III
The Sequent Calculus for $\mathcal{ALC}$

III.1 A Sequent Calculus for logic$\mathcal{ALC}$

The Sequent Calculus for $\mathcal{ALC}$ that it is shown in Figures III.1 and III.2 considers the extension of the language $\mathcal{ALC}$ presented in Section II.1 for labeled concepts. The labels are a list of existencial or universal quantified roles names. Its syntax is as following:

$$L \rightarrow \forall R, L \mid \exists R, L \mid \emptyset$$
$$\phi_{lc} \rightarrow L \phi_{c}$$

where $R$ stands for atomic role names, $L$ for list of labels and $\phi_{c}$ for $\mathcal{ALC}$ concept descriptions defined in Section II.1.

Each labeled $\mathcal{ALC}$ concept has a straightforward $\mathcal{ALC}$ concept equivalent. For example, the $\mathcal{ALC}$ concept $\exists R_2.\forall Q_2.\exists R_1.\forall Q_1.\alpha$ has the same semantics of the labeled concept $\exists R_2.\forall Q_2.\exists R_1.\forall Q_1.\alpha$.

In other words, the list of labels are just the roles prefix of a concept. Labels are syntactic artifacts of our system, which means that labeled concepts and its equivalent $\mathcal{ALC}$ have the same semantics. The system was designed to be extended to description logics with role constructors and subsumptions of roles. This is one of the main reasons to use roles-as-labels in its formulation. Besides that, whenever roles are promoted to labels the rules of the calculus can compose or decompose concept description preserving role prefix stored as labels. In that way, labels are a kind of “context” where concept manipulation occurs.

Given that any labeled concept has an equivalent $\mathcal{ALC}$ concept, the semantics of a labeled concept can be given with the support of a formal transformation of labeled concepts into $\mathcal{ALC}$ concepts. We defined the function $\sigma : \phi_{lc} \rightarrow \phi_{c}$ that takes a labeled $\mathcal{ALC}$ concept an returns a $\mathcal{ALC}$ concept. Considering $\alpha$ an $\mathcal{ALC}$ concept description, the function $\sigma$ is recursively defined as:
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\[
\begin{align*}
\sigma (\emptyset \alpha) &= \alpha \\
\sigma (^{\forall R,L} \alpha) &= \forall R. \sigma (^{L} \alpha) \\
\sigma (^{\exists R,L} \alpha) &= \exists R. \sigma (^{L} \alpha)
\end{align*}
\]

Given \( \sigma \), the semantics of a labeled concept \( \gamma \) is given by \( \sigma (\gamma)^T \).

We define \( \Delta \Rightarrow \Gamma \) as a sequent where \( \Delta \) and \( \Gamma \) are finite sequences of labeled concepts. The natural interpretation of the sequent \( \Delta \Rightarrow \Gamma \) is the \( \mathcal{ALC} \) formula:

\[
\bigcap_{\delta \in \Delta} \sigma (\delta) \subseteq \bigcup_{\gamma \in \Gamma} \sigma (\gamma)
\]

The \( \text{SC}_{\mathcal{ALC}} \) system is presented in Figures III.1 and III.2. In all rules of the figures, the greek letters \( \alpha \) and \( \beta \) stands for \( \mathcal{ALC} \) concepts (formulas without labels), \( \gamma_i \) and \( \delta_i \) stands for labeled concepts, \( \Gamma_i \) and \( \Delta_i \) for lists of labeled concepts. For a clean presentation, the lists of labels are omitted whenever they are not used in the rule, this is the case of all structural rules in Figure III.1. The notation \( L \Gamma \) has to be taken as a list of labeled formulas of the form \( L \gamma_1, \ldots, L \gamma_k \) for all \( \gamma \in \Gamma \). The notation \( ^{+\forall R} \gamma \) (resp. \( ^{+\exists R} \gamma \)) which can also be used with list of labeled concepts, \( ^{+\forall \Gamma} \) (resp. \( ^{+\exists \Gamma} \)), means the addition of a label, \( \forall R \) or \( \exists R \) of a given role \( R \), in front of the list of labels of \( \gamma \), respectively in all \( \gamma \in \Gamma \). Finally, we write \( ^{3L} \alpha \) to denote that all labels of \( L \) are existential quantificated and \( ^{\forall L} \alpha \) whenever all labels are universal quantificated (value restricted).

Considering the labeled formula \( L \alpha \), the notation \( ^{L} \beta \) denotes exchanging the universal roles occurring in \( L \) for existential roles and vice-versa in a consistent way. Thus, if \( \beta \equiv -\alpha \) them the formulas will be a negation each other. For example, \( ^{+\forall R.3Q} \beta \) is \( 3R.\forall Q.\neg \alpha \).

The system ought to be used by applying propositional rules, then the introduction of labels and then the quantification rules. This procedure will derive a normal derivation. Example 1 was taken from [7] and is useful to give an idea of how the rules of the \( \text{SC}_{\mathcal{ALC}} \) system can be used.

**Example 1** Given the \( \mathcal{ALC} \) subsumption axiom:

\[
\exists \text{child}. \top \sqcap \forall \text{child}. \neg (\exists \text{child}. \neg \text{Doctor}) \sqsubseteq \exists \text{child}. \forall \text{child}. \text{Doctor} \quad (1)
\]

In \( \text{SC}_{\mathcal{ALC}} \), we can prove that it is a theorem with the proof:
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\[
\begin{align*}
\alpha \Rightarrow \alpha \\
\Delta \Rightarrow \Gamma & \quad \text{weak-l} \\
\Delta, \delta \Rightarrow \Gamma & \quad \text{weak-r} \\
\Delta, \delta, \delta \Rightarrow \Gamma & \quad \text{contraction-l} \\
\Delta \Rightarrow \Gamma, \gamma & \quad \text{contraction-r} \\
\Delta_1, \delta_1, \delta_2, \Delta_2 \Rightarrow \Gamma & \quad \text{perm-l} \\
\Delta_1 \Rightarrow \Gamma, \gamma_1, \gamma_2, \Gamma_2 & \quad \text{perm-r} \\
\Delta_1 \Rightarrow \Gamma_1, L, \Delta \Rightarrow \Gamma_2 & \quad \text{cut} \\
\end{align*}
\]

Figure III.1: The System SC\(_{\text{ALC}}\): structural rules

\[
\begin{align*}
\Delta, \Delta, L, \forall R \alpha \Rightarrow \Gamma & \quad \forall-l \\
\Delta, L, (\forall R \alpha)L_2 \Rightarrow \Gamma & \quad \forall-r \\
\Delta, \forall L \alpha, \forall L \beta \Rightarrow \Gamma & \quad \forall-l \\
\Delta, \forall L (\alpha \cap \beta) \Rightarrow \Gamma & \quad \forall-r \\
\Delta, \exists L \alpha \Rightarrow \Gamma & \quad \exists-l \\
\Delta, \exists L (\exists R \alpha) \Rightarrow \Gamma & \quad \exists-r \\
\Delta, \exists L \alpha, \exists L \beta \Rightarrow \Gamma & \quad \exists-l \\
\Delta, \exists L (\alpha \sqcup \beta) \Rightarrow \Gamma & \quad \exists-r \\
\Delta \Rightarrow \Gamma, \neg L \alpha & \quad \neg-l \\
\Delta \Rightarrow \Gamma, \neg L \alpha \Rightarrow \Gamma & \quad \neg-r \\
\delta \Rightarrow \Gamma & \quad \text{prom-}\exists \\
\+ \forall \Rightarrow + \forall \Gamma & \quad \text{prom-}\forall \\
\end{align*}
\]

Figure III.2: The System SC\(_{\text{ALC}}\): logical rules
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III.2 $\text{SC}_{\text{ALC}}$ Soundness

The soundness of $\text{SC}_{\text{ALC}}$ is proved by taking into account the intuitive meaning of each sequent and establishing that the truth preservation holds. From Section III.1, a sequent $\Delta \Rightarrow \Gamma$ is equivalent in meaning to the $\text{ALC}$ formula:

$$\prod_{\delta \in \Delta} \sigma(\delta) \subseteq \bigcup_{\gamma \in \Gamma} \sigma(\gamma)$$

A sequent is defined to be valid or a tautology if and only if its corresponding $\text{ALC}$ formula is.

When using the calculus, the usual axioms of a particular DL theory (TBox or an ontology) of the form $C \subseteq D$ should be taken as sequents $C \Rightarrow D$. Labeled formulas occur only during the proof procedure, since they are in practical terms taken as intermediate data.

**Theorem 11 (SC$_{\text{ALC}}$ is sound)** Considering $\Omega$ a set of sequents, a theory or a TBox, let a $\Omega$-proof be any $\text{SC}_{\text{ALC}}$ proof in which sequents from $\Omega$ are permitted as initial sequents (in addition to the logical axioms). The soundness of $\text{SC}_{\text{ALC}}$ states that if a sequent $\Delta \Rightarrow \Gamma$ has a $\Omega$-proof, then $\Delta \Rightarrow \Gamma$ is satisfied by every interpretation which satisfies $\Omega$. That is,

$$\text{if } \Omega \vdash_{\text{SC}_{\text{ALC}}} \Delta \Rightarrow \Gamma \text{ then } \Omega \models \prod_{\delta \in \Delta} \sigma(\delta) \subseteq \bigcup_{\gamma \in \Gamma} \sigma(\gamma)$$

In the proof of Theorem 11 we will write $\Delta^\mathcal{I}$ as an abbreviation for the set interpretation of the conjunction of concepts in $\Delta$, that is, $\bigcap_{\delta \in \Delta} \sigma(\delta)^\mathcal{I}$, and $\Gamma^\mathcal{I}$ as an abbreviation for the set interpretation of the disjunction of the concepts in $\Gamma$, $\bigcup_{\gamma \in \Gamma} \sigma(\gamma)^\mathcal{I}$. 
During the proof below, we will use many times the axioms and facts from Section II.6.

**Proof:** We prove Theorem 11 by induction on the length of the $\Omega$-proofs. The length of a $\Omega$-proof is the number of applications for any derivation rule of the calculus in a top-down approach.

**Base case** Proofs with length zero are proofs $\Omega \vdash \Delta \Rightarrow \Gamma$ where $\Delta \Rightarrow \Gamma$ occurs in $\Omega$. In that case, it is easy to see that the theorem holds.

For the initial sequents, logical axioms like $C \Rightarrow C$, it is easy to see that $\sigma(C)^I \subseteq \sigma(C)^I$ for every interpretation $I$ since every set is a subset of itself.

**Induction hypothesis** As inductive hypothesis, we will consider that for proofs of length $n$ the theorem holds. It is now sufficient to show that each of the derivation rules preserves the truth. That is, if the premises holds, the conclusion must also hold.

**Cut rule** Given the sequents $\Delta_1 \Rightarrow \Gamma_1, L C$ and $L C, \Delta_2 \Rightarrow \Gamma_2$ then, by hypothesis, we know that they are valid and so

$$\bigcap_{\delta \in \Delta_1} \sigma(\delta)^I \subseteq \bigcup_{\gamma \in \Gamma_1} \sigma(\gamma)^I \cup \sigma(L C)^I$$

and

$$\sigma(L C)^I \cap \bigcap_{\delta \in \Delta_2} \sigma(\delta)^I \subseteq \bigcup_{\gamma \in \Gamma_2} \sigma(\gamma)^I$$

Let $\Delta_1^I = \bigcap_{\delta \in \Delta_1} \sigma(\delta)^I$, $\Gamma_1^I = \bigcup_{\gamma \in \Gamma_1} \sigma(\gamma)^I$, $\Delta_2^I = \bigcap_{\delta \in \Delta_2} \sigma(\delta)^I$, $\Gamma_2^I = \bigcup_{\gamma \in \Gamma_2} \sigma(\gamma)^I$ and $X = \sigma(L C)^I$. Now we must show that the application of the cut rule preserves the set inclusion. In other words, given $\Delta_1^I \subseteq (\Gamma_1^I \cup X)$ and $(X \cap \Delta_2^I) \subseteq \Gamma_2^I$, we must have $(\Delta_1^I \cap \Delta_2^I) \subseteq (\Gamma_1^I \cup \Gamma_2^I)$. What is easy to show using the standard set theory.

**Rules weak-l and weak-r** Given the sequent $\Delta \Rightarrow \Gamma$, by the inductive hypothesis we know that

$$\Delta^I \subseteq \Gamma^I$$

By set theory, $\Delta^I \cap X \subseteq \Gamma^I$ and $\Delta^I \subseteq \Gamma^I \cup X$ for any set $X$ interpretation of a labeled concept $\alpha$. In the first case, we have the interpretation of $\Delta, \alpha \Rightarrow \Gamma$. In the second case, we have the interpretation of $\Delta \Rightarrow \Gamma, \alpha$. This is sufficient to show the soundness of both rules.
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**Rules perm-l and perm-r** By the definition of the meaning of a sequent and its semantics, it is easy to see that both rules are sound. Note that the order of the formulas in both sides of a sequent do not change the sequent semantics.

**Rules prom-∀ and prom-∃** The soundness of rule prom-∃ if easily proved using the Fact 1 and Axiom 3. The soundness of rule prom-∀ is proved using Fact 2 and the Axiom 1.

**Rules ∀-r, ∀-l, ∃-r and ∃-l** From the definition of σ function, we know that in all those four rules, both the premises and the conclusions have, given a interpretation function, the same semantics.

**Rules □-l and □-r** In order to prove the soundness of those rules we need the ALC Axiom 1 that states the distributivity of the universal quantified constructor over the conjunction. Moreover, we must observe that both rules have an important proviso. That is, they are restricted to details only with labeled concepts were all labels are universal quantified. This restriction permit us to apply the Axiom 1 inductively.

Taking the sequent $\Delta, L\alpha, L\beta \Rightarrow \Gamma$ valid as hypothesis, we have:

$$\Delta^I \cap \sigma(L\alpha)^I \cap \sigma(L\beta)^I \subseteq \Gamma^I$$

To show that the rule (□-l) is sound, We must prove that $\Delta, L\alpha \land L\beta \Rightarrow \Gamma$ is also valid. In other words, that $\Delta^I \cap \sigma(L\alpha \land L\beta)^I \subseteq \Gamma^I$ holds. What is true by the definition of σ, the Axiom 1 and the rule proviso which allows us to apply the Axiom 1 over the list of labels $L$.

Now consider the rule (□-r). By induction hypothesis, the sequents $\Delta \Rightarrow \Gamma, L\alpha$ and $\Delta \Rightarrow \Gamma, L\beta$ are valid, and so,

$$\Delta^I \subseteq \Gamma^I \cup \sigma(L\alpha)^I \quad \text{and} \quad \Delta^I \subseteq \Gamma^I \cup \sigma(L\beta)^I$$

holds for all interpretations $^I$. Now, suppose the application of the rule (□-r) over the two sequent above. We must show that $\Delta \Rightarrow \Gamma, L(\alpha \land L\beta)$ is also valid, that is,

$$\Delta^I \subseteq \Gamma^I \cup \sigma(L(\alpha \land L\beta))^I$$

holds. But by basic set theory we have

$$\Delta^I \subseteq ((\Gamma^I \cup \sigma(L\alpha)^I) \cap (\Gamma^I \cup \sigma(L\beta)^I))$$
And by distributive law

$$\Delta^T \subseteq \Gamma^T \cup (\sigma^T(\lambda) \cap \sigma^T(\beta))$$

Finally, by the definition of $\sigma$, the Axiom (1) and the rule proviso, we can conclude that the rule conclusion if valid.

**Rules $\sqcup$-l and $\sqcup$-r** In both rules the proviso is that the labels list $L$ must contain only existential quantified roles. The soundness of both rules are proved with the support of this proviso and the Axiom 3, applied inductively over the labels lists.

As inductive hypothesis the sequents $\Delta, \lambda \Rightarrow \Gamma$ and $\Delta, \beta \Rightarrow \Gamma$ are valid. That is, given $\Delta^T = \bigcap_{\delta \in \Delta} \sigma^T(\delta)$ and $\Gamma^T = \bigcup_{\gamma \in \Gamma} \sigma^T(\gamma)$, we know that

$$\Delta^T \cap \sigma^T(\lambda) \subseteq \Gamma^T \quad \text{and} \quad \Delta^T \cap \sigma^T(\beta) \subseteq \Gamma^T$$

holds. Now considering the application of the rule ($\sqcup$-l) over the two sequents above we must prove that the resulting sequent $\Delta, \alpha \sqcup \beta \Rightarrow \Gamma$ is also valid:

$$\Delta^T \cap \sigma^T(\alpha \sqcup \beta) \subseteq \Gamma^T$$

Following from the hypothesis and basic set theory we know that if $\Delta^T \cap X_1 \subseteq \Gamma^T$ and $\Delta^T \cap X_2 \subseteq \Gamma^T$ then $(\Delta^T \cap X_1) \cup (\Delta^T \cap X_2) \subseteq \Gamma^T$ what gives us

$$\Delta^T \cap (\sigma^T(\lambda) \cup \sigma^T(\beta)) \subseteq \Gamma^T$$

and by the Axiom 3 applied inductively over the list $L$ we have the desired semantics of the resulting sequent:

$$\Delta^T \cap (\sigma^T(\alpha \sqcup \beta)) \subseteq \Gamma^T$$

For rule ($\sqcup$-r) the inductive hypothesis is that $\Delta \Rightarrow \Gamma, \alpha, \beta$ is valid. And so, the following statement must holds:

$$\Delta^T \subseteq \Gamma^T \cup \sigma^T(\lambda) \cap \sigma^T(\beta)$$

Now by the Axiom 3 we can rewrite to

$$\Delta^T \subseteq \Gamma^T \cup \sigma^T(\alpha \sqcup \beta)$$

What is the semantics of the rule conclusion.
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Rules $\neg l$ and $\neg r$ Given a concept $^L\alpha$ and an interpretation $^I$, we define the set $X = \sigma(^L\alpha)^I$ and the interpretation of its negation, $\sigma(^L\neg\alpha)^I$, will be the set $\overline{X} = \Delta^I \setminus X$.

For rule ($\neg l$), the inductive hypothesis is that the premise $\Delta \Rightarrow \Gamma, ^L\alpha$ is valid. Which means that $\Delta^I \subseteq (\Gamma^I \cup X)$. From the basic set theory this implies that $(\Delta^I \cap \overline{X}) \subseteq \Delta^I$, which is the interpretation of the conclusion.

For rule ($\neg r$), the inductive hypothesis is that the premise $\Delta, ^L\alpha \Rightarrow \Gamma$ is valid. Which means that $(\Delta^I \cap X) \subseteq \Gamma^I$. From the basic set theory this implies that $\Delta^I \subseteq (\Gamma^I \cup \overline{X})$, the interpretation of the conclusion as desired.

III.3 The Completeness of $\text{SC}_{\mathcal{ALC}}$

We show the relative completeness of $\text{SC}_{\mathcal{ALC}}$ regarding the axiomatic presentation of $\mathcal{ALC}$ from Section II.6. Since $\mathcal{ALC}$ formulas are not labeled, the completeness must take into account only formulas with empty list of labels. Proceeding in this way, the $\mathcal{ALC}$ sequent calculus deduction rules without labels behave exactly as the sequent calculus rules for classical propositional logic. Thus, in order to prove that $\text{SC}_{\mathcal{ALC}}$ is complete, we have only to derive the axioms above.

The derivation of the rule of necessitation is accomplished by

\[
\begin{align*}
\Rightarrow \alpha & \quad \text{prom-v} \\
\Rightarrow \forall R. \alpha & \quad \text{v_r} \\
\Rightarrow \forall R. \alpha & 
\end{align*}
\]

The derivation of the Axiom 1 is obtained from the following derivations. First we consider the case:

\[
\forall R. (\alpha \sqcap \beta) \subseteq \forall R. \alpha \sqcap \forall R. \beta
\]

\[
\begin{array}{c}
\forall R. \alpha \Rightarrow \forall R. \alpha \quad \text{prom-\forall} \\
\forall R. \beta \Rightarrow \forall R. \beta \quad \text{weak-l} \\
\forall R. \alpha \sqcap \forall R. \beta \Rightarrow \forall R. \alpha \quad \text{\forall_r} \\
\forall R. \alpha \sqcap \forall R. \beta \Rightarrow \forall R. \beta \quad \text{\forall_l} \\
\forall R. (\alpha \sqcap \beta) \Rightarrow \forall R. \alpha \sqcap \forall R. \beta \\
\forall R. (\alpha \sqcap \beta) \Rightarrow \forall R. (\alpha \sqcap \beta) \\
\end{array}
\]

Finally, we prove the subsumption from right to left:

\[
\forall R. \alpha \sqcap \forall R. \beta \subseteq \forall R. (\alpha \sqcap \beta)
\]

by
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III.4 The cut-elimination theorem

In this section we adopt the usual terminology of proof theory for sequent calculus presented in [13, 66]. We follow Gentzen’s original proof for cut elimination with the introduction of the mix rule.

Let \( \delta \) be a labeled formula. An inference of the following form is called mix with respect to \( \psi \), a labeled concept:

\[
\frac{\Delta_1 \Rightarrow \Gamma_1 \quad \Delta_2 \Rightarrow \Gamma_2}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2} (\psi)
\]

where both \( \Gamma_1 \) and \( \Delta_2 \) contain the formula \( \delta \), and \( \Gamma_1^* \) and \( \Delta_2^* \) are obtained from \( \Gamma_1 \) and \( \Delta_2 \) respectively by deleting all the occurrences of \( \delta \) in them.

But in order to obtain an easier presentation of our cut elimination we introduce four additional rules of inference called quasi-mix rules.

\[
\frac{L_\delta \Rightarrow \Gamma_1 \quad \Delta_2 \Rightarrow \Gamma_2}{\exists L_\delta, \Delta_1^*, \Delta_2^* \Rightarrow +\exists R \Gamma_1, \Gamma_2} (L_\alpha, \exists R L_\alpha)
\]

\[
\frac{\Delta_1 \Rightarrow L_\alpha \quad \Delta_2 \Rightarrow \Gamma_2}{+\forall R \Delta_1, \Delta_2^* \Rightarrow \Gamma_2} (L_\alpha, \forall R L_\alpha)
\]

\[
\frac{\Delta_1 \Rightarrow L_\alpha \quad \Delta_2 \Rightarrow \Gamma_2}{\forall R \Delta_1, \Delta_2^* \Rightarrow \Gamma_2} (L_\alpha, \forall R L_\alpha)
\]

\[
\frac{\Delta_1 \Rightarrow \Gamma_1 \quad \Delta_2 \Rightarrow L_\gamma}{\Delta_1, +\forall R \Delta_2^* \Rightarrow \Gamma_1, \forall R L_\gamma} (L_\alpha, \forall R L_\alpha)
\]

where in each rule, the tuple of concepts on the right indicates the two mix formulas of this inference rule. \( \Gamma_1 \), the list of formulas on the right from the left-side premise, contains the first projection of the tuple, \( \Delta_2 \), the list of formulas on the left from the right-side premise, contains the second projection. \( \Gamma_1^* \) and \( \Delta_2^* \) are obtained from \( \Gamma_1 \) and \( \Delta_2 \) by deleting all occurrences of the first and second tuple’s projection, respectively. The notation \( +\exists R \Delta \) (resp. \( +\forall R \Delta \)) means the addition of \( \exists R \) (resp. \( \forall R \)) on the list of labels of all \( \delta \in \Delta \).

By the definitions of mix and quasi-mix rules, the mix rule is a special case of quasi-mix rules in which both mix formulas in the tuple are equal. Therefore, we can also consider a quasi-mix the mix rule.
**Definition 12 (The SC\(_{\text{ALC}}^*\) system)** We call SC\(_{\text{ALC}}^*\) the new system obtained from SC\(_{\text{ALC}}\) by replacing the cut rule by the quasi-mix (and mix) rules.

**Lemma 13** The systems SC\(_{\text{ALC}}\) and SC\(_{\text{ALC}}^*\) are equivalent, that is, a sequent is SC\(_{\text{ALC}}\)-provable if and only if that sequent is also SC\(_{\text{ALC}}^*\)-provable.

**Proof:** The four quasi-mix rules are derived from inferences where the promotional rules (prom-\(\forall\) and prom-\(\exists\)) are applied just before a mix rule. In that way, one can transformed all the applications of quasi-mix rule into a sequence of prom-\(\forall\) or prom-\(\exists\) followed by mix rules applications. All applications of mix rule can than be replaced by applications of cut rule provide that all the repetitions of the cut formula in the upper sequents being first transformed into just one occurrence on each sequent. This is easily done by one or more application of the contraction and permutation rules.

To illustrate the process, let us consider an application of a quasi-mix rule in the SC\(_{\text{ALC}}^*\)-proof fragment bellow where \(\Pi_n\) are proof fragments. The double-line labeled with “perm*; contract*” means the application of rule permutation one or more times followed by one or more applications of contraction rule.

\[
\begin{array}{c}
\Delta_1 \Rightarrow \Gamma_1 \\
\Pi_1 \\
\hline
\Delta_1 \Rightarrow \Gamma_1, L\alpha \Rightarrow \Gamma_2 \quad (3R.L\alpha, L\alpha) \\
\Pi_2 \\
\hline
\Delta_1 \Rightarrow \Gamma_1^*, +3R\Gamma_2 \\
\Pi_3 \\
\end{array}
\]

And its corresponding SC\(_{\text{ALC}}\)-proof:

\[
\begin{array}{c}
\Delta_1 \Rightarrow \Gamma_1 \\
\Pi_1 \\
\hline
\Delta_1 \Rightarrow \Gamma_1^*, 3 RL\alpha \\
\permutation^*; \contraction^* \\
\hline
\Delta_1 \Rightarrow \Gamma_1^*, +3R\Gamma_2 \quad \text{prom-}\exists \\
\Pi_3 \\
\end{array}
\]

By the proof of Lemma 13, a derivation \(\Pi\) of \(\Delta \Rightarrow \Gamma\) in SC\(_{\text{ALC}}\) with cuts can be transformed in a derivation \(\Pi'\) of \(\Delta \Rightarrow \Gamma\) in SC\(_{\text{ALC}}^*\) with quasi-mixes (and mixes). So that, it is sufficient to show that the quasi-mix (and mix) rules are redundant in SC\(_{\text{ALC}}^*\), since a proof in SC\(_{\text{ALC}}^*\) without quasi-mix (and mix) is at the same time a proof in SC\(_{\text{ALC}}\) without cut.

**Definition 14 (SC\(_{\text{ALC}}^T\) system)** SC\(_{\text{ALC}}\) was defined with initial sequents of the form \(\alpha \Rightarrow \alpha\) with \(\alpha\) a ALC concept definition (logical axiom). However, it is often convenient to allow for other initial sequents. So if \(T\) is a set of...
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sequent of the form $\Delta \Rightarrow \Gamma$, where $\Delta$ and $\Gamma$ are sequences of $\mathcal{ALC}$ concept descriptions (non-logical axioms), we define $\text{SC}^T_{\mathcal{ALC}}$ to be the proof system defined like $\text{SC}_{\mathcal{ALC}}$ but allowing initial sequents to be from $T$ too.

The Definition 14 can be extend to the system $\text{SC}^*_\mathcal{ALC}$ in the same way, obtaining the systems $\text{SC}^*_T\mathcal{ALC}$.

**Definition 15 (Free-quasi-mix free proof)** Let $P$ be an $\text{SC}^*_T\mathcal{ALC}$-proof. A formula occurring in $P$ is anchored (by an $T$-sequent) if it is a direct descendent of a formula from $T$ occurring in an initial sequent. A quasi-mix inference in $P$ is anchored if either:

(i) the mix formulas are not atomic and at least one of the occurrences of the mix formulas in the upper sequents is anchored, or

(ii) the mix formulas are atomic and both of the occurrences of the mix formulas in the upper sequents are anchored.

A quasi-mix inference which is not anchored is said to be free. A proof $P$ is free-quasi-mix free if it contains no free quasi-mixes.

Given that a mix is a special quasi of quasi-mix, the Definition 15 can also be used to define free mixes. If a proof $P$ is free-quasi-mix free it is also free-mix free.

**Theorem 16 (Free-quasi-mix Elimination)** Let $T$ be a set of sequents. If $\text{SC}^*_T\mathcal{ALC} \vdash \Delta \Rightarrow \Gamma$ then there is a free-quasi-mix free $\text{SC}^*_T\mathcal{ALC}$-proof of $\Delta \Rightarrow \Gamma$.

Theorem 16 is a consequence of the following lemma.

**Lemma 17** If $P$ is a proof of $S$ (in $\text{SC}^*_T\mathcal{ALC}$) which contains only one free-quasi-mix, occurring as the last inference, then $S$ is provable without any free-quasi-mix.

Theorem 16 is obtained from Lemma 17 by simple induction over the number of quasi-free-mix occurring in a proof $P$.

We can now concentrate our attention on Lemma 17. First we define three scalars as a measure of the complexity of the proof. The *grade* of a formula $^{L}\alpha$ is defined as the number of logical symbols of $\alpha$ (denoted by $g(L\alpha)$). The *label-degree* of a formula $^{L}\alpha$ is defined as $ld(L\alpha) = |L|$ where $|L|$ means the length of the list $L$.

Let $P$ be a proof containing only one quasi-mix as its last inference:
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The grade of a quasi-mix is

$$g(\gamma, \gamma') = g(\gamma) + g(\gamma')$$

Given that, the grade of a mix (a special quasi of quasi-mix) is the double of the grade of the mix formula.

In a similar way, the label-degree of a quasi-mix is

$$ld(\gamma, \gamma') = ld(\gamma) + ld(\gamma')$$

and the label-degree of a mix is again the double of the label-degree of the mix formula.

We say that the grade of $P$ (denote by $g(P)$) and the label-degree of $P$ (denoted by $ld(P)$) is the grade and label-degree of that quasi-mix.

We refer to the left and right sequents as $S_1$ and $S_2$ respectively, and to the lower sequent as $S$. We call a thread in $P$ a left (or right) thread if it contains the left (or right) upper sequent of the quasi-mix $J$. The rank of the thread $F$ in $P$ is defined as the number of consecutive sequents, counting upward from the left (right) upper sequent of $J$, that contains $\gamma$ ($\gamma'$) in its succedent (antecedent). Since the left (right) upper sequent always contains the mix formulas, the rank of a thread in $P$ is at least 1. The rank of a thread $F$ in $P$ is denoted by $\text{rank}(F; P)$ and is defined as follows:

$$\text{rank}_l(P) = \max\{\text{rank}(F; P)\},$$

where $F$ ranges over all the left threads in $P$, and

$$\text{rank}_r(P) = \max\{\text{rank}(F; P)\},$$

where $F$ ranges over all the right threads in $P$. The rank of $P$ is defined as

$$\text{rank}(P) = \text{rank}_l(P) + \text{rank}_r(P),$$

where $\text{rank}(P) \geq 2$.

Proof: We prove Lemma 17 by lexicographically induction on the ordered triple (grade, label-degree, rank) of the proof $P$. We divide the proof into two main cases, namely $\text{rank} = 2$ and $\text{rank} > 2$ (regardless of the grade and label-degree).
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**Case 1: rank = 2** We shall consider several cases according to the form of the proofs of the upper sequents of the quasi-mix.

1.1) The left upper sequent $S_1$ is an logical initial sequent. There are several cases to be examined.

   a) $P$ has the form:

   $\begin{array}{c}
   \alpha \Rightarrow \alpha \\
   \Delta_2 \Rightarrow \Gamma_2 \\
   \end{array}
   \rightleftarrows
   \begin{array}{c}
   \exists R_\alpha, \Delta_2^* \Rightarrow \Gamma_2 \\
   \end{array}$

   \( (\alpha, \exists R_\alpha) \)

   We can easily obtain the same end-sequent without using the quasi-mix as follows: \(^1\)

   $\begin{array}{c}
   \Delta_2 \Rightarrow \Gamma_2 \\
   \end{array}
   \rightleftarrows
   \begin{array}{c}
   \exists R_\alpha, \ldots, \exists R_\alpha, \Delta_2^* \Rightarrow \Gamma_2 \\
   \end{array}$

   \( \text{perm}^* \)

   \( \exists R_\alpha, \Delta_2^* \Rightarrow \Gamma_2 \)

   \( \text{contract}^* \)

   All other cases, that it, other quasi-mix occurrences in a similar proof format, are treated in a similar way. Note also that a logical initial sequent can only have ALC formulas on both sides of the sequent.

1.2) The right upper sequent $S_2$ is an logical initial sequent. Similar as Case 1 above.

1.3) $S_1$ or $S_2$ (or both) are non-logical initial sequents. In this case, it is obvious that the quasi-mix is not a free and it will be not eliminated.

1.4) Neither $S_1$ nor $S_2$ are initial sequents, and $S_1$ is the lower sequent of a structural inference $J_1$. Since $\text{rank}_i(P) = 1$, the mix formula $\psi$ cannot appear in the succedent of the premiss of $J_1$, that is, $J_1$ must be the weak-r that introduced $\psi$. Again there are several cases to be examined for each possible quasi-mix rule used.

   a) Let us consider the quasi-mix case $(\La, \exists R_\alpha, \La) :

   \begin{array}{c}
   \delta \Rightarrow \Gamma_1 \\
   \Delta_2 \Rightarrow \Gamma_2 \\
   \end{array}
   \rightleftarrows
   \begin{array}{c}
   \exists R_\delta, \Delta_2^* \Rightarrow + \exists R_\Gamma_1, \Gamma_2 \\
   \end{array}$

   \( (\La, \exists R_\alpha, \La) \)

   where $\Gamma_1$ does not contain $\La$. We can eliminate the quasi-mix as follows:

\(^1\)The notation contract* (perm*) means zero or more applications of contraction (permutation) rule.
All other cases are treated in a similar way.

1.5) The same conditions that hold for Case 4 but with $S_2$ as the lower sequent of structural inference instead of $S_1$. As in Case 4.

1.6) Neither $S_1$ nor $S_2$ are an initial sequents and $S_1$ is the lower sequent of a prom-$\exists$ rule application and $J$ is a mix rule application.

$$
\begin{align*}
& P_1 \\
& \delta \Rightarrow \Gamma_1 \\
& +3R\delta \Rightarrow +3R\Gamma_1 \quad \text{prom-1} \\
& +3R\delta \Rightarrow +3R\Gamma_1 \\
& \Delta^*_2, +3R\delta \Rightarrow +3R\Gamma_1, \Delta^*_2 \quad \text{perm*} \\
& +3R\delta, \Delta^*_2 \Rightarrow +3R\Gamma_1, \Gamma_2 \\
& +3R\delta, \Delta^*_2 \Rightarrow +3R\Gamma_1, \Gamma_2
\end{align*}
$$

where by assumption none of the proofs $P_n$ for $n \in \{1,2\}$ contain a mix or quasi-mix. Moreover, $\Gamma_1$ does not contain $+3R\gamma$ since $\text{rank}_l(P) = 1$. That is, the prom-$\exists$ rule introduced the mix formula of $J$. We can replace the application of the mix rule by an application of quasi-mix rule as follows:

$$
\begin{align*}
& P_1 \\
& \delta \Rightarrow \Gamma_1 \\
& J \quad \text{prom-$\exists$} \\
& +3R\delta \Rightarrow +3R\Gamma_1 \\
& +3R\delta, \Delta^*_2 \Rightarrow +3R\Gamma_1, \Gamma_2 \\
& \Delta^*_2 \Rightarrow \Delta^*_2 \Rightarrow +3R\Gamma_1, \Gamma_2 \\
& +3R\gamma, \Delta^*_2 \Rightarrow +3R\Gamma_1, \Gamma_2 \\
& \Delta^*_2 \Rightarrow +3R\Gamma_1, \Gamma_2
\end{align*}
$$

The new quasi-mix rule has label-degree less than the label-degree of the original mix rule, $\text{ld}(+3R\gamma, +3R\gamma)$. So by the induction hypothesis, we can obtain a proof which contains no mixes.

1.7) Similar case as above with $S_1$ being lower sequent of a prom-$\forall$ or $S_2$ being lower sequent of prom-$\exists$ or prom-$\forall$. We apply similar transformation of mix application into quasi-mix rules applications. Always “moving” the mix upward into the direction of the prom-$\forall$ or prom-$\exists$ inference.

1.8) Both $S_1$ and $S_2$ are lower sequents of logical inferences and $\text{rank}_l(P) = \text{rank}_r(P) = 1$, $J$ being a mix with the mix formula $\gamma$ of each side being the principal formula of the logical inference. We use induction on the grade, distinguishing several cases according to the outermost logical symbol of $\gamma$:

i) The outermost logical symbol is $\cap$. $P$ has the form:
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$\begin{align*}
P_1 & \quad P_2 & \quad P_3 \\
\Delta_1 \Rightarrow \Gamma_1, L\alpha & \quad \Delta_2 \Rightarrow \Gamma_1, L\beta & \quad \Delta_3 \Rightarrow \Gamma_2, (L\alpha \land L\beta) \\
\Delta_1 \Rightarrow \Gamma_1, (L\alpha \land L\beta) & \quad \Delta_2 \Rightarrow \Gamma_1, (L\alpha \land L\beta) & \quad \Delta_3 \Rightarrow \Gamma_2, (L\alpha \land L\beta) \\
\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2
\end{align*}$

where by assumption none of the proofs $P_n$ for $n \in \{1, 2, 3\}$ contain a quasi-mix. We transform $P$ into:

$\begin{align*}
P_2 & \quad P_1 & \quad P_3 \\
\Delta_1 \Rightarrow \Gamma_1, L\beta & \quad \Delta_2 \Rightarrow \Gamma_1, L\alpha & \quad \Delta_3 \Rightarrow \Gamma_2, (L\alpha \land L\beta) \\
\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2
\end{align*}$

which contains two mix but both with grade less than $g(L\alpha \land L\beta)$. So by induction hypothesis, we can obtain a proof which contains no mixes. Note that the mix $(L\alpha)$ is now the last inference rule of a proof which contains no mix. Given that, this mix can be omitted using the transformations defined above.

ii) The outermost logical symbol is $\perp$. In this case $S_1$ and $S_2$ must be lower sequents of $\perp$-r and $\perp$-l rule, respectively:

$\begin{align*}
P_1 & \quad P_2 & \quad P_3 \\
\Delta_1 \Rightarrow \Gamma_1, L\alpha, L\beta & \quad \Delta_2 \Rightarrow \Gamma_2 & \quad \Delta_3 \Rightarrow \Gamma_2, (L\alpha \land L\beta) \\
\Delta_1 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1 \Rightarrow \Gamma_1, \Gamma_2
\end{align*}$

where, by hypothesis, none of the proofs $P_n$ for $n \in \{1, 2, 3\}$ contain a quasi-mix. This proof can be transformed into:

$\begin{align*}
P_1 & \quad P_2 & \quad P_3 \\
\Delta_1 \Rightarrow \Gamma_1, L\alpha, L\beta & \quad \Delta_2 \Rightarrow \Gamma_2 & \quad \Delta_3 \Rightarrow \Gamma_2, (L\alpha \land L\beta) \\
\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 & \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2
\end{align*}$

This proof contains two mix, but both with grade less than $g(L\alpha \land L\beta)$. So by the induction hypothesis, we can obtain a proof which contains no mixes. As mentioned above, the new created mixes are now the last inference rule of proofs which contains no mix.

iii) The outermost logical symbol is $\forall$. In this case $S_1$ and $S_2$ must be lower sequents of $\forall$-r and $\forall$-l rule, respectively. $P$ is:

$\begin{align*}
P_1 & \quad P_2 \\
\Delta_1 \Rightarrow \Gamma_1, L\forall R\alpha & \quad \Delta_2 \Rightarrow \Gamma_2 & \quad \Delta_3 \Rightarrow \Gamma_2, (L\forall R\alpha) \\
\Delta_1 \Rightarrow \Gamma_2 & \quad \Delta_1 \Rightarrow \Gamma_2 & \quad \Delta_1 \Rightarrow \Gamma_2
\end{align*}$
which again by hypothesis, none of the proofs \( P_n \) for \( n \in \{1, 2\} \) contain a mix. These proof can be transformed into:

\[
\begin{align*}
\Gamma & \vdash_{L,R} \alpha & \Delta_1 \Rightarrow \Gamma_1, \Delta_2 \Rightarrow \Gamma_2 \\
\Delta_1, \Delta_2 & \Rightarrow \Gamma_1, \Gamma_2
\end{align*}
\]

which contains one mix with grade less than \( g(\forall R. \alpha) \). So by induction hypothesis, we can obtain a proof which contains no mixes.

iv) The outermost logical symbol is \( \exists \). The treatment is similar to the case above.

v) The outermost logical symbol is \( \neg \) and \( P \) is:

\[
\begin{align*}
\Delta_1, \neg \alpha & \Rightarrow \Gamma_1 \\
\Delta_1 & \Rightarrow \Gamma_1, \neg \alpha \\
\Delta_2, \neg \alpha & \Rightarrow \Gamma_2
\end{align*}
\]

This proof can be transformed into:

\[
\begin{align*}
\Delta_1, \neg \alpha & \Rightarrow \Gamma_1 \\
\Delta_2 & \Rightarrow \Gamma_2, \neg \alpha \\
\Delta_1, \Delta_2 & \Rightarrow \Gamma_1, \Gamma_2
\end{align*}
\]

which contains one mix with grade less than \( g(\neg \alpha) \). So by the induction hypothesis, we can obtain a proof which contains no mixes.

1.9) Both \( S_1 \) and \( S_2 \) are lower sequents of logical inferences, \( rank_l(P) = rank_r(P) = 1 \) and \( J \) being a quasi-mix \((\gamma, \exists R \gamma)\) where the mix formulas on each side is the principal formula of the logical inferences. Let us here present just the case \( \uplus \). In this case \( S_1 \) and \( S_2 \) must be lower sequents of \( \uplus \) and \( \sqcap \) rule, respectively:

\[
\begin{align*}
P_1 & \\
\delta & \Rightarrow \Gamma_1, \neg \alpha, L \beta & \Delta_1 & \Rightarrow \Gamma_1, \neg \alpha, L \beta \\
\uplus \rightarrow & \delta \Rightarrow \Gamma_1, \neg \alpha, L \beta & \Delta_2 & \Rightarrow \Gamma_1, \neg \alpha, L \beta
\end{align*}
\]

This proof can be transformed into:

\[
\begin{align*}
P_1 & \\
\delta & \Rightarrow \Gamma_1, \neg \alpha, L \beta & \Delta_2 & \Rightarrow \Gamma_1, \neg \alpha, L \beta
\end{align*}
\]
which again contains one mix and one quasi-mix, but both with grade less than the grade of quasi-mix on $P$. So by the induction hypothesis, we can obtain a proof which contains no quasi-mixes at all. All other cases of outermost logical symbol in quasi-mix inferences can be obtained in a similar way.

**Case 2: rank > 2, i.e., rank₁(P) > 1 and/or rankᵣ(P) > 1** The induction hypothesis is that from every proof $Q$ which contains a quasi-mix only as the last inference, and which satisfies either $g(Q) < g(P)$, or $g(Q) = g(P)$ and rank$(Q) < rank(P)$, we can eliminate the application of the quasi-mix.

2.1) rankᵣ(P) > 1

2.1.1) Let us consider a quasi-mix of the form $(L_α, ³R, L_α)$ in which $Γ₂$ contains $³R, L_α$ or $L_δ$ is $L_α$. In this case, we construct a new proof as follows:

$$
\frac{L_δ \Rightarrow Γ₁}{³R, L_δ \Rightarrow +³RΓ₁} \text{ prom} \Rightarrow \frac{³R, L_δ \Rightarrow +³RΓ₁, ³R, L_α}{Γ₂} \text{ perm}; \text{ contract} \Rightarrow \frac{³R, L_δ, Δ₂ \Rightarrow +³RΓ₁, Γ₂}{Γ₂} \text{ weak}; \text{ perm} \Rightarrow
$$

where the assumption $Γ₂$ contains $³R, L_α$ were used in the last inference to construct $Γ₂$. When $Δ₁$ is $L_α$, we construct a new proof as follows:

$$
\frac{Δ₂ \Rightarrow Γ₂}{³R, L_α, Δ₂ \Rightarrow Γ₂} \text{ perm}; \text{ weak} \Rightarrow \frac{³R, L_α, Δ₂ \Rightarrow +³RΓ₁, Γ₂}{Γ₂} \text{ perm}; \text{ weak} \Rightarrow
$$

2.1.2) $S₂$ is the lower sequent of a inference $J₂$, where $J₂$ is not a logical inference whose principal formula is $δ$. We will consider just the case where the quasi-mix is of the form $(δ, +³Rδ)$, the other cases of quasi-mix can be treated in a similar way. $P$ has the form:

$$
P_1 \quad \frac{P_2}{Δ₁ \Rightarrow Γ₁} \quad \frac{Φ \Rightarrow Ψ}{Δ₂ \Rightarrow Γ₂} \quad J₂ \quad (δ, +³Rδ)
$$

where $P₁$ and $P₂$ contain no quasi-mixes and $Φ$ contains at least one occurrence of $+³Rδ$. We first consider the proof $P'ₙ$:  

$$
P_1 \quad \frac{P_2}{Δ₁ \Rightarrow Γ₁} \quad \frac{Φ \Rightarrow Ψ}{Δ₂ \Rightarrow Γ₂} \quad (δ, +³Rδ)
$$
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\( g(P) = g(P') \), \( \text{rank}_i(P') = \text{rank}_i(P) \) and \( \text{rank}_r(P') = \text{rank}_r(P) - 1 \). Thus, by the induction hypothesis, the final sequent in \( P' \) is provable without quasi-mix. Given that, we can now construct a proof:

\[
\begin{align*}
& \Delta_1, \Phi^* \Rightarrow \Gamma_1^*, \Psi_{\text{perm}^*} \\
& \Phi^*, \Delta_1 \Rightarrow \Gamma_1^*, \Psi_{J_2} \\
& \Delta_2^*, \Delta_1 \Rightarrow \Gamma_1^*, \Gamma_2^* \\
\end{align*}
\]

In the case that the auxiliary formula in \( J_2 \) in \( P \) is a mix in \( \Phi \), we need an additional weakening before \( J_2 \) in the last proof.

2.1.3) \( \Delta_1 \) contains no \( \delta \)'s, \( S_2 \) is the lower sequent of a logical inference whose principal formula is \( \delta \) and \( J \) is a mix rule inference. We have to consider several cases according to the outermost logical symbol of \( \delta \):

i) The outermost logical symbol of \( \delta \) is \( \sqcap \). The last part of \( P \) is of the form:

\[
\begin{align*}
P_1 & \quad P_2 \\
J & \quad \Delta_1 \Rightarrow \Gamma_1 \\
& \quad \Delta_2, L\alpha, L\beta \Rightarrow \Gamma_2, J_2 \\
& \quad \Delta_1^*, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2^* (\sqcap (\alpha \sqcap \beta)) \\
\end{align*}
\]

Now let us consider the proof \( Q \):

\[
\begin{align*}
P_1 & \quad P_2 \\
J & \quad \Delta_1 \Rightarrow \Gamma_1 \\
& \quad \Delta_2, L\alpha, L\beta \Rightarrow \Gamma_2, (\sqcap (\alpha \sqcap \beta)) \\
& \quad \Delta_1^*, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2^* \\
\end{align*}
\]

assuming that \( L(\alpha \sqcap \beta) \) is in \( \Delta_2 \). Note that \( g(Q) = g(P) \), \( \text{rank}_i(Q) = \text{rank}_i(P) \) and \( \text{rank}_r(Q) = \text{rank}_r(P) - 1 \). Hence by the induction hypothesis, the end-sequent of \( Q \) is provable without a mix. Let us call such proof \( Q' \) and consider the following proof \( P' \):

\[
\begin{align*}
P_1 & \quad \Delta_1, L\alpha, L\beta \Rightarrow \Gamma_1, \Gamma_2, J_2 \\
J & \quad \Delta_1 \Rightarrow \Gamma_1 \\
& \quad \Delta_1^*, \Delta_2^* \Rightarrow \Gamma_1^*, \Gamma_2^* (\sqcap (\alpha \sqcap \beta)) \\
& \quad \Delta_1, \Delta_2 \Rightarrow \Gamma_1^*, \Gamma_2^* \\
\end{align*}
\]

Given that, \( g(P') = g(P) \), \( \text{rank}_i(P') = \text{rank}_i(P) \) and \( \text{rank}_r(P') = 1 \) (for \( \Delta_1 \) contains no occurrences of \( L(\alpha \sqcap \beta) \)) by the induction hypothesis the end-sequent of \( P' \) is provable without a mix, and so is the end-sequent of \( P \).

ii) The outermost logical symbol of \( \delta \) is \( \sqcup \). Let us consider a proof \( P \) whose last part is of the form:
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$$\begin{array}{c}
P_1 \quad \frac{P_2}{P_3} \\
\Delta_1 \Rightarrow \Gamma_1 \quad \Delta_2, \overline{L} \alpha \Rightarrow \Gamma_2 \quad \Delta_2, \overline{L} \beta \Rightarrow \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} (\alpha \sqcup \beta) \Rightarrow \Gamma_1, \Gamma_2 \\
(\overline{L} (\alpha \sqcup \beta))
\end{array}$$

Assuming that $\overline{L} (\alpha \sqcup \beta)$ is in $P_1$ and $P_2$, consider the proof $Q_1$:

$$\begin{array}{c}
P_1 \quad \frac{P_2}{Q_1'} \\
\Delta_1 \Rightarrow \Gamma_1 \quad \Delta_2, \overline{L} \alpha \Rightarrow \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} (\alpha \sqcup \beta) \Rightarrow \Gamma_1, \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} \beta \Rightarrow \Gamma_1, \Gamma_2 \\
(\overline{L} (\alpha \sqcup \beta))
\end{array}$$

and $Q_2$:

$$\begin{array}{c}
P_1 \quad \frac{P_3}{Q_2'} \\
\Delta_1 \Rightarrow \Gamma_1 \quad \Delta_2, \overline{L} \beta \Rightarrow \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} (\alpha \sqcup \beta) \Rightarrow \Gamma_1, \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} \alpha \Rightarrow \Gamma_1, \Gamma_2 \\
(\overline{L} (\alpha \sqcup \beta))
\end{array}$$

We note that $g(Q_1) = g(Q_2) = g(P)$, $\text{rank}_l(Q_1) = \text{rank}_l(Q_2) = \text{rank}_l(P)$ and $\text{rank}_r(Q_1) = \text{rank}_r(Q_2) < \text{rank}_r(P)$. Hence, by the induction hypothesis, the end-sequents of $P_1$ and $P_2$ are provable without a $\text{mix}$. Let us consider new proofs without $\text{mix}$ $Q_1'$ and $Q_2'$ in the construction of $P'$:

$$\begin{array}{c}
P_1 \\
\Delta_1 \Rightarrow \Gamma_1 \\
\Delta_1, \Delta_2, \overline{L} \alpha \Rightarrow \Gamma_1, \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} \beta \Rightarrow \Gamma_1, \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} (\alpha \sqcup \beta) \Rightarrow \Gamma_1, \Gamma_2 \\
(\overline{L} (\alpha \sqcup \beta))
\end{array}$$

Then, $g(P') = g(P)$, $\text{rank}_l(P') = \text{rank}_l(P)$ and $\text{rank}_r(P') = 1$, since $\Delta_1$ and $\Delta_2$ do not contain $\overline{L} (\alpha \sqcup \beta)$. By the induction hypothesis the end-sequent of $P'$ is provable without a $\text{mix}$.

iii) The outermost logical symbol of $\delta$ is $\forall$. That is, the mix formula is of the form $\overline{L} (\forall R. \alpha)$. Let us consider the proof $P$:

$$\begin{array}{c}
P_1 \\
\Delta_1 \Rightarrow \Gamma_1 \\
\Delta_2, \overline{L} (\forall R. \alpha) \Rightarrow \Gamma_2 \\
\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2 \\
(\overline{L} (\forall R. \alpha))
\end{array}$$

where $\overline{L} (\forall R. \alpha)$ occurs on $\Delta_2$ since $\text{rank}_r(P) > 1$. Let us consider a proof $Q$ as follows:

$$\begin{array}{c}
P_1 \\
\Delta_1 \Rightarrow \Gamma_1 \\
\Delta_2, \overline{L} (\forall R. \alpha) \Rightarrow \Gamma_2 \\
\Delta_1, \Delta_2, \overline{L} R \alpha \Rightarrow \Gamma_1, \Gamma_2 \\
(\overline{L} (\forall R. \alpha))
\end{array}$$

Note that $\text{grade}(Q) = \text{grade}(P)$, $\text{rank}_l(Q) = \text{rank}_l(P)$ and $\text{rank}_r(Q) = \text{rank}_r(P) - 1$. So, by the induction hypothesis on
can obtain a proof $Q'$ with the same end-sequent as $Q$ without quasi-mix inferences. Now consider the new proof $P'$:

\[
\begin{array}{c}
P_1 \\
\Delta_1 \Rightarrow \Gamma_1 \\
\Delta_1, \Delta_2, L^R \alpha \Rightarrow \Gamma_1^*, \Gamma_2 \\
\Delta_1, \Delta_2^*, L^\lor \alpha \Rightarrow \Gamma_1^*, \Gamma_2 \\
\end{array}
\]

Now we have

\[
g(P') = g(P) \quad \text{and} \quad rank_l(P') = rank_l(P)
\]

and \( rank_r(P') = 1 \) (for $\Delta_1$ and $\Delta_2^*$ do not contain $L^\lor \alpha$).

Thus the end-sequent of $P'$ (the same of $P$) is provable without quasi-mix by the induction hypothesis.

The remaining cases where $\delta$ is of the form $L^\exists R. \alpha$ and $L^\neg \alpha$ are treated in a similar way.

2.1.4) The same conditions that hold for 2.1.3 but $J$ is a quasi-mix rule inference. We have to consider several cases according to the outermost logical symbol of $\delta$. All the cases are treated in a similar way of cases 2.1.3.

2.2) $rank_r(P) = 1$ and $rank_l(P) > 1$. This case is proved as in case 2.1 above.

\[\blacksquare\]