IV
Comparing SC$_{\mathcal{ALC}}$ with other $\mathcal{ALC}$ Deduction Systems

The structural subsumption algorithm is restricted to a quite inexpressive language. Simple Tableaux based algorithms generally fails to provide short proofs. On the other hand, the later has an useful property, it returns a counter-model from an unsuccessful proof. A counter-model, that is, an interpretation that falsifies the premise, is a quite useful object to a knowledge-base engineer.

In Section IV.1 we compare SC$_{\mathcal{ALC}}$ with the structual subsumption algorithm. In Section IV.2 we show how to extend SC$_{\mathcal{ALC}}$ in order to be able to construct a counter-model from unsuccessful proofs. In this way, SC$_{\mathcal{ALC}}$ can be compared with Tableaux algorithms, indeed. In fact the system that will be presented in the section, SC$^{\square}{}_{\mathcal{ALC}}$, is a structural-free sequent calculus designed to provide sequent proofs without considering backtracking during the proof-construction from conclusion to axioms. Nevertheless, two secondary results are obtained from Section IV.2:

1. In Section III.3 a relative completeness of SC$_{\mathcal{ALC}}$ regarding the axiomatic presentation of $\mathcal{ALC}$ is shown. In Section IV.2 we present an alternative proof of SC$_{\mathcal{ALC}}$ completeness. The method used in this section is a basis for constructing a proof of SC$_{\mathcal{ALCQI}}$ completeness.

2. Since the results of Section IV.2 are obtained from a SC$_{\mathcal{ALC}}$ without cut-rule, we are actually proving the completeness of SC$_{\mathcal{ALC}}$ without the cut-rule. Given that, the results can also be considered an alternative method of cut-elimination for the SC$_{\mathcal{ALC}}$ presented in Section III.4, where we followed Gentzen's original proof for cut elimination.

IV.1 Comparing $\mathcal{SALC}$ with the Structural Subsumption algorithm

The *structural subsumption algorithms* (**SSA**) presented in [1] compare the syntactic structure of two normalized concept descriptions in order to verify if the first one is subsumed by the second one. In order to compare deductions in SC$_{\mathcal{ALC}}$ with deductions in **SSA**, we have just to observe that each step taken by a bottom-up construction of a SC$_{\mathcal{ALC}}$ proof corresponds to a step of the **SSA** algorithm towards the concepts matching. Moreover, **SSA** can deal with concepts expressed in $\mathcal{ALN}$ language ($\mathcal{AL}$ augmented with number restrictions). In other hands, SC$_{\mathcal{ALC}}$ can deal with concepts expressed in $\mathcal{ALC}$ and will be extended in Chapter VI to deal with $\mathcal{ALCQI}$.

For a concrete example, let us consider the SC$_{\mathcal{ALC}}$ proof below where A and B stands for atomic concepts and C and D for normalized concepts.

$$\dfrac{\dfrac{\dfrac{\dfrac{A_1 \Rightarrow B_1}{\forall R_1.C_1, A_1 \Rightarrow B_1}}{A_1, \forall R_1.C_1 \Rightarrow B_1} \qquad \dfrac{\dfrac{\dfrac{{}^{R_1}C_1 \Rightarrow {}^{S_1}D_1}{{}^{R_1}C_1 \Rightarrow \forall S_1.D_1}}{\forall R_1.C_1 \Rightarrow \forall S_1.D_1}}{A_1, \forall R_1.C_1 \Rightarrow \forall S_1.D_1}}{A_1, \forall R_1.C_1 \Rightarrow B_1 \sqcap \forall S_1.D_1}}{A_1 \sqcap \forall R_1.C_1 \Rightarrow B_1 \sqcap \forall S_1.D_1}$$

The deduction above deals with two normalized concepts, $A_1 \sqcap \forall R_1.C_1$ and $B_1 \sqcap \forall S_1.D_1$. It would conclude the subsumption (sequent) whenever the top-sequents ensure also their respective subsumptions. This is just what the recursive procedure of **SSA** does.

IV.2 Obtaining counter-models from unsuccessful proof trees

The SC$_{\mathcal{ALC}}$ system rules are not *deterministic*. That is, if rules are applied in the wrong order, we can fail to obtain a proof of an $\mathcal{ALC}$ theorem. For instance, consider the fully expanded proof tree presented in the Example 2. The initial sequent denotes a subsumption proved valid by the Example 1 (page 23). Despite that, reading bottom-up, from the sequent

$$^{\exists child}\top, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow {}^{\exists child, \forall child}Doctor$$

the rule weak-l was applied to allow the application of rule prom-∃. But the weak-l rule removed the wrong concept from the sequent, which turned the proof impossible to be finished, that is, the top sequent is not an axiom. Given that, in order to obtain a counter-model from unsuccessful proofs, we must consider not only one of the possibles fully expanded proof trees but all of them. In other words, one possible fully expanded proof tree of a given sequent is not a sufficient evidence that this sequent is not a theorem.

Example 2 *An unsuccessful proof of a valid sequent in* SC$_{\mathcal{ALC}}$*:*

$$
\begin{array}{cl}
\top \Rightarrow {}^{\forall child}Doctor & \\ \hline
{}^{\exists child}\top \Rightarrow {}^{\exists child,\forall child}Doctor & \textit{prom-}\exists \\ \hline
{}^{\exists child}\top, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow {}^{\exists child,\forall child}Doctor & \textit{weak-l} \\ \hline
{}^{\exists child}\top child, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow {}^{\exists child}\forall child.Doctor & \forall\textit{-r} \\ \hline
{}^{\exists child}\top, {}^{\forall child}\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \exists\textit{-r} \\ \hline
{}^{\exists child}\top, \forall child.\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \forall\textit{-l} \\ \hline
\exists child.\top, \forall child.\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \exists\textit{-l} \\ \hline
\exists child.\top \sqcap \forall child.\neg(\exists child.\neg Doctor) \Rightarrow \exists child.\forall child.Doctor & \sqcap\textit{-l}
\end{array}
$$

To better illustrate the problem with obtaining a counter-model from a fully expanded proof trees, consider Example 3 (page 44) which does not hold for concepts A and B in general.

Example 3 *Two possibly fully expanded proof trees for the invalid subsumption:*

$$\exists R.A \sqcap \exists R.B \sqsubseteq \exists R.(A \sqcap B)$$

$$
\begin{array}{cl}
B \Rightarrow A \qquad B \Rightarrow B & \\ \hline
B \Rightarrow A \sqcap B & \sqcap\textit{-r} \\ \hline
{}^{\exists R}B \Rightarrow {}^{\exists R}A \sqcap B & \textit{prom-}\exists \\ \hline
{}^{\exists R}A, {}^{\exists R}B \Rightarrow {}^{\exists R}A \sqcap B & \textit{weak-l} \\ \hline
{}^{\exists R}A, {}^{\exists R}B \Rightarrow \exists R.A \sqcap B & \exists\textit{-r} \\ \hline
{}^{\exists R}A, \exists R.B \Rightarrow \exists R.A \sqcap B & \exists\textit{-l} \\ \hline
\exists R.A, \exists R.B \Rightarrow \exists R.A \sqcap B & \exists\textit{-l} \\ \hline
\exists R.A \sqcap \exists R.B \Rightarrow \exists R.A \sqcap B & \sqcap\textit{-l}
\end{array}
\qquad
\begin{array}{cl}
A \Rightarrow A \qquad A \Rightarrow B & \\ \hline
A \Rightarrow A \sqcap B & \sqcap\textit{-r} \\ \hline
{}^{\exists R}A \Rightarrow {}^{\exists R}A \sqcap B & \textit{prom-}\exists \\ \hline
{}^{\exists R}A, {}^{\exists R}B \Rightarrow {}^{\exists R}A \sqcap B & \textit{weak-l} \\ \hline
{}^{\exists R}A, {}^{\exists R}B \Rightarrow \exists R.A \sqcap B & \exists\textit{-r} \\ \hline
{}^{\exists R}A, \exists R.B \Rightarrow \exists R.A \sqcap B & \exists\textit{-l} \\ \hline
\exists R.A, \exists R.B \Rightarrow \exists R.A \sqcap B & \exists\textit{-l} \\ \hline
\exists R.A \sqcap \exists R.B \Rightarrow \exists R.A \sqcap B & \sqcap\textit{-l}
\end{array}
$$

Given a fully expanded proof tree, in the attempt to construct a counter-model for the bottom sequent, the process should have to start from the most top sequents, not axioms, going into the direction of the bottom sequent adjusting the model at each rule application in order to guarantee that in each step, if the model being constructed does not satisfy the premiss, it should not satisfy the conclusion. In this way we would have an algorithm to construct a counter-model for any fully expanded proof tree.

In Example 3, let us first consider the fully expanded proof tree on the left, if we start from the logical axiom $B \sqsubseteq B$ it would be not possible to construct any counter-model for it. But starting from $B \Rightarrow A$ we can easly construct an interpretation $\mathcal{I}$ where $B^{\mathcal{I}} \not\sqsubseteq A^{\mathcal{I}}$. But this is not sufficient to negate the bottom sequent. Basically, from top-down, when we get into the point to consider the application of rule weak-l, we must note that the formula introduced on the left force us to include one more restriction in the counter-model $\mathcal{I}$ being constructed. $\mathcal{I}$ not only has to guarantee $B^{\mathcal{I}} \not\sqsubseteq A^{\mathcal{I}}$ but also $A^{\mathcal{I}} \not\sqsubseteq B^{\mathcal{I}}$. The derivation on the right would let us conclude this same restrictions in the inverse order. The two derivations are basically the two possibles choices of formulas in the application of weak-l rule.

One important property of weak and promotional rules is that they are not double-sound. A rule is said double-sound if it is not only truth-preserving from the premiss to conclusion but also from the conclusion to its premiss. Regarding the top-bottom construction of counter-model, this means that in the adjustment of the counter-model $\mathcal{I}$ being constructed, the fact that $\mathcal{I}$ does not satisfy the premiss of a weak rule application does not guarantee that it does not satisfy its conclusion. Moreover, the introduced formula by the weak rule can be arbitrary complex making the adjustment of the counter-model not trivial nor modular.

Let us consider the system $\mathrm{SC}\square_{\mathcal{ALC}}$, a conservative extension of $\mathrm{SC}_{\mathcal{ALC}}$ presented in Figure IV.1. $\mathrm{SC}\square_{\mathcal{ALC}}$ sequents are expressions of the form $\Delta \Rightarrow \Gamma$ where Δ and Γ are *sets* of labeled concepts (possibly frozen). A frozen concept α is represented as $[\alpha]^n$ where n is the index (context identifier) of the frozen concept. The notation $[\Delta]^n$ means that each $\delta \in \Delta$ is frozen with the same index (i.e. $\{[\delta]^n \mid \delta \in \Delta\}$). Given a $\mathrm{SC}\square_{\mathcal{ALC}}$ sequent with the general form 1, we call each pair (Δ_k, Γ_k) a *context* in the sequent.

$$\Delta_1, [\Delta_2]^1, \ldots, [\Delta_n]^{n-1} \Rightarrow \Gamma_1, [\Gamma_2]^1, \ldots, [\Gamma_n]^{n-1} \tag{1}$$

$\mathrm{SC}\square_{\mathcal{ALC}}$ does not have permutation, contraction or the *cut* rule from $\mathrm{SC}_{\mathcal{ALC}}$. Reading bottom-up, the weak rules of $\mathrm{SC}\square_{\mathcal{ALC}}$ *save* the context of the proof before removing a concept from the lefthand (antecedent) or righthand side (succedent) of the sequent and the frozen-exchange changes the contexts during a proof construction. Considering that in $\mathrm{SC}\square_{\mathcal{ALC}}$ the sequents are constructed by two sets (not lists) of concepts, weak rules are still necessary only to allow the application of promotional rules.

The notation ${}^{+\forall R}\Gamma$ or ${}^{+\exists R}\Gamma$ denotes the addition of the Role R existentialy or universaly quantified in the front of each list of labels of all formulas of Γ. In rules $\sqcap$-{l,r}, $\sqcup$-{l,r}, $\forall$-{l,r}, $\exists$-{l,r} and in the axiom, Δ and Γ stand for labeled concepts frozen or not. In the promotional, frozen-exchange and weak rules we have to distinguish the frozen concepts from the non-frozen ones. We use the notation $[\Delta]$ (resp. $[\Gamma]$) to denote the set of all frozen concepts in the sequent regardless their index. The index k must be in all rules a fresh one.

In rule frozen-exchange, all formulas in Δ_2 and Γ_2 cannot be the conclusion of any rule application except the frozen-exchange. This proviso is not actually necessary to guarantee the soundness of the system, it is more a strategy for proof constructions. The idea is to postpone the exchange of contexts until no other rule can reduce the current active context, avoiding unnecessary swapping of contexts.

$$\frac{}{\Delta, \delta \Rightarrow \delta, \Gamma}$$

$$\frac{[\Delta, \delta]^k, \Delta \Rightarrow \Gamma, [\Gamma]^k}{\Delta, \delta \Rightarrow \Gamma}\ \text{weak-l} \qquad \frac{[\Delta]^k, \Delta \Rightarrow \Gamma, [\Gamma, \gamma]^k}{\Delta \Rightarrow \Gamma, \gamma}\ \text{weak-r}$$

$$\frac{\Delta, {}^{L,\forall R}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}(\forall R.\alpha)L_2 \Rightarrow \Gamma}\ \forall\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\forall R}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}(\forall R.\alpha)}\ \forall\text{-r}$$

$$\frac{\Delta, {}^{L,\exists R}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}(\exists R.\alpha) \Rightarrow \Gamma}\ \exists\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\exists R}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}(\exists R.\alpha)}\ \exists\text{-r}$$

$$\frac{\Delta, {}^{\forall L}\alpha, {}^{\forall L}\beta \Rightarrow \Gamma}{\Delta, {}^{\forall L}(\alpha \sqcap \beta) \Rightarrow \Gamma}\ \sqcap\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{\forall L}\alpha \quad \Delta \Rightarrow \Gamma, {}^{\forall L}\beta}{\Delta \Rightarrow \Gamma, {}^{\forall L}(\alpha \sqcap \beta)}\ \sqcap\text{-r}$$

$$\frac{\Delta, {}^{\exists L}\alpha \Rightarrow \Gamma \quad \Delta, {}^{\exists L}\beta \Rightarrow \Gamma}{\Delta, {}^{\exists L}(\alpha \sqcup \beta) \Rightarrow \Gamma}\ \sqcup\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{\exists L}\alpha, {}^{\exists L}\beta}{\Delta \Rightarrow \Gamma, {}^{\exists L}(\alpha \sqcup \beta)}\ \sqcup\text{-r}$$

$$\frac{\Delta \Rightarrow \Gamma, {}^{\neg L}\alpha}{\Delta, {}^{L}\neg\alpha \Rightarrow \Gamma}\ \neg\text{-l} \qquad \frac{\Delta, {}^{\neg L}\alpha \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, {}^{L}\neg\alpha}\ \neg\text{-r}$$

$$\frac{[\Delta], {}^{L}\delta \Rightarrow \Gamma, [\Gamma_1]}{[\Delta], {}^{\exists R,L}\delta \Rightarrow {}^{+\exists R}\Gamma, [\Gamma_1]}\ \text{prom-}\exists \qquad \frac{[\Delta_1], \Delta \Rightarrow {}^{L}\gamma, [\Gamma]}{[\Delta_1], {}^{+\forall R}\Delta \Rightarrow {}^{\forall R,L}\gamma, [\Gamma]}\ \text{prom-}\forall$$

$$\frac{[\Delta], [\Delta_2]^k, \Delta_1 \Rightarrow \Gamma_1, [\Gamma_2]^k, [\Gamma]}{[\Delta], \Delta_2, [\Delta_1]^n \Rightarrow [\Gamma_1]^n, \Gamma_2, [\Gamma]}\ \text{frozen-exchange}$$

Figure IV.1: The System SC$^{\Box}{}_{\mathcal{ALC}}$

In Section III.1, we presented the natural interpretation of a sequent $\Delta \Rightarrow \Gamma$ in $\mathrm{SC}_{\mathcal{ALC}}$ as the $\mathcal{ALC}$ formula

$$\bigsqcap_{\delta \in \Delta} \sigma(\delta) \sqsubseteq \bigsqcup_{\gamma \in \Gamma} \sigma(\gamma)$$

Given an interpretation function $\cdot^{\mathcal{I}}$ we write $\mathcal{I} \models \Delta \Rightarrow \Gamma$, if and only if,

$$\bigcap_{\delta \in \Delta} \sigma(\delta)^{\mathcal{I}} \subseteq \bigcup_{\gamma \in \Gamma} \sigma(\gamma)^{\mathcal{I}}$$

Now we have to extend that definition to give the *semantics* of the sequents with (indexed) frozen concepts.

Definition 18 (Satisfability of frozen-labeled sequents) *Let $\Delta \Rightarrow \Gamma$ be a sequent with its succedent and antecedent having formulas that range over labeled concepts and frozen labeled concepts. This sequent has the general form of 1. Let $(\mathcal{I}_1, \ldots, \mathcal{I}_n)$ be a tuple of interpretations. We say that this tuple satisfies $\Delta \Rightarrow \Gamma$ if and only if, one of the following clauses holds:*

$$\mathcal{I}_1 \models \Delta_1 \Rightarrow \Gamma_1 \qquad \mathcal{I}_2 \models \Delta_2 \Rightarrow \Gamma_2 \quad \ldots \quad \mathcal{I}_n \models \Delta_n \Rightarrow \Gamma_n \tag{2}$$

That is, the first projection should satisfy the set of non-frozen formulas. The second projection should satisfy the set of frozen formulas with the minimum index and so on. The sequent $\Delta \Rightarrow \Gamma$ is not satisfiable by a tuple of interpretations, if and only if, no interpretation in the tuple satisfy its corresponding context.

Before proceeding to present the procedure to obtain counter-models from $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$-proofs, we must introduce Lemma 19 showing that $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$ is a conservative extension of $\mathrm{SC}_{\mathcal{ALC}}$.

Lemma 19 *Consider $\Delta \Rightarrow \Gamma$ a $\mathrm{SC}_{\mathcal{ALC}}$ sequent. If P is a proof of $\Delta \Rightarrow \Gamma$ in $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$ then it is possible to construct a proof P' of $\Delta \Rightarrow \Gamma$ in $\mathrm{SC}_{\mathcal{ALC}}$.*

Proof:

Each application of a frozen-exchange rule correspond to a shift of contexts during the bottom-up proof construction process. To proof Lemma 19 we need a two-steps procedure to: (1) remove all frozen-exchange applications of a given proof (a proof in $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$ without any frozen-exchange application is naturally translated to a proof in $\mathrm{SC}_{\mathcal{ALC}}$); (2) replace the weak rules of $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$ by their counterparts in $\mathrm{SC}_{\mathcal{ALC}}$.

We show that a proof P in $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$ can always be transformed into a proof P' in $\mathrm{SC}^{\Box}_{\mathcal{ALC}}$ without any frozen-exchange rule applications by induction over the number of applications of frozen-exchange occurring in a proof P. Let us consider a topmost application of rule frozen-exchange in P, where reading bottom-up, the frozen-exchange rule recover a context that was frozen by the γ rule that can be a weak-l or weak-r rule.

$$\dfrac{\dfrac{\begin{array}{c}\Delta, \alpha \Rightarrow \alpha, \Gamma \\ \Pi_1 \\ [\Delta], [\Delta_1'']^j, \Delta_1 \Rightarrow \Gamma_1, [\Gamma_1'']^j, [\Gamma]\end{array}}{\begin{array}{c}[\Delta], [\Delta_1]^k \Delta_1'' \Rightarrow \Gamma_1'', [\Gamma_1,]^k, [\Gamma] \\ \Pi_2 \\ [\Delta], [\Delta_1]^k \Delta_1' \Rightarrow \Gamma_1', [\Gamma_1,]^k, [\Gamma]\end{array}} \text{frozen-exchange}}{[\Delta], \Delta_1 \Rightarrow \Gamma_1, [\Gamma]} \gamma$$

We can obtain P' bellow by simple discarding the proof fragment Π_2.

$$\begin{array}{c}\Delta, \alpha \Rightarrow \alpha, \Gamma \\ \Pi_1 \\ [\Delta], \Delta_1 \Rightarrow \Gamma_1, [\Gamma]\end{array}$$

Applying recursively the transformations above from top to bottom we obtain a proof in $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$ without any frozen-exchange rule application. Note also that this procedure will remove any *branch* created between the rule that introduced the frozen-formulas and the removed frozen-exchange application.

Given a frozen-exchange free $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$-proof, to obtain a $\mathrm{SC}_{\mathcal{ALC}}$-proof, we have only to drop out the frozen concepts and substitute weak-{l,r} rules application of $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$ for their counter-parts in $\mathrm{SC}_{\mathcal{ALC}}$.

Let us consider the weak-l case, rule weak-r can be dealt similarly. Given the $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$-proof fragment below containing the top most application of rule weak-r:

$$\dfrac{\begin{array}{c}\Pi \\ [\Delta], [\Delta_1]^k \Rightarrow \Gamma_2, [\gamma, \Gamma_2]^k, [\Gamma]\end{array}}{[\Delta], \Delta_1 \Rightarrow \gamma, \Gamma_2, [\Gamma]} \text{weak-r}$$

From the fragment above, we construct:

$$\dfrac{\begin{array}{c}\Pi \\ \Delta_1 \Rightarrow \Gamma_2\end{array}}{\Delta_1 \Rightarrow \gamma, \Gamma_2} \text{weak-r}$$

Applying recursively the transformations above from top to bottom we obtain a proof in $\mathrm{SC}_{\mathcal{ALC}}$ from a proof in $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$. ■

Let us give a precise definition of *fully expanded proof tree.* A fully expanded proof tree of $\Delta \Rightarrow \Gamma$ is a tree having $\Delta \Rightarrow \Gamma$ as root, each internal node being a sequent premise of a valid $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$ rule application, and each leaf being either a $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$ axiom (initial sequent) or a top-sequent (not axiom) with not necessarily only atomic concepts. A sequent is a top-sequent if and only if it does not contain *reducible contexts*. A reducible contexts is a context that if active could be further reduced. In the following lemmas we are interested in fully expanded proof trees that are not $\mathrm{SC}^{\square}{}_{\mathcal{ALC}}$ proofs.

If we consider a particular strategy of rule applications, any fully expanded proof tree will have a special form called *normal form*. The following are the main properties of this strategy:

1. It is a fair strategy of rules applications that avoid infinite loops of, for instance, frozen-exchange applications swapping contexts or unnecessary repetition of proof fragments;

2. Promotional rules will be applied whenever possible, that is, they have high priority over the other rules;

3. The strategy will discard contexts created by the successive application of weak rules and avoid further applications of weak rules once it is possible to detected that they will not be useful to obtain an initial sequent. For instance, from a sequent $\Delta \Rightarrow \Gamma$, where Δ and Γ only contain atomic concept names without any common concept name, we know that using weak rules we would not obtain an initial sequent. Moreover, weak rules will be used with the unique purpose of enabling promotion rules applications.

A more insightful definition of the last item above would be possible if we replace the weak rules in $\mathrm{SC}^{\square}{}_{\mathcal{ALC}}$ by the *weak** rule below.

$$\frac{[\Delta'], [\Delta, \Delta_1]^k, \Delta \Rightarrow \Gamma, [\Gamma_1, \Gamma]^k, [\Gamma']}{[\Delta'], \Delta, \Delta_1 \Rightarrow \Gamma_1, \Gamma, [\Gamma']} \; \scriptstyle{weak^*}$$

Lemma 20 *The weak** *rule is a derived rule in* $\mathrm{SC}^{\square}{}_{\mathcal{ALC}}$.

Proof: To prove Lemma 20, given a derivation fragment Π with a *weak** rule application, we show how to replace it by successive weak-l and weak-r applications. Without lost of generality, let us consider one special case of *weak** freezing two concepts of both sides of a sequent.

$$\frac{\begin{array}{c}\Pi' \\ [\Delta, \delta_1, \delta_2]^k, \Delta \Rightarrow \Gamma, [\gamma_1, \gamma_2, \Gamma]^k\end{array}}{\Delta, \delta_1, \delta_2 \Rightarrow \gamma_1, \gamma_2, \Gamma} \; weak^*$$

The corresponding fragment Π_1 is presented below. The context k is now followed by the contexts $k+1, k+2, k+3$.

$$
\begin{array}{cl}
\Pi' & \\
[\Delta,\delta_1,\delta_2]^k,[\Delta,\delta_2]^{k+1},[\Delta]^{k+2},[\Delta]^{k+3},\Delta \Rightarrow \Gamma,[\gamma_1,\gamma_2,\Gamma]^k,[\gamma_1,\gamma_2,\Gamma]^{k+1},[\gamma_1,\gamma_2,\Gamma]^{k+2},[\gamma_2,\Gamma]^{k+3} & \\ \hline
[\Delta,\delta_1,\delta_2]^k,[\Delta,\delta_2]^{k+1},[\Delta]^{k+2},\Delta \Rightarrow \gamma_2,\Gamma,[\gamma_1,\gamma_2,\Gamma]^k,[\gamma_1,\gamma_2,\Gamma]^{k+1},[\gamma_1,\gamma_2,\Gamma]^{k+2} & \text{weak-r} \\ \hline
[\Delta,\delta_1,\delta_2]^k,[\Delta,\delta_2]^{k+1},\Delta \Rightarrow \gamma_1,\gamma_2,\Gamma,[\gamma_1,\gamma_2,\Gamma]^k,[\gamma_1,\gamma_2,\Gamma]^{k+1} & \text{weak-r} \\ \hline
[\Delta,\delta_1,\delta_2]^k,\Delta,\delta_2 \Rightarrow \gamma_1,\gamma_2,\Gamma,[\gamma_1,\gamma_2,\Gamma]^k & \text{weak-l} \\ \hline
\Delta,\delta_1,\delta_2 \Rightarrow \gamma_1,\gamma_2,\Gamma & \text{weak-l}
\end{array}
$$

Applying recursively the transformations above from top to bottom we obtain a *weak**-free proof in $\mathrm{SC}^{[]}_{\mathcal{ALC}}$. ■

We introduced the *weak** rule to avoid *dispensable contexts* during the bottom-up proof search procedure. Using the strategy suggested above, we only apply the weak rules in order to allow further application of promotional rules. The idea is that we don't need to save unnecessary contexts that are variants of already saved contexts.

Example 4 *Consider the fully expanded proof tree* Π *having sequent 3 as root.*

$$\exists R.A \sqcap \exists R.B \Rightarrow \exists R.(A \sqcap B) \tag{3}$$

$$
\begin{array}{cl}
[A]^2,[\ldots]^3,B \Rightarrow A,[\ldots]^3,[B]^2 \qquad [A]^2,[\ldots]^3,B \Rightarrow B,[\ldots]^3,[B]^2 & \\ \hline
[A]^2,[{}^{\exists R}A,{}^{\exists R}B]^3,B \Rightarrow A\sqcap B,[{}^{\exists R}(A\sqcap B)]^3,[B]^2 & \sqcap\text{-}r \\ \hline
[A]^2,[{}^{\exists R}A,{}^{\exists R}B]^3,{}^{\exists R}B \Rightarrow {}^{\exists R}(A\sqcap B),[{}^{\exists R}(A\sqcap B)]^3,[B]^2 & \textit{prom-}\exists \\ \hline
[A]^2,{}^{\exists R}A,{}^{\exists R}B \Rightarrow {}^{\exists R}(A\sqcap B),[B]^2 & \textit{weak}^* \\ \hline
[{}^{\exists R}A,{}^{\exists R}B]^1,A \Rightarrow B,[{}^{\exists R}(A\sqcap B)]^1 \qquad [\ldots]^1,A \Rightarrow A,[\ldots]^1 & \textit{f-exch} \\ \hline
[\ldots]^1,A \Rightarrow A\sqcap B,[\ldots]^1 & \sqcap\text{-}r \\ \hline
[{}^{\exists R}A,{}^{\exists R}B]^1,{}^{\exists R}A \Rightarrow {}^{\exists R}(A\sqcap B),[{}^{\exists R}(A\sqcap B)]^1 & \textit{prom-}\exists \\ \hline
{}^{\exists R}A,{}^{\exists R}B \Rightarrow {}^{\exists R}(A\sqcap B) & \textit{weak}^* \\ \hline
{}^{\exists R}A,{}^{\exists R}B \Rightarrow \exists R.(A\sqcap B) & \exists\text{-}r \\ \hline
{}^{\exists R}A,\exists R.B \Rightarrow \exists R.(A\sqcap B) & \exists\text{-}l \\ \hline
\exists R.A,\exists R.B \Rightarrow \exists R.(A\sqcap B) & \exists\text{-}l \\ \hline
\exists R.A\sqcap\exists R.B \Rightarrow \exists R.(A\sqcap B) & \sqcap\text{-}l
\end{array}
$$

We can split Π *in three fragments named* Π_1, Π_2 *and* Π_3. *The fragments are separated by weak** *and frozen-exchanges. Fragments* Π_2 *and* Π_3 *correspond to the two different ways to apply the weak** *in the top-sequent of the fragment* Π_1.

$$
\Pi_1 \equiv \quad
\begin{array}{cl}
\Pi_2 & \\
{}^{\exists R}A,{}^{\exists R}B \Rightarrow {}^{\exists R}(A\sqcap B) & \\ \hline
{}^{\exists R}A,{}^{\exists R}B \Rightarrow \exists R.(A\sqcap B) & \exists\text{-}r \\ \hline
{}^{\exists R}A,\exists R.B \Rightarrow \exists R.(A\sqcap B) & \exists\text{-}l \\ \hline
\exists R.A,\exists R.B \Rightarrow \exists R.(A\sqcap B) & \exists\text{-}l \\ \hline
\exists R.A\sqcap\exists R.B \Rightarrow \exists R.(A\sqcap B) & \sqcap\text{-}l
\end{array}
$$

$$\Pi_2 \equiv \quad \dfrac{\dfrac{\begin{matrix}\Pi_3 \\ [\ldots]^1, A \Rightarrow B, [\ldots]^1\end{matrix} \qquad [\ldots]^1, A \Rightarrow A, [\ldots]^1}{[\ldots]^1, A \Rightarrow A \sqcap B, [\ldots]^1}\ \sqcap\text{-}r}{[{}^{\exists R}A, {}^{\exists R}B]^1, {}^{\exists R}A \Rightarrow {}^{\exists R}(A \sqcap B), [{}^{\exists R}(A \sqcap B)]^1}\ \textit{prom-}\exists$$

$$\Pi_3 \equiv \quad \dfrac{\dfrac{[A]^2, [\ldots]^3, B \Rightarrow A, [\ldots]^3, [B]^2 \qquad [A]^2, [\ldots]^3, B \Rightarrow B, [\ldots]^3, [B]^2}{[A]^2, [\ldots]^3, B \Rightarrow A \sqcap B, [\ldots]^3, [B]^2}\ \sqcap\text{-}r}{[A]^2, [{}^{\exists R}A, {}^{\exists R}B]^3, {}^{\exists R}B \Rightarrow {}^{\exists R}(A \sqcap B), [{}^{\exists R}(A \sqcap B)]^3, [B]^2}\ \textit{prom-}\exists$$

Regarding the contexts created during the proof, contexts 1 and 3 were not turned active yet, they are called auxiliary contexts, they were created during the bottom-up proof construction to save a proof state to further activation and transformation with the system rules, if necessary. Context 1 was used but context 3 was not. Context 2 is the top-sequent of fragment Π_2*, saved after been reduced. The idea is that from the fragments* Π_2 *and* Π_3 *we can construct a counter-model for the root sequent of* Π*.*

Lemma 21 *If* P *is a fully expanded proof-tree in* $\mathrm{SC}^{\square}_{\mathcal{ALC}}$ *with sequent* S *as root (conclusion) and if* P *is in the normal form, from any top-sequent not initial (non-axiom), one can construct a counter-model for* S*.*

Proof: To prove Lemma 21 we must first identify all possible top-sequents in $\mathrm{SC}^{\square}_{\mathcal{ALC}}$. If weak rules are not allowed during the derivation, all top-sequents in $\mathrm{SC}^{\square}_{\mathcal{ALC}}$ would have the general form of 4.

$$\underbrace{A_1, \ldots, A_n}_{\Delta_1}, \underbrace{{}^{\forall R_1, L_1}B_1, \ldots, {}^{\forall R_m, L_m}B_m}_{\Delta_2} \Rightarrow \underbrace{C_1, \ldots, C_l}_{\Delta_3}, \underbrace{{}^{\exists R_1, L_1}D_1, \ldots, {}^{\exists R_p, L_p}D_p}_{\Delta_4} \quad (4)$$

where we group the concepts into four sets $\Delta_1, \Delta_2, \Delta_3$ and Δ_4. $A_{1,n}$ and $C_{1,l}$ are sets of atomic concepts. In Δ_2, $B_{1,m}$ are atomic concepts or disjunctions of concepts (not necessarily atomic). In Δ_4, $D_{1,p}$ are atomic concepts or conjunctions of concepts (not necessarily atomic).

To see that no other rule of $\mathrm{SC}^{\square}_{\mathcal{ALC}}$, rather than weak, could be apply in a sequent like 4, one has just to observe that: (1) the ⊓-r and ⊔-l rules provisos are blocking the decomposition of the conjunctions and disjunctions; and (2) the prom-∀ (prom-∃) rule cannot be applied due the lack of a universal (existential) quantified concept on the right (left).

Nevertheless, with the presence of *weak** rule and considering the strategy for construct normal derivations, *weak** can always be applied to top-sequents like 4 reducing them to the simpler cases below. For each one, we will see that it is possible to construct a counter-model.

Case $\Delta_1 \Rightarrow \Delta_3$ That is, a sequent $A_1, \ldots, A_n \Rightarrow C_1, \ldots, C_l$ without labeled concept, it is easy to construct a counter-model $\mathcal{I}$ such that there exist an element $a \in (A_1 \sqcap \ldots \sqcap A_n)^{\mathcal{I}}$ and $a \notin (C_1 \sqcup \ldots \sqcup C_l)^{\mathcal{I}}$.

Case $\Delta_2 \Rightarrow$ We can construct a counter-model $\mathcal{I}$ such that there exist an element $a \in ({}^{\forall R_1, L_1}B_1 \sqcap \ldots \sqcap {}^{\forall R_m, L_m}B_m)^{\mathcal{I}}$. The right side of a sequent is interpreted as a disjunction, so that, if empty, its semantics for any interpretation function is the empty set. If we consider the simplified case where all roles (labels) are equal, that is ${}^{\forall R, L_1}B_1, \ldots, {}^{\forall R, L_m}B_m \Rightarrow$, we only need to provide a new element a without fillers in R, that is, $\exists x(a, x) \notin R^{\mathcal{I}}$. For the general case, where the most external roles on each concept can be different, the element a cannot have fillers in any of the roles. That is, $\forall R$ occuring in front of the list of labels in Δ_2, $\exists x(a, x) \notin R^{\mathcal{I}}$. We must mention that even if one of the concepts in Δ_2 is $\top$ or $\bot$, we can always construct $\mathcal{I}$.

Case $\Rightarrow \Delta_4$ We can construct a counter-model such that $\mathcal{I} \not\models \Rightarrow \Delta_4$. From the natural interpretation of a sequent, we know that an interpretation will not satisfy this case when there exist at least one element $a \notin ({}^{\exists R_1, L_1}D_1 \sqcup \ldots \sqcup {}^{\exists R_p, L_p}D_p)^{\mathcal{I}}$. Since the left side of a sequent is interpreted as a conjunction, if empty, its semantics for any interpretation function is the universe set of the interpretation. Once more, let us first consider the case where all existential roles are equal, ${}^{\exists R, L_1}D_1 \sqcup \ldots \sqcup {}^{\exists R, L_p}D_p$. We only need to provide an element a without fillers in R. If we have different roles in the sequent, a can not have fillers in any of them.

Case $\Delta_2 \Rightarrow \Delta_4$ This case can be reduced for the two cases above. We can always provide an element $a \in \Delta_2^{\mathcal{I}}$ (by second case) and $a \notin \Delta_4^{\mathcal{I}}$ (by third case). In both cases, a will be a fresh element without fillers in any R, for all R most external labels of Δ_2 and Δ_4. ■

Lemma 22 *If P is a weak*-free proof fragment with at least one top-sequent not initial and having S as the bottom sequent. That is, a fragment where no weak rule were applied. If $\mathcal{I}$ is a counter-model for one of its top-sequents, There is $\mathcal{I}'$ that is a counter-model for S.*

Proof: We prove Lemma 22 by case analysis considering each possible rule application and showing how to extend an interpretation that is counter-model of the premiss to be a counter-model of the conclusion.

Cases $\forall$***-{l,r} and*** $\exists$***-{l,r}*** In these rules the premiss and conclusion have the same semantics, that is, a counter-model for its premiss is also a counter-model for its conclusions.

Cases $\sqcup$***-{l,r} and*** $\sqcap$***-{l,r}*** Let us first consider the rule $\sqcup$-l. Let $\mathcal{I}$ be an interpretation counter-model for at least one of the premiss. That is, $(\Delta \sqcap {}^{\exists L}\alpha)^{\mathcal{I}} \not\subset \Gamma^{\mathcal{I}}$ or $(\Delta \sqcap {}^{\exists L}\beta)^{\mathcal{I}} \not\subset \Gamma^{\mathcal{I}}$. If any of these cases holds, we have $(\Delta \sqcap {}^{\exists L}\alpha)^{\mathcal{I}} \cup (\Delta \sqcap {}^{\exists L}\beta)^{\mathcal{I}} \not\subset \Gamma^{\mathcal{I}}$ and by the distributivity of the intersection over the union $(\Delta \sqcap ({}^{\exists L}\alpha \sqcup {}^{\exists L}\beta))^{\mathcal{I}} \not\subset \Gamma^{\mathcal{I}}$, which is semantically equivalent to conclusion of the rule: $(\Delta \sqcap ({}^{\exists L}\alpha \sqcup \beta))^{\mathcal{I}} \not\subset \Gamma^{\mathcal{I}}$. Case $\sqcap$-r would be proved in the same way by showing that if $A \not\subset B \cup D$ or $A \not\subset C \cup D$ then $A \not\subset (B \cap C) \cup D$. Rules $\sqcap$-l and $\sqcup$-r are even simpler given the natural interpretation of the sequents. Basically, we are using the results of Section III.2 which shows that these rules are double-sound.

Case $\neg$***-l and*** $\neg$***-r*** First rule $\neg$-r where δ a labeled concept and $\neg\delta$ its negation. Let us consider a interpretation $\mathcal{I}$ such that $\mathcal{I} \not\models \Delta, \delta \Rightarrow \Gamma$. So we have an element $a \in (\Delta \sqcap \delta)^{\mathcal{I}}$ and $a \notin \Gamma^{\mathcal{I}}$. Thus, $a \in \delta^{\mathcal{I}}$ and so, $a \notin (\neg\delta)^{\mathcal{I}}$. Consequently, $a \notin (\neg\delta \sqcup \Gamma)^{\mathcal{I}}$ as desired. The case of rule $\neg$-l is similar.

Case prom-$\exists$ Assume that we have $\mathcal{I} \not\models \delta \Rightarrow \Gamma$. So we have an element $b \in \delta^{\mathcal{I}}$ and $b \notin \Gamma^{\mathcal{I}}$. We now construct $\mathcal{I}'$ extending $\mathcal{I}$ with one more new element a in the domain and the tuple $(a, b) \in R^{\mathcal{I}}$. In this way, we obtain the necessary condition to $\mathcal{I}' \not\models {}^{+\exists R}\delta \Rightarrow {}^{+\exists R}\Gamma$ which is $a \in {}^{+\exists R}\delta^{\mathcal{I}}$ and $a \notin {}^{+\exists R}\Gamma^{\mathcal{I}}$ since a is a fresh element.

Case prom-$\forall$ Assume that we have $\mathcal{I} \not\models \Delta \Rightarrow \gamma$. Once more, we have an element $b \in \Delta^{\mathcal{I}}$ and $b \notin \gamma^{\mathcal{I}}$. We construct $\mathcal{I}'$ as in the case above, introducing one new element a in the domain and the tuple $(a, b) \in R^{\mathcal{I}}$. Since a is a fresh element with just one filler in R, we guarantee by construction that $a \in {}^{+\forall R}\Delta^{\mathcal{I}}$ and $a \notin {}^{+\forall R}\gamma^{\mathcal{I}}$ and so, $\mathcal{I}' \not\models {}^{+\forall R}\Delta \Rightarrow {}^{+\forall R}\gamma$. Alternatively, we can also introduce in $\mathcal{I}'$ the element a without any filler in R to guarantee that $\mathcal{I}'$ will also be a counter-model for the conclusion. ■

Lemmas 21 and 22 guarantee that from the top-sequents we can construct counter-models and extend them in fragments $weak^*$-free. The following lemma states that we can merge counter-models of proof fragments with top-sequents that are not axioms.

Lemma 23 *Given a weak* application with a conclusion S, reading top-down, this application has two proof fragments with roots S_1 and S_2, their premise and the context that was frozen. If there are interpretations $\mathcal{I}_1$ and $\mathcal{I}_2$ such that $\mathcal{I}_1 \not\models S_1$ and $\mathcal{I}_2 \not\models S_2$ then there is $\mathcal{I}$ such that $\mathcal{I} \not\models S$.*

Proof: Without lost of generality, we can consider 5 a general format for sequents conclusion of *weak** application. Remember that if we use the strategy define previous, *weak** will only be applied in order to permit promotional rules applications. The case with two existential quantified concepts on the left and two universal quantified concepts on the right will be sufficient to tread all possible combinations. The result of this proof can be easily generalized.

$$\Delta, {}^{\forall R,L_1}\alpha_1, {}^{\exists R,L_2}\alpha_2, {}^{\exists R,L_3}\alpha_3 \Rightarrow \Gamma, {}^{\forall R,L_4}\alpha_4, {}^{\forall R,L_5}\alpha_5, {}^{\exists R,L_6}\alpha_6 \tag{5}$$

To prove Lemma 23, we have to consider each possible pair of proof fragments that a *weak** rule can combine in a top-down construction. In addition, we assume as hypothesis that for both fragments we already constructed a counter-model for its roots – from Lemmas 21 and 22.

1. $S \equiv \Delta, {}^{\exists R,L_2}\alpha_2 \Rightarrow \Gamma, {}^{\exists R,L_6}\alpha_6$. From the hypothesis, we have $\mathcal{I}_1 \not\models \Delta \Rightarrow \Gamma$ and $\mathcal{I}_2 \not\models {}^{\exists R,L_2}\alpha_2 \Rightarrow {}^{\exists R,L_6}\alpha_6$, that is, $\Delta^{\mathcal{I}_1} \not\subset \Gamma^{\mathcal{I}_1}$ and ${}^{\exists R,L_2}\alpha_2{}^{\mathcal{I}_2} \not\subset {}^{\exists R,L_6}\alpha_6{}^{\mathcal{I}_2}$. We create an interpretation $\mathcal{I} = \mathcal{I}_1 \uplus \mathcal{I}_2$, a disjoint union of $\mathcal{I}_1$ and $\mathcal{I}_2$. Now, from $\mathcal{I}_1$ we select an element $a \in \Delta^{\mathcal{I}_1}$ and $a \notin \Gamma^{\mathcal{I}_1}$ that must exist by hypothesis. From $\mathcal{I}_2$ we select an element $b \in \alpha_2^{\mathcal{I}_2}$ and $b \notin \alpha_6^{\mathcal{I}_2}$ that must exist by hypothesis. Now In $\mathcal{I}$ we add $(a, b) \in R^{\mathcal{I}}$ and we guarantee that $(\Delta \sqcap {}^{\exists R,L_2}\alpha_2)^{\mathcal{I}} \not\subset (\Gamma \sqcup {}^{\exists R,L_6}\alpha_6)^{\mathcal{I}}$.

2. $S \equiv \Delta, {}^{\forall R,L_1}\alpha_1 \Rightarrow \Gamma, {}^{\forall R,L_5}\alpha_5$. By hypothesis, we have $\mathcal{I}_1 \not\models \Delta \Rightarrow \Gamma$ and $\mathcal{I}_2 \not\models {}^{\forall R,L_1}\alpha_1 \Rightarrow {}^{\forall R,L_5}\alpha_5$, that is, $\Delta^{\mathcal{I}_1} \not\subset \Gamma^{\mathcal{I}_1}$ and ${}^{\forall R,L_1}\alpha_1{}^{\mathcal{I}_2} \not\subset {}^{\forall R,L_5}\alpha_5{}^{\mathcal{I}_2}$. We create the interpretation $\mathcal{I}$ as in the previous case, $\mathcal{I} = \mathcal{I}_1 \uplus \mathcal{I}_2$. From $\mathcal{I}_1$ we select an element $a \in \Delta^{\mathcal{I}_1}$ and $a \notin \Gamma^{\mathcal{I}_1}$. From $\mathcal{I}_2$ we select an element $b \in \alpha_1^{\mathcal{I}_2}$ and $b \notin \alpha_5^{\mathcal{I}_2}$. In $\mathcal{I}$ we add $(a, b) \in R^{\mathcal{I}}$ and we guarantee that $(\Delta \sqcap {}^{\forall R,L_1}\alpha_1)^{\mathcal{I}} \not\subset (\Gamma \sqcup {}^{\forall R,L_5}\alpha_5)^{\mathcal{I}}$.

3. $S \equiv {}^{\exists R,L_2}\alpha_2, {}^{\exists R,L_3}\alpha_3 \Rightarrow {}^{\exists R,L_6}\alpha_6$. By hypothesis, we have $\mathcal{I}_1 \not\models {}^{\exists R,L_2}\alpha_2 \Rightarrow {}^{\exists R,L_6}\alpha_6$ and $\mathcal{I}_2 \not\models {}^{\exists R,L_3}\alpha_3 \Rightarrow {}^{\exists R,L_6}\alpha_6$. We create the interpretation $\mathcal{I}$ as in the previous case, $\mathcal{I} = \mathcal{I}_1 \uplus \mathcal{I}_2$. From $\mathcal{I}_1$ we have $a \in ({}^{\exists R,L_2}\alpha_2)^{\mathcal{I}_1}$, and thus, an $(a, b) \in R^{\mathcal{I}_1}$ with $b \in \alpha_2^{\mathcal{I}_1}$. From $\mathcal{I}_2$ we have $b \in ({}^{\exists R,L_3}\alpha_3)^{\mathcal{I}_2}$, and thus, an $(b, c) \in R^{\mathcal{I}_2}$ with $c \in \alpha_3^{\mathcal{I}_2}$. We create now a fresh element d and add in $R^{\mathcal{I}}$ the set $\{(d, b), (d, c)\}$. We have guarantee that $d \in ({}^{\exists R,L_2}\alpha_2 \sqcap {}^{\exists R,L_3}\alpha_3)^{\mathcal{I}}$ and $d \notin ({}^{\exists R,L_6}\alpha_6)^{\mathcal{I}}$. Note that $b \notin ({}^{\exists R,L_6}\alpha_6)^{\mathcal{I}}$ (resp. c) by hypothesis.

4. If we consider $\forall R.\alpha \equiv \neg\exists R.\neg\alpha$, cases $S \equiv {}^{\exists R,L_2}\alpha_2, {}^{\forall R,L_1}\alpha_1 \Rightarrow {}^{\exists R,L_6}\alpha_6, {}^{\forall R,L_4}\alpha_4$ and $S \equiv {}^{\forall R,L_1}\alpha_1 \Rightarrow {}^{\forall R,L_4}\alpha_4, {}^{\forall R,L_5}\alpha_5$ has been already considered.

$\blacksquare$