V
A Natural Deduction for $\mathcal{ALC}$

In this chapter we present a Natural Deduction (**ND**) system for $\mathcal{ALC}$, named $\mathrm{ND}_{\mathcal{ALC}}$. We briefly discuss the motivation and the basic considerations behind the design of $\mathrm{ND}_{\mathcal{ALC}}$. We also prove the completeness, soundness and the normalization theorem for $\mathrm{ND}_{\mathcal{ALC}}$.

It is quite well-known the fact that Natural Deduction (**ND**) proofs in intuitionistic logic (**IL**) have computational content. This content can be explicitly read from the typed λ-calculus term associated to each proof. Moreover, to each normalization step that can be applied in the proof, there is a corresponding β-reduction in its associated typed λ-term. This is known as the Curry-Howard isomorphism (**CH-ISO**) between **ND** and the typed λ-calculus [30]. For classical logic this isomorphism does not hold any more. However, there are some attempts to justify weak or modified forms of this isomorphism for classical logic (see [5] and [3] for example).

It seems to exist some connections between the computational content of a proof and its ability to provide good structures to explanation extraction from proofs. In fact, an algorithm is one of the most precise arguments to explain how to obtain a result out of some inputs. Given that, translating algorithms according the propositions-as-types **CH-ISO** we should obtain a quite good argument establishing the conclusion from the premises. Despite the fact that for classical logic the **CH-ISO** is not well-established at all, we still argue in favour of ND proofs instead of Sequent Calculus (**SC**) in order to provide good explanations. One of the main points in favour of **ND** is the fact that it is single-conclusion and provides, in this way, a direct chain of inferences linking the propositions in the proof. It is worth noting that there is more than one **ND** normal proof related to the same cut-free **SC** proof. It is mainly because of this fact that a (cut-free) **SC** proof is related to more than one **ND** proof. We believe that explanations should be as specific as their proof-theoretical counterparts.

V.1 The $\mathrm{ND}_{\mathcal{ALC}}$ System

Figure V.1 shows the system called $\mathrm{ND}_{\mathcal{ALC}}$. Despite the use of labeled formulas, the main non-standard feature of $\mathrm{ND}_{\mathcal{ALC}}$ is the fact that it is defined on two kind of "formulas", namely *concept formulas* and *subsumptions of concepts*.

$$\frac{{}^{L^\forall}(\alpha \sqcap \beta)}{{}^{L^\forall}\alpha}\ \sqcap\text{-e} \qquad \frac{{}^{L^\forall}(\alpha \sqcap \beta)}{{}^{L^\forall}\beta}\ \sqcap\text{-e} \qquad \frac{{}^{L^\forall}\alpha \quad {}^{L^\forall}\beta}{{}^{L^\forall}(\alpha \sqcap \beta)}\ \sqcap\text{-i}$$

$$\frac{{}^{L^\exists}(\alpha \sqcup \beta) \quad \begin{array}{c}[{}^{L^\exists}\alpha]\\ \vdots \\ \gamma\end{array} \quad \begin{array}{c}[{}^{L^\exists}\beta]\\ \vdots \\ \gamma\end{array}}{\gamma}\ \sqcup\text{-e} \qquad \frac{{}^{L^\exists}\alpha}{{}^{L^\exists}(\alpha \sqcup \beta)}\ \sqcup\text{-i} \qquad \frac{{}^{L^\exists}\beta}{{}^{L^\exists}(\alpha \sqcup \beta)}\ \sqcup\text{-i}$$

$$\frac{{}^{L}\alpha \quad {}^{\neg L}\neg\alpha}{\bot}\ \neg\text{-e} \qquad \frac{\begin{array}{c}[{}^{L}\alpha]\\ \vdots \\ \bot\end{array}}{{}^{\neg L}\neg\alpha}\ \neg\text{-i} \qquad \frac{\begin{array}{c}[{}^{\neg L}\neg\alpha]\\ \vdots \\ \bot\end{array}}{{}^{L}\alpha}\ \bot_c$$

$$\frac{{}^{L}\exists R.\alpha}{{}^{L,\exists R}\alpha}\ \exists\text{-e} \qquad \frac{{}^{L,\exists R}\alpha}{{}^{L}\exists R.\alpha}\ \exists\text{-i} \qquad \frac{{}^{L}\forall R.\alpha}{{}^{L,\forall R}\alpha}\ \forall\text{-e}$$

$$\frac{{}^{L,\forall R}\alpha}{{}^{L}\forall R.\alpha}\ \forall\text{-i} \qquad \frac{{}^{L_1}\alpha \quad {}^{L_1}\alpha \sqsubseteq {}^{L_2}\beta}{{}^{L_2}\beta}\ \sqsubseteq\text{-e} \qquad \frac{\begin{array}{c}[{}^{L_1}\alpha]\\ \vdots \\ {}^{L_2}\beta\end{array}}{{}^{L_1}\alpha \sqsubseteq {}^{L_2}\beta}\ \sqsubseteq\text{-i}$$

$$\frac{{}^{L}\alpha}{{}^{\forall R,L}\alpha}\ Gen$$

Figure V.1: The Natural Deduction system for $\mathcal{ALC}$

If $\Phi_1, \Phi_2 \vdash \Psi$ is an inference rule involving only concept formulas then it states that whenever the premises are taken as non-empty collections of individuals the conclusion is taken as non-empty too. Particularly, providing any DL-interpretation for the premise concepts, if a is an individual belonging to both interpreted concepts then it also belongs to the interpreted conclusion. On the other hand, a subsumption $\Phi \sqsubseteq \Psi$ has no concept associate to it. It states, instead, a truth-value statement, depending on whether the interpretation of Φ is included in the corresponding interpretation of Ψ. In terms of a logical system, DL has no concept internalizing $\sqsubseteq$. As we will see on the next section, this imposes quite particular features on the form of the normal proofs in $\mathrm{ND}_{\mathcal{ALC}}$.

In the rule $\sqsubseteq$-i, ${}^{L_1}\alpha \sqsubseteq {}^{L_2}\beta$ depends only on the assumption ${}^{L_1}\alpha$ and no other hypothesis. The proviso to the application of rule *Gen* application is that the premise ${}^{L}\alpha$ does not depend on any hypothesis. In $\bot_c$-rule, ${}^{L}\alpha$ has to be different from $\bot$. In some rules the list of labels L has a superscript, $L^{\forall}$ or $L^{\exists}$. This notation means that the list of labels L should contain only $\forall R$ (resp. $\exists R$) labels. When L has not superscript, any kind of label is allowed.

The semantics of $\text{ND}_{\mathcal{ALC}}$ follows the $\mathcal{ALC}$ semantics presented in Section II.1, that is, is given by an *interpretation*. However, since $\text{ND}_{\mathcal{ALC}}$ deals with two different kind of formulas, we must define how an interpretation satisfies both kinds.

Definition 24 *Let $\Omega = (\mathcal{C}, \mathcal{S})$ be a tuple composed by a set of labeled concepts $\mathcal{C} = \{\alpha_1, \ldots, \alpha_n\}$ and a set of subsumption $\mathcal{S} = \{\gamma_1^1 \sqsubseteq \gamma_2^1, \ldots, \gamma_1^k \sqsubseteq \gamma_2^k\}$. We say that an interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ satisfies Ω and write $\mathcal{I} \models \Omega$ whenever:*

1. *$\mathcal{I} \models \mathcal{C}$, which means $\bigcap_{\alpha \in \mathcal{C}} \sigma(\alpha)^{\mathcal{I}} \neq \emptyset$; and*
2. *$\mathcal{I} \models \mathcal{S}$, which means that for all $\gamma_1^i \sqsubseteq \gamma_2^i \in \mathcal{S}$, we have $\sigma(\gamma_1^i)^{\mathcal{I}} \subseteq \sigma(\gamma_2^i)^{\mathcal{I}}$.*

We adopted the standard notation $\Omega \vdash F$ if exists a deduction Π with conclusion F (concept or subsumption) from Ω as set of hypothesis.

V.2 $\text{ND}_{\mathcal{ALC}}$ Soundness

Lemma 25 *Let Π be a deduction in $\text{ND}_{\mathcal{ALC}}$ of F with all hypothesis in $\Omega = (\mathcal{C}, \mathcal{S})$, then if F is a concept:*

$$\mathcal{S} \models \left(\bigsqcap_{A \in \mathcal{C}} A\right) \sqsubseteq F$$

and if F is a subsumption $A_1 \sqsubseteq A_2$:

$$\mathcal{S} \models \left(\bigsqcap_{A \in \mathcal{C}} A\right) \sqcap A_1 \sqsubseteq A_2$$

With the sake of clear presentation in the following proof we adopt some special notations. We will write $\forall L.\alpha$ to abbreviate $\forall R_1.\ldots.\forall R_n.\alpha$ when $L = \forall R_1.\ldots.\forall R_n$. The labelled concept ${}^{L}\alpha$ will be taken as equivalent to its $\mathcal{ALC}$ correspondent concept $\sigma({}^{L}\alpha)$. [1] Letters γ and δ stand for labelled concepts while α and β stand for $\mathcal{ALC}$ concepts. We take $\mathcal{C}$ as $\bigsqcap_{A \in \mathcal{C}} A$. We will aso use many times the axioms presented in Section II.6.

Proof: The proof of Lemma 25 is done by induction on the height of the proof tree Π, represented by $\mid \Pi \mid$.

[1] In Section III.1 the reader can find the definition of σ function and labeled formulas.

Base case If $\mid \Pi \mid = 1$ then $\Omega \vdash {}^{L}\alpha$ is such that ${}^{L}\alpha$ is in Ω. In that case, is easy to see that Lemma 25 holds since by basic set theory $(A \cap B) \subseteq A$ for all A and B.

Rule $\sqcap$***-e*** By induction hypothesis, if $\overset{\Pi_1}{{}^{L}(\alpha \sqcap \beta)}$ is a derivation with all hypothesis in $\{\mathcal{C}, \mathcal{S}\}$ then $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}(\alpha \sqcap \beta)$. From the definition of labeled concepts and Axiom 1 we can rewrite to $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}\alpha \sqcap {}^{L}\beta$ which from basic set theory guarantee $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}\alpha$.

Rule $\sqcap$***-i*** Let us consider the two derivations $\overset{\Pi_1}{{}^{L}\alpha}$ and $\overset{\Pi_2}{{}^{L}\beta}$ with all hypothesis in $\{\mathcal{C}_1, \mathcal{S}_1\}$ and $\{\mathcal{C}_2, \mathcal{S}_2\}$. By induction hypothesis, (1) $\mathcal{S}_1 \models \mathcal{C}_1 \sqsubseteq {}^{L}\alpha$ an (2) $\mathcal{S}_2 \models \mathcal{C}_2 \sqsubseteq {}^{L}\beta$. Now let us consider the deduction

$$\frac{\begin{array}{cc}\Pi_1 & \Pi_2 \\ {}^{L}\alpha & {}^{L}\beta\end{array}}{{}^{L}(\alpha \sqcap \beta)}$$

with all hypothesis in $\{\mathcal{C}_1 \cup \mathcal{C}_2, \mathcal{S}_1 \cup \mathcal{S}_2\}$. It is easy to see that from (1) and (2) $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq {}^{L}\alpha$ and $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq {}^{L}\beta$. From basic set theory we may write $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq {}^{L}\alpha \sqcap {}^{L}\beta$ and finally from Axiom 1 we get the desired result $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq {}^{L}(\alpha \sqcap \beta)$.

Rules $\sqcup$***-i*** Again by induction hypothesis, if $\overset{\Pi_1}{{}^{L}\alpha}$ is a derivation with all hypothesis in $\{\mathcal{C}, \mathcal{S}\}$ then $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}\alpha$. Using basic set theory we can rewrite to $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}\alpha \sqcup {}^{L}\beta$ and using Axiom 3 to $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}(\alpha \sqcup \beta)$.

Rule ($\sqcup$***-e)*** By induction hypothesis, if

$$\begin{array}{c}\Pi_1 \\ {}^{L}(\alpha \sqcup \beta),\end{array} \qquad \begin{array}{c}[{}^{L}\alpha] \\ \Pi_2 \\ \gamma\end{array} \quad \text{and} \quad \begin{array}{c}[{}^{L}\beta] \\ \Pi_3 \\ \gamma\end{array}$$

are derivations with hypothesis in $\{\mathcal{C}, \mathcal{S}\}$, $\{{}^{L}\alpha, \mathcal{S}\}$ and $\{{}^{L}\beta, \mathcal{S}\}$, respectively. Then, $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}(\alpha \sqcup \beta)$, $\mathcal{S} \models {}^{L}\alpha \sqsubseteq \gamma$ and $\mathcal{S} \models {}^{L}\beta \sqsubseteq \gamma$. From set theory $\mathcal{S} \models ({}^{L}\alpha \sqcup {}^{L}\beta) \sqsubseteq \gamma$ and from Axiom 3, $\mathcal{S} \models {}^{L}(\alpha \sqcup \beta) \sqsubseteq \gamma$. Now by the transitivity of set inclusion we can get the desired result $\mathcal{S} \models \mathcal{C} \sqsubseteq \gamma$.

Rules $\forall$***-i,*** $\forall$***-e,*** $\exists$***-i and*** $\exists$***-e*** They are sound since the premises and conclusions have the same semantics.

Rule* $\neg$*-e By induction hypothesis, if

$$\begin{array}{c}\Pi_1\\ {}^{L}\alpha\end{array} \quad \text{and} \quad \begin{array}{c}\Pi_2\\ {}^{\neg L}\neg\alpha\end{array}$$

are derivation with hypothesis in $\{\mathcal{C}_1, \mathcal{S}_1\}$ and $\{\mathcal{C}_2, \mathcal{S}_2\}$ we know that $\mathcal{S}_1 \models \mathcal{C}_1 \sqsubseteq {}^{L}\alpha$ and $\mathcal{S}_2 \models \mathcal{C}_2 \sqsubseteq {}^{\neg L}\neg\alpha$. Now consider the deduction

$$\dfrac{\begin{array}{c}\Pi_1\\ {}^{L}\alpha\end{array} \quad \begin{array}{c}\Pi_2\\ {}^{\neg L}\neg\alpha\end{array}}{\bot}$$

with hypothesis in $\{\mathcal{S}_1 \cup \mathcal{S}_2, \mathcal{C}_1 \cup \mathcal{C}_2\}$. By inductive hypothesis we can write $\mathcal{S}_1 \cup \mathcal{S}_2 \models \mathcal{C}_1 \sqsubseteq {}^{L}\alpha$ and $\mathcal{S}_2 \cup \mathcal{S}_2 \models \mathcal{C}_2 \sqsubseteq {}^{\neg L}\neg\alpha$. Now, from the fact that $\mathcal{ALC}$ semantics states ${}^{L}\alpha$ and ${}^{\neg L}\neg\alpha$ as two disjoint sets, we have $\mathcal{C}_1 \sqcap \mathcal{C}_2 = \emptyset$ and we can write $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq \bot$ as desired.

Rule* $\neg$*-i If $\{\mathcal{C}, \mathcal{S}\}$ holds all the hypothesis of the deduction $\begin{array}{c}{}^{L}\alpha\\ \Pi_2\\ \bot\end{array}$ then by induction hypothesis $\mathcal{S} \models \mathcal{C} \sqcap {}^{L}\alpha \sqsubseteq \bot$ (taking $\bot$ as its semantics counterpart, namely, the empty set). From basic set theory $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{\neg L}\neg\alpha$ as desired.

***Rule* $\bot_c$** The argument is similar from above.

Rule* $\sqsubseteq$*-e By induction hypothesis, if $\begin{array}{c}\Pi_1\\ \gamma\end{array}$ and $\begin{array}{c}\Pi_2\\ \gamma \sqsubseteq \delta\end{array}$ are deduction with hypothesis in $\{\mathcal{C}_1, \mathcal{S}_1\}$ and $\{\mathcal{C}_2, \mathcal{S}_2\}$, we have (1) $\mathcal{S}_1 \models \mathcal{C}_1 \sqsubseteq \gamma$ and (2) $\mathcal{S}_2 \models \mathcal{C}_2 \sqcap \gamma \sqsubseteq \delta$. Let us now consider the application of rule $\sqsubseteq$-e to construct the derivation

$$\dfrac{\begin{array}{c}\Pi_1\\ \gamma\end{array} \quad \begin{array}{c}\Pi_2\\ \gamma \sqsubseteq \delta\end{array}}{\delta}$$

with hypothesis in $\{C_1 \cup C_2, S_1 \cup S_2\}$. From (2) and $\mathcal{ALC}$ semantics we can conclude $\mathcal{S}_1 \cup \mathcal{S}_2 \models \mathcal{C}_2 \sqcap \gamma \sqsubseteq \delta$. Finally, from basic set theory $\mathcal{C}_1 \sqcap \mathcal{C}_2 \sqsubseteq \mathcal{C}_2$ we obtain $\mathcal{S}_1 \cup \mathcal{S}_2 \models \mathcal{C}_1 \sqcap \mathcal{C}_2 \sqsubseteq \delta$.

Rule* $\sqsubseteq$*-i By induction hypothesis, if $\begin{array}{c}\gamma\\ \Pi_1\\ \delta\end{array}$ is a deduction with hypothesis in $\{\mathcal{C}, \mathcal{S}\}$ then $\mathcal{S} \models \mathcal{C} \sqsubseteq \delta$ and we conclude $\mathcal{S} \models \mathcal{C}^{-} \sqcap \gamma \sqsubseteq \delta$ where $\mathcal{C}^{-}$ is $\mathcal{C} - \{\gamma\}$.

Rule *Gen* Let Π be a proof of ${}^{L}\alpha$ following from an empty set of hypothesis, we may write $\vdash {}^{L}\alpha$. That is, ${}^{L}\alpha$ is a DL-tautology or $\sigma({}^{L}\alpha)^{\mathcal{I}} \equiv \Delta^{\mathcal{I}}$. From the necessitation rule from Section II.6, whenever a concept C is a DL-tautology, for any given R, the concept $\forall R.C$ will be also. For that, we can conclude that ${}^{\forall R,L}\alpha$ for any given R will be also a tautology. Remember that ${}^{\forall R,L}\alpha \equiv \forall R.\sigma({}^{L}\alpha)$. ■

Let us now state the main theorem of this section.

Theorem 26 $\text{ND}_{\mathcal{ALC}}$ *is sound regarding the standard semantics of* $\mathcal{ALC}$.

$$\textit{if} \quad \Omega \vdash \gamma \quad \textit{then} \quad \Omega \models \gamma$$

where $\Omega = (\mathcal{C}, \mathcal{S}))$ *is a tuple composed by a set of labeled concepts (*C*) and subsumptions (*$\mathcal{S}$*).*

Proof: It follows directly from Lemma 25. ■

V.3 $\text{ND}_{\mathcal{ALC}}$ Completeness

We use the same strategy from Section III.3 to prove $\text{ND}_{\mathcal{ALC}}$ completeness. That is, we show how the axiomatic presentation of $\mathcal{ALC}$ can be derived in $\text{ND}_{\mathcal{ALC}}$.

Theorem 27 $\text{ND}_{\mathcal{ALC}}$ *is complete regarding the standard semantics of* $\mathcal{ALC}$.

Proof: The DL rule of generalization

$$\frac{\vdash \alpha}{\vdash \forall R.\alpha}$$

is a derived rule of $\text{ND}_{\mathcal{ALC}}$, for supposing $\vdash \alpha$ implies the existence of a proof (without hypothesis) Π of α. We prove $\forall R.\alpha$, without any new hypothesis by means of the following schema:

$$\cfrac{\cfrac{\begin{array}{c}\Pi \\ \vdots \\ \alpha\end{array}}{{}^{R}\alpha}\,Gen}{\forall R.\alpha}\,\forall\text{-i}$$

The following proofs justifies in $\text{ND}_{\mathcal{ALC}}$ the $\mathcal{ALC}$ axiom $\forall R.(A \sqcap B) \equiv (\forall R.A \sqcap \forall R.B)$, where $\alpha \equiv \beta$ is an abbreviation for $\alpha \sqsubseteq \beta$ and $\beta \sqsubseteq \alpha$, having obvious $\equiv$ elimination and introduction rules, based on $\sqsubseteq$ elimination and introduction rules.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall R.(A \sqcap B)]}{{}^{\forall R}(A \sqcap B)}\,\forall\text{-e}}{{}^{\forall R}A}\,\sqcap\text{-e}}{\forall R.A}\,\forall\text{-i} \qquad \dfrac{\dfrac{\dfrac{[\forall R.(A \sqcap B)]}{{}^{\forall R}(A \sqcap B)}\,\forall\text{-e}}{{}^{\forall R}B}\,\sqcap\text{-e}}{\forall R.B}\,\forall\text{-i}}{\forall R.A \sqcap \forall R.B}\,\sqcap\text{-i}}{\forall R.(A \sqcap B) \sqsubseteq \forall R.A \sqcap \forall R.B}\,\sqsubseteq\text{-i}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall R.A \sqcap \forall R.B]}{\forall R.A}\,\sqcap\text{-e}}{{}^{\forall R}A}\,\forall\text{-e} \qquad \dfrac{\dfrac{[\forall R.A \sqcap \forall R.B]}{\forall R.B}\,\sqcap\text{-e}}{{}^{\forall R}B}\,\forall\text{-e}}{{}^{\forall R}(A \sqcap B)}\,\sqcap\text{-i}}{\forall R.(A \sqcap B)}\,\forall\text{-i}}{\forall R.A \sqcap \forall R.B \sqsubseteq \forall R.(A \sqcap B)}\,\sqsubseteq\text{-i}$$

$\text{ND}_{\mathcal{ALC}}$ is a conservative extension of the classical propositional calculus. To see that, let Δ be a set of formulas of the form $\{\gamma_1, \ldots, \gamma_k, \alpha_1 \to \beta_1, \ldots, \alpha_n \to \beta_n\}$, where each γ_i, α_i and β_i are propositional formulas and α_i and β_i do not have any occurrence of $\to$. One can easily verify that any propositional classical consequence $\Delta \models \alpha$ is justified by a proof in classical **ND**. Now trasform this proof into a proof in $\text{ND}_{\mathcal{ALC}}$ by replacing each $\to$ by $\sqsubseteq$.

Since $\text{ND}_{\mathcal{ALC}}$ is a conservative extension of the classical propositional **ND** system that has the generalization as a derived rule, and, proves axiom $\forall R.(A \sqcap B) \equiv (\forall R.A \sqcap \forall R.B)$, we have the completeness for $\text{ND}_{\mathcal{ALC}}$ by a relative completeness to the axiomatic presentation of $\mathcal{ALC}$. ■

V.4 Normalization theorem for $\text{ND}_{\mathcal{ALC}}$

In this section we prove the normalization theorem for $\text{ND}_{\mathcal{ALC}}$. It is worth nothing that the usual reductions for obtaining a normal proof in classical propositional logic also applies to $\text{ND}_{\mathcal{ALC}}$. Thus, the first thing to observe is that we follow Prawitz's [49] approach incremented by Seldin's [62] permutation rules for the classical absurdity $\bot_c$. That is, using a set of permutative rules, we move any application of $\bot_c$-rule downwards the conclusion. After this transformation we end up with a proof having in each *branch* at most one $\bot_c$-rule application as the last rule of it.

In order to move the absurdity rule downwards the conclusion and also to have a more succinct proof we restrict the language to the fragment $\{\neg, \forall, \sqcap, \sqsubseteq\}$. This will not limit our results since any $\mathcal{ALC}$ formula can be rewritten in an equivalent one in this restricted fragment. We shall consider the system $\text{ND}^{-}{}_{\mathcal{ALC}}$ obtained from $\text{ND}_{\mathcal{ALC}}$ by removing from $\text{ND}_{\mathcal{ALC}}$ $\sqcup$-rules and $\exists$-rules. The Proposition 28 states that the system $\text{ND}^{-}{}_{\mathcal{ALC}}$ is essentially just a syntactic variation of $\text{ND}_{\mathcal{ALC}}$ system.

Proposition 28 *The* $\text{ND}_{\mathcal{ALC}}$ $\sqcup$*-rules and* $\exists$*-rules are derived in* $\text{ND}^{-}{}_{\mathcal{ALC}}$.

Proof: Considering the concept description ${}^{L}\alpha \sqcup \beta$ being defined by ${}^{L}\neg(\neg\alpha \sqcap \neg\beta)$ and the concept description ${}^{L}\exists R.\alpha$ by ${}^{L}\neg\forall R.\neg\alpha$.

The rules ($\sqcup$-i) can be derived as follows:

$$\dfrac{\dfrac{{}^{L}\alpha \quad \dfrac{\left[{}^{\neg L}(\neg\alpha \sqcap \neg\beta)\right]^{1}}{{}^{\neg L}\neg\alpha}\,\sqcap\text{-e}}{\bot}\,\neg\text{-e}}{{}^{L}\neg(\neg\alpha \sqcap \neg\beta)}\,\neg\text{-i} \qquad \dfrac{\dfrac{{}^{L}\beta \quad \dfrac{\left[{}^{\neg L}(\neg\alpha \sqcap \neg\beta)\right]^{1}}{{}^{\neg L}\neg\beta}\,\sqcap\text{-e}}{\bot}\,\neg\text{-e}}{{}^{L}\neg(\neg\alpha \sqcap \neg\beta)}\,\neg\text{-i}$$

where L contains only existencial quantified labels. $\neg L$ as described in Section III.1, is the negation of L, that is, universal quantified are changed to existential quantified and vice-versa. We note that rule $\sqcup$-i proviso requires that L contains only existential quantified labels, what makes the rule $\sqcap$-e proviso satisfied since $\neg L$ will only contains universal quantified labels. The rule $\sqcup$-e can also be derived:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\begin{matrix}[{}^{L}\alpha]\\ \vdots\\ \gamma\end{matrix} \quad [\neg\gamma]}{\bot}}{{}^{\neg L}\neg\alpha} \quad \dfrac{\dfrac{\begin{matrix}[{}^{L}\beta]\\ \vdots\\ \gamma\end{matrix} \quad [\neg\gamma]}{\bot}}{{}^{\neg L}\neg\beta}}{{}^{\neg L}(\neg\alpha \sqcap \neg\beta)} \qquad {}^{L}\neg(\neg\alpha \sqcap \neg\beta)}{\bot}}{\gamma}$$

For rules ($\exists$-i) and ($\exists$-e), it is worth noting that $\text{ND}^{-}{}_{\mathcal{ALC}}$ does not restrict the occurrence of existential labels, only the existential constructor of $\mathcal{ALC}$. In other words, we have just reused the $\mathcal{ALC}$ constructors $\forall$ and $\exists$ to "type" the labels and keep track of the original role quantification when it is promoted to label. Nevertheless, although the confusion could be avoided if we adopted $\neg\forall R$ instead of $\exists R$ in the labels of $\text{ND}^{-}{}_{\mathcal{ALC}}$ concepts, for clear presentation we choose to allow $\exists R$ on $\text{ND}^{-}{}_{\mathcal{ALC}}$ concept's labels.

$$\dfrac{\dfrac{{}^{L,\exists R}\alpha \quad \dfrac{\left[{}^{\neg L}\forall R.\neg\alpha\right]}{{}^{(\neg L),\forall R}\neg\alpha}}{\bot}}{{}^{L}\neg\forall R.\neg\alpha} \qquad \dfrac{\dfrac{\dfrac{\left[{}^{(\neg L),\forall R}\neg\alpha\right]}{{}^{\neg L}\forall R.\neg\alpha} \quad {}^{L}\neg\forall R.\neg\alpha}{\bot}}{{}^{L,\exists R}\alpha}$$

■

In the sequel, we adopt Prawitz's [50] terminologies such as: formula-tree, deductions or derivations, rule application, minor and major premises, *threads*, *branches* and so on. Nevertheless some terminologies have different definition in our system, in that case, we will present that definition.

A *branch* in a $\text{ND}_{\mathcal{ALC}}$ or $\text{ND}^{-}{}_{\mathcal{ALC}}$ deduction is an initial part $\alpha_1, \alpha_2, \ldots, \alpha_n$ of a thread such that α_n is either (i) the first formula occurrence in the thread that is a minor premise of an application of $\sqsubseteq$-e or (ii) the last formula occurrence of a thread (the end-formula of the deduction) if there is no such premise in the thread.

Given a deduction Π on $\text{ND}_{\mathcal{ALC}}$ or $\text{ND}^{-}{}_{\mathcal{ALC}}$, we define the *height* of a formula occurrence α in Π inductively:

- if α is the end-formula of Π (conclusion), then $h(\alpha) = 0$;
- if α is a premise of a rule application, say λ, in Π and is not the end-formula of Π, then $h(\alpha) = h(\beta) + 1$ where β is the conclusion of λ.

In a similar matter we can define the height of a *rule application* in a deduction.

A *maximal formula* is a formula occurrence that is consequence of an introduction rule and the major premise of an elimination rule. A maximal $\sqsubseteq$-formula in a proof Π is a maximal formula that is a subsumption.

Lemma 29 *Let Π be a proof of α (concept or subsumption of concepts) from Δ in $\text{ND}^{-}{}_{\mathcal{ALC}}$. Then there is a proof Π' without maximal $\sqsubseteq$-formulas.*

Proof: We prove Lemma 29 by induction over the number of maximal $\sqsubseteq$-formulas occurrences. We apply a sequence of reductions choosing always a highest maximal $\sqsubseteq$-formula occurence in the proof tree. In the reduction shown below we note that α cannot be a subsumpption, so that, the reduction application will never introduce new maximal $\sqsubseteq$-formulas. In other words, we cannot have nested subsumptions, subsumptions are not concepts.

$$\dfrac{\begin{matrix}\Pi_1\\ \alpha\end{matrix} \quad \dfrac{\begin{matrix}[\alpha]\\ \Pi_2\\ \beta\end{matrix}}{\alpha \sqsubseteq \beta}}{\beta} \quad \rhd \quad \begin{matrix}\Pi_1\\ [\alpha]\\ \Pi_2\\ \beta\end{matrix}$$

■

Lemma 30 (Moving $\bot_c$ downwards on branches) *If $\Omega \vdash_{\text{ND}^{-}{}_{\mathcal{ALC}}} \alpha$, then there is a deduction Π in $\text{ND}^{-}{}_{\mathcal{ALC}}$ of α from Ω where each branch in Π has at most one application of $\bot_c$-rule and, whenever it has one, it is one of the following cases: (i) the last rule applied in this branch; (ii) its conclusion is the premisse of a $\sqsubseteq$-i application, being this $\sqsubseteq$-i the last rule applied in the branch.*

Proof: Let Π be a deduction in $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$ of α (subsumption of concepts or concept) from a set of hypothesis Δ. Let λ be an application of a $\bot_c$-rule in Π with $h(\lambda) = d$ such that there is no other application of $\bot_c$-rule above λ. Let us consider each possible rule application immediately below λ. For each case, we show how one can exchange the rules decreasing the height of λ.

Rule $\forall$-e

$$
\cfrac{\cfrac{\begin{array}{c}[{}^{\neg L}\neg\forall R.\alpha]\\ \vdots\\ \bot\end{array}}{{}^{L}\forall R.\alpha}}{{}^{L,\forall R}\alpha}
\quad \rhd \quad
\cfrac{\begin{array}{c}\cfrac{\cfrac{\cfrac{[{}^{L}\forall R.\alpha]}{{}^{L,\forall R}\alpha} \quad [{}^{\neg L,\exists R}\neg\alpha]}{\bot}}{{}^{\neg L}\neg\forall R.\alpha}\\ \vdots\\ \bot\end{array}}{{}^{L,\forall R}\alpha}
$$

Rule $\forall$-i

$$
\cfrac{\cfrac{\begin{array}{c}[{}^{\neg L,\exists R}\neg\alpha]\\ \vdots\\ \bot\end{array}}{{}^{L,\forall R}\alpha}}{{}^{L}\forall R.\alpha}
\quad \rhd \quad
\cfrac{\begin{array}{c}\cfrac{\cfrac{\cfrac{[{}^{L,\forall R}\alpha]}{{}^{L}\forall R.\alpha} \quad [{}^{\neg L}\neg\forall R.\alpha]}{\bot}}{{}^{\neg L,\exists R}\neg\alpha}\\ \vdots\\ \bot\end{array}}{{}^{L}\forall R.\alpha}
$$

Rule $\sqcap$-i

$$
\cfrac{\cfrac{\begin{array}{c}{}^{\exists L}\neg\alpha\\ \vdots\\ \bot\end{array}}{{}^{\forall L}\alpha} \quad \begin{array}{c}\Pi\\ {}^{\forall L}\beta\end{array}}{{}^{\forall L}(\alpha \sqcap \beta)}
\quad \rhd \quad
\cfrac{\begin{array}{c}\cfrac{\cfrac{\cfrac{[{}^{\forall L}\alpha]^{2} \quad \begin{array}{c}\Pi\\ {}^{\forall L}\beta\end{array}}{{}^{\forall L}(\alpha \sqcap \beta)} \quad \left[{}^{\exists L}\neg(\alpha \sqcap \beta)\right]^{1}}{\bot}}{{}^{\exists L}\neg\alpha}\;2\\ \vdots\\ \bot\end{array}}{{}^{\forall L}(\alpha \sqcap \beta)}\;1
$$

Rule $\sqcap$-e

$$
\cfrac{\cfrac{\begin{array}{c}{}^{\exists L}\neg(\alpha \sqcap \beta)\\ \vdots\\ \bot\end{array}}{{}^{\forall L}(\alpha \sqcap \beta)}}{{}^{\forall L}\alpha}
\quad \rhd \quad
\cfrac{\begin{array}{c}\cfrac{\cfrac{\left[{}^{\exists L}\neg\alpha\right]^{2} \quad \cfrac{\left[{}^{\forall L}(\alpha \sqcap \beta)\right]^{1}}{{}^{\forall L}\alpha}}{\bot}}{{}^{\exists L}\neg(\alpha \sqcap \beta)}\;1\\ \vdots\\ \bot\end{array}}{{}^{\forall L}\alpha}\;2
$$

Rule ¬-e

$$\dfrac{\dfrac{\begin{array}{c}[{}^{\neg L}\neg\alpha]\\ \vdots\\ \bot\end{array}}{{}^{L}\alpha} \qquad \begin{array}{c}[\Delta]\\ \Pi\\ {}^{\neg L}\neg\alpha\end{array}}{\bot} \qquad \rhd \qquad \begin{array}{c}\dfrac{\dfrac{[{}^{L}\alpha] \qquad \begin{array}{c}[\Delta]\\ \Pi\\ {}^{\neg L}\neg\alpha\end{array}}{\bot}}{{}^{\neg L}\neg\alpha}\\ \vdots\\ \bot\end{array}$$

One must observe that in all reductions above, the conclusion of $\bot_c$ rule application is the premise of the rule considered in each case. That is why the ¬-i rule was not considered, if so, the conclusion of $\bot_c$ rule would have to be a $\bot$, wish is prohibit by the restriction on $\bot_c$-rule.

Rule ⊑-e

$$\dfrac{\dfrac{\begin{array}{c}[\neg\alpha]\\ \Pi_1\\ \bot\end{array}}{\alpha} \qquad \begin{array}{c}\Pi_2\\ \alpha \sqsubseteq \beta\end{array}}{\beta} \qquad \rhd \qquad \begin{array}{c}\dfrac{\dfrac{\dfrac{[\alpha]^1 \qquad \begin{array}{c}\Pi_2\\ \alpha \sqsubseteq \beta\end{array}}{\beta} \qquad [\neg\beta]^2}{\bot}}{\neg\alpha}\,1\\ \Pi_1\\ \dfrac{\bot}{\beta}\,2\end{array}$$

■

The reductions below will be used in the induction step in Theorem 31.

Let Π be a deduction of α from Ω which contains a maximal formula occurrence F. We say that Π' is a reduction of Π at F if we obtain Π' by removing F using the reductions below. Since F clearly can not be atomic, each reduction refers to a possible principal sign of F. If the principal sign of F is ψ, then Π' is said to be a ψ-reduction of Π. In each case, one can easily verify that Π' obtained is still a deduction of α from Ω.

⊓-***reduction***

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ {}^{\forall L}\alpha\end{array} \qquad \begin{array}{c}\Pi_2\\ {}^{\forall L}\beta\end{array}}{{}^{\forall L}(\alpha \sqcap \beta)}}{{}^{\forall L}\alpha} \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ {}^{\forall L}\alpha\end{array}$$

∀-***reduction***

$$\dfrac{\dfrac{\begin{array}{c}\Pi_1\\ {}^{L,\forall R}\alpha\end{array}}{{}^{L}\forall R.\alpha}}{{}^{L,\forall R}\alpha} \qquad \rhd \qquad \begin{array}{c}\Pi_1\\ {}^{L,\forall R}\alpha\end{array}$$

¬-reduction

$$\dfrac{\dfrac{\begin{array}{c}[{}^{L}\alpha]\\ \Pi_1\\ \bot\end{array}}{{}^{\neg L}\neg\alpha} \quad \begin{array}{c}\Pi_2\\ {}^{L}\alpha\end{array}}{\bot} \quad \rhd \quad \begin{array}{c}\Pi_2\\ [{}^{L}\alpha]\\ \Pi_1\\ \bot\end{array}$$

The fact that DL has no concept internalizing $\sqsubseteq$ imposes quite particular features on the form of the normal proofs in $\mathrm{ND}_{\mathcal{ALC}}$.

A $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$ deduction is called *normal* when it does not have maximal formula occurrences. Theorem 31 shows how we can construct a normal deduction in $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$.

Consider a deduction Π in $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$. Applying Lemma 29 we obtain a new deduction Π' without any maximal $\sqsubseteq$-formulas. Then we apply Lemma 30 to reduce the number of applications of $\bot_c$-rule on each branch and moving the remaining downwards to the end of each branch. Without loss of generality we can from now on consider any deduction in $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$ as having no maximal $\sqsubseteq$-formula and at most one $\bot_c$-rule application per branch, namely, the last one application in the branch.

Theorem 31 (normalization of $\mathrm{ND}_{\mathcal{ALC}}$**)** *If* $\Omega \vdash_{\mathrm{ND}^{-}{}_{\mathcal{ALC}}} \alpha$*, then there is a normal deduction in* $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$ *of* α *from* Ω*.*

Proof: Let Π be a deduction in $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$ having the form remarked in the previous paragraph. Consider the pair (d, n) where d is the maximum degree among the maximal formulas, and n is the number of maximal formulas with degree d. We proceed the normalization proof by induction on the lexicographic pair (d, n).

Let F be one of the highest maximal formula with degree d and consider each possible case according the principal sign of F.

If F has as principal sign $\sqcap$, applying the $\sqcap$-reduction we get a new deduction Π_1 with complexity (d_1, n_1). We now have $d_1 \leq d$, depending on the existence of other maximal $\sqcap$-formulas on Π. If $d_1 = d$, then necessarily $n_1 < n$. The cases where the principal sign of F is $\neg$ or $\forall$ are similar. Two facts can be observed. First, the $\sqsubseteq$-reduction will not be used anymore, since Π does not have any remaining maximal $\sqsubseteq$-formula. Second, although the $\neg$-reduction can increase the number of maximal formulas, those maximal formulas will undoubtedly have degree less than d, so that, we indeed have $(d_1, n_1) < (d, n)$. So induction hypothesis we have that Π_1 is normalizable and so is Π for each principal sign considered. ■

As we have already mentioned $\mathrm{ND}_{\mathcal{ALC}}$ has no concept internalization $\sqsubseteq$. This imposes quite particular form of the normal proofs in $\mathrm{ND}^{-}{}_{\mathcal{ALC}}$. Consider a thread in a deduction Π in $\text{ND}^{-}{}_{\mathcal{ALC}}$, such that no element of the thread is a minor premise of $\sqsubseteq$-e rule. We shall see that if Π is normal, the thread can be divided into two parts. There is one formula occurrence A in the thread such that all formula occurrences in the thread above A are premises of applications of elimination rules and all formula occurrences below A in the thread (except the last one) are premises of applications of introduction rules. Therefore, in the first part of the thread, we start from the top-most formula an decrease the complexity of that until A. In the second part of the thread we pass to more and more complex formulas. Given that, A is said thus the minimum formula in the thread. Moreover, each branch on Π has at most one application of $\bot_c$ rule as its last rule application.

Normalization is important since form it one can provide complete procedure to produce canonical proofs in $\mathcal{ALC}$. Canonical proofs are important regarding explaining theoremhood.