrow \Gamma_2}\ \text{perm*; weak*}}{{}^{\exists R,L}\alpha, \Delta_2^* \Rightarrow {}^{+\exists R}\Gamma_1^*, \Gamma_2}\ \text{perm*; weak*}$$

2.1.2) S_2 is the lower sequent of a inference J_2, where J_2 is not a logical inference whose principal formula is δ. We will consider just the case where the *quasi-mix* is of the form $(\delta, {}^{+\exists R}\delta)$, the other cases of quasi-mix can be treated in a similar way. P has the form:

$$\dfrac{\begin{array}{c} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{array} \quad \dfrac{\begin{array}{c} P_2 \\ \Phi \Rightarrow \Psi \end{array}}{\Delta_2 \Rightarrow \Gamma_2}\ J_2}{\Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2}\ (\delta, {}^{+\exists R}\delta)$$

where P_1 and P_2 contain no *quasi-mixes* and Φ contains at least one occurrence of ${}^{+\exists R}\delta$. We first consider the proof P':

$$\dfrac{\begin{array}{c} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{array} \quad \begin{array}{c} P_2 \\ \Phi \Rightarrow \Psi \end{array}}{\Delta_1, \Phi^* \Rightarrow \Gamma_1^*, \Psi}\ (\delta, {}^{+\exists R}\delta)$$

$g(P) = g(P')$, $rank_l(P') = rank_l(P)$ and $rank_r(P') = rank_r(P) - 1$. Thus, by the induction hypothesis, the final sequent in P' is provable without *quasi-mix*. Given that, we can now construct a proof:

$$\dfrac{\dfrac{\begin{matrix} P' \\ \Delta_1, \Phi^* \Rightarrow \Gamma_1^*, \Psi \end{matrix}}{\Phi^*, \Delta_1 \Rightarrow \Gamma_1^*, \Psi}\ \text{perm}^*}{\Delta_2^*, \Delta_1 \Rightarrow \Gamma_1^*, \Gamma_2}\ J_2$$

In the case that the auxiliary formula in J_2 in P is a mix in Φ, we need an additional weakening before J_2 in the last proof.

2.1.3) Δ_1 contains no δ's, S_2 is the lower sequent of a logical inference whose principal formula is δ and J is a *mix* rule inference. We have to consider several cases according to the outermost logical symbol of δ:

i) The outermost logical symbol of δ is $\sqcap$. The last part of P is of the form:

$$J\ \dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} P_2 \\ \Delta_2, {}^L\alpha, {}^L\beta \Rightarrow \Gamma_2 \end{matrix}}{\Delta_2, {}^L(\alpha \sqcap \beta) \Rightarrow \Gamma_2}\ J_2}{\Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2}\ \left({}^L(\alpha \sqcap \beta)\right)$$

Now let us consider the proof Q:

$$J\ \dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \begin{matrix} P_2 \\ \Delta_2, {}^L\alpha, {}^L\beta \Rightarrow \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2^*, {}^L\alpha, {}^L\beta \Rightarrow \Gamma_1^*, \Gamma_2}\ \left({}^L(\alpha \sqcap \beta)\right)$$

assuming that ${}^L(\alpha \sqcap \beta)$ is in Δ_2. Note that $g(Q) = g(P)$, $rank_l(Q) = rank_l(P)$ and $rank_r(Q) = rank_r(P) - 1$. Hence by the induction hypothesis, the end-sequent of Q is provable without a mix. Let us call such proof Q' and consider the following proof P':

$$J\ \dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} Q' \\ \Delta_1, \Delta_2^*, {}^L\alpha, {}^L\beta \Rightarrow \Gamma_1^*, \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2^*, {}^L(\alpha \sqcap \beta) \Rightarrow \Gamma_1^*, \Gamma_2}\ J_2}{\Delta_1, \Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_1^*, \Gamma_2}\ \left({}^L(\alpha \sqcap \beta)\right)$$

Given that, $g(P') = g(P)$, $rank_l(P') = rank_l(P)$ and $rank_r(P') = 1$ (for Δ_1 contains no occurences of ${}^L(\alpha \sqcap \beta)$) by the induction hypothesis the end-sequent of P' is provable without a mix, and so is the end-sequent of P.

ii) The outermost logical symbol of δ is $\sqcup$. Let us consider a proof P whose last part is of the form:

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} P_2 \\ \Delta_2, {}^{L}\alpha \Rightarrow \Gamma_2 \end{matrix} \qquad \begin{matrix} P_3 \\ \Delta_2, {}^{L}\beta \Rightarrow \Gamma_2 \end{matrix}}{\Delta_2, {}^{L}(\alpha \sqcup \beta) \Rightarrow \Gamma_2}}{\Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2} \; ({}^{L}(\alpha \sqcup \beta))$$

Assuming that ${}^{L}(\alpha \sqcup \beta)$ is in P_1 and P_2, consider the proof Q_1:

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \begin{matrix} P_2 \\ \Delta_2, {}^{L}\alpha \Rightarrow \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2^*, {}^{L}\alpha \Rightarrow \Gamma_1^*, \Gamma_2} \; ({}^{L}(\alpha \sqcup \beta))$$

and Q_2:

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \begin{matrix} P_3 \\ \Delta_2, {}^{L}\beta \Rightarrow \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2^*, {}^{L}\beta \Rightarrow \Gamma_1^*, \Gamma_2} \; ({}^{L}(\alpha \sqcup \beta))$$

We note that $g(Q_1) = g(Q_2) = g(P)$, $rank_l(Q_1) = rank_l(Q_2) = rank_l(P)$ and $rank_r(Q_1) = rank_r(Q_2) < rank_r(P)$. Hence, by the induction hypothesis, the end-sequents of P_1 and P_2 are provable without a *mix*. Let us consider new proofs without *mix* Q_1' and Q_2' in the construction of P':

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} Q_1' \\ \Delta_1, \Delta_2^*, {}^{L}\alpha \Rightarrow \Gamma_1^*, \Gamma_2 \end{matrix} \qquad \begin{matrix} Q_2' \\ \Delta_1, \Delta_2^*, {}^{L}\beta \Rightarrow \Gamma_1^*, \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2^*, {}^{L}(\alpha \sqcup \beta) \Rightarrow \Gamma_1^*, \Gamma_2} \; \sqcup\text{-l}}{\Delta_1, \Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_1^*, \Gamma_2} \; ({}^{L}(\alpha \sqcup \beta))$$

Then, $g(P') = g(P)$, $rank_l(P') = rank_l(P)$ and $rank_r(P') = 1$, since Δ_1 and Δ_2^* do not contain ${}^{L}(\alpha \sqcup \beta)$. By the induction hypothesis the end-sequent of P' is provable without a *mix*.

iii) The outermost logical symbol of δ is $\forall$. That is, the mix formula is of the form ${}^{L}(\forall R.\alpha)$. Let us consider the proof P:

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \dfrac{\begin{matrix} P_2 \\ \Delta_2, {}^{L,R}\alpha \Rightarrow \Gamma_2 \end{matrix}}{\Delta_2, {}^{L}(\forall R.\alpha) \Rightarrow \Gamma_2} \; \forall\text{-l}}{\Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2} \; ({}^{L}\forall R.\alpha)$$

where ${}^{L}\forall R.\alpha$ occurs on Δ_2 since $rank_r(P) > 1$. Let us consider a proof Q as follows:

$$\dfrac{\begin{matrix} P_1 \\ \Delta_1 \Rightarrow \Gamma_1 \end{matrix} \qquad \begin{matrix} P_2 \\ \Delta_2, {}^{L,R}\alpha \Rightarrow \Gamma_2 \end{matrix}}{\Delta_1, \Delta_2^*, {}^{L,R}\alpha \Rightarrow \Gamma_1^*, \Gamma_2} \; ({}^{L}\forall R.\alpha)$$

Note that $grade(Q) = grade(P)$, $rank_l(Q) = rank_l(P)$ and $rank_r(Q) = rank_r(P) - 1$. So, by the induction hypothesis on can obtain a proof Q' with the same end-sequent as Q without *quasi-mix* inferences. Now consider the new proof P':

$$\dfrac{\dfrac{\dfrac{P_1}{\Delta_1 \Rightarrow \Gamma_1} \qquad \dfrac{\dfrac{Q'}{\Delta_1, \Delta_2^*, {}^{L,R}\alpha \Rightarrow \Gamma_1^*, \Gamma_2}}{\Delta_1, \Delta_2^*, {}^{L}\forall R.\alpha \Rightarrow \Gamma_1^*, \Gamma_2}\,\forall\text{-l}}{\Delta_1, \Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_1^*, \Gamma_2}\,({}^{L}\forall R.\alpha)}{\Delta_1, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2}\,\text{contract}^*$$

Now we have

$$g(P') = g(P) \quad \text{and} \quad rank_l(P') = rank_l(P)$$

and $rank_r(P') = 1$ (for Δ_1 and Δ_2^* do not contain ${}^{L}\forall R.\alpha$). Thus the end-sequent of P' (the same of P) is provable without *quasi-mix* by the induction hypothesis.

The remaining cases where δ is of the form ${}^{L}\exists R.\alpha$ and ${}^{L}\neg\alpha$ are treated in a similar way.

2.1.4) The same conditions that hold for 2.1.3 but J is a *quasi-mix* rule inference. We have to consider several cases according to the outermost logical symbol of δ. All the cases are treated in a similar way of cases 2.1.3.

2.2) $rank_r(P) = 1$ and $rank_l(P) > 1$. This case is proved as in case 2.1 above.

■