VI
Towards a proof theory for $\mathcal{ALCQI}$

Some pratical applications require a more expressive DL. For instance, if we want to formalize and reasoning over ER or UML diagrams using DL, we will need to move to $\mathcal{ALCQI}$ [4, 17, 15, 14, 16].

In this chapter we present a Sequent Calculus and a Natural Deduction for $\mathcal{ALCQI}$ description logic. These calculi are the first step towards extensions for the previously presented systems to more expressive description logics. In Section VII.3, we present a pratical use of the $\text{ND}_{\mathcal{ALCQI}}$ for reasoning over an UML diagram.

VI.1 $\mathcal{ALCQI}$ Introduction

$\mathcal{ALCQI}$ is an extension of $\mathcal{ALC}$ with number restrictions and inverse roles.

$$
\begin{aligned}
C ::= &\ \bot \mid A \mid \neg C \mid C_1 \sqcap C_2 \mid C_1 \sqcup C_2 \mid \exists R.C \mid \forall R.C \mid \leq nR.C \mid \geq nR.C \\
R ::= &\ P \mid P^-
\end{aligned}
$$

where A stands for atomic concepts and R for atomic roles. Some of the above operators can be mutually defined: (i) $\bot$ for $A \sqcap \neg A$; (ii) $\top$ for $\neg\bot$; (iii) $\geq kR.C$ for $\neg(\leq k-1R.C)$; (iv) $\leq kR.C$ for $\neg(\geq k+1R.C)$; (v) $\exists R.C$ for $\geq 1R.C$.

An $\mathcal{ALCQI}$ theory is a finite set of inclusion assertions of the form $C_1 \sqsubseteq C_2$. The semantics of $\mathcal{ALCQI}$ constructors and theories is analogous to that of $\mathcal{ALC}$. The semantics for qualified number restrictions are presented in Section II.3. The semantics of inverse roles is:

$$(P^-)^{\mathcal{I}} = \{(a, a') \in \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \mid (a', a) \in P^{\mathcal{I}}\}$$

The next sections presents a sequent calculus for $\mathcal{ALCQI}$ named $\text{SC}_{\mathcal{ALCQI}}$. In Section VI.2 we present the system and in Section VI.3 we prove its soundness. The proof of $\text{SC}_{\mathcal{ALCQ}}$ completeness should be obtained following the same strategy used for $\text{SC}_{\mathcal{ALC}}$. A version of $\text{SC}_{\mathcal{ALCQ}}$ can be designed along the same basic idea used to design the $SC^{\square}_{\mathcal{ALC}}$. Afterwards, provision of counter-example from fully expanded trees that are not proofs must be obtained.

VI.2 The Sequent Calculus for $\mathcal{ALCQI}$

The $SC_{\mathcal{ALCQI}}$ sequent calculus is a conservative extension of $SC_{\mathcal{ALC}}$ system to deal with qualified number restriction. The syntax for labeled concepts is modified to accept upper (at-most) and lower (at-least) bounds labels:

$$
\begin{aligned}
LB &::= \forall R \mid \exists R \mid \leq nR \mid \geq nR \\
R &::= P \mid P^{-} \\
L &::= LB, L \mid \emptyset \\
\phi_{lc} &::= {}^{L}\phi_{c}
\end{aligned}
$$

where n range over natural numbers, R over atomic role names and C over $\mathcal{ALCQI}$ concepts.

The translation of $SC_{\mathcal{ALCQI}}$ labeled concept to their $\mathcal{ALCQI}$ concept counterpart is straightforward. That is, we can easily extend the definiton of the σ function presented in Section III.1. For instance, ${}^{\geq nR}\alpha$ is equivalent of $\geq nR.\alpha$ and ${}^{\leq nR}\alpha$ is equivalent of $\leq nR.\alpha$. Finally, we observe that $\mathcal{ALCNI}$ is trivially obtained from $\mathcal{ALCQI}$ if we restrict qualified number restriction labels only to the $\top$ concept.

The $SC_{\mathcal{ALCQI}}$ system is presented at Figures VI.1, VI.2, VI.3 and VI.4 where L , stands for list of labels. In some rules, we superscribe the list of labels with the kind of labels allowed on it. For example, in rule $\sqcap$-l, we retrict L to contain only $\forall R$ or $\geq nR$ labels. We use the notation $L^{\forall\leq}$. Moreover, for easier understading, some provisos regarding the order relation between the number n and m are presented on the left of some rules. The provisos of rules $\forall$-r, $\forall$-l, prom-$\exists$, prom-$\forall$, $\sqcup$-l and $\sqcup$-r are the same presented in Section III.1. Moreover, we have the following additional provisos:

- Rules $\neg$-l and $\neg$-r, the list of labels L cannot have number restrictions $\leq nR$ nor $\geq nR$ for any R;
- Rule $\sqcap$-l, L cannot have $\leq nR$ nor $\exists R$ labels;
- Rule $\sqcap$-r, L cannot have $\geq nR$ nor $\exists R$ labels;
- Rule $\sqcup$-l, L cannot have $\geq nR$ nor $\forall R$ labels;
- Rule $\sqcup$-r, L cannot have $\leq nR$ nor $\forall R$ labels;

– Rule prom-$\geq$, for all ${}^{L}\delta \in \Delta$, L must only contain $\geq nR$ or $\forall R$ labels. For all ${}^{L}\gamma \in \Gamma$, L must only contain $\geq nR$ or $\exists R$ labels.

$$\frac{}{\alpha \Rightarrow \alpha} \qquad\qquad \frac{}{\bot \Rightarrow \alpha}$$

$$n \leq m \; \frac{}{{}^{\leq nR,L}\alpha \Rightarrow {}^{\leq mR,L}\alpha} \qquad\qquad n \geq m \; \frac{}{{}^{\geq nR,L}\alpha \Rightarrow {}^{\geq mR,L}\alpha}$$

Figure VI.1: The System SC$_{\mathcal{ALCQI}}$: the axioms

$$\frac{\Delta \Rightarrow \Gamma}{\Delta, \delta \Rightarrow \Gamma}\text{ weak-l} \qquad\qquad \frac{\Delta \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, \gamma}\text{ weak-r}$$

$$\frac{\Delta, \delta, \delta \Rightarrow \Gamma}{\Delta, \delta \Rightarrow \Gamma}\text{ contraction-l} \qquad\qquad \frac{\Delta \Rightarrow \Gamma, \gamma, \gamma}{\Delta \Rightarrow \Gamma, \gamma}\text{ contraction-r}$$

$$\frac{\Delta_1, \delta_1, \delta_2, \Delta_2 \Rightarrow \Gamma}{\Delta_1, \delta_2, \delta_1, \Delta_2 \Rightarrow \Gamma}\text{ perm-l} \qquad\qquad \frac{\Delta \Rightarrow \Gamma_1, \gamma_1, \gamma_2, \Gamma_2}{\Delta \Rightarrow \Gamma_1, \gamma_2, \gamma_1, \Gamma_2}\text{ perm-r}$$

$$\frac{\Delta_1 \Rightarrow \Gamma_1, {}^{L}\alpha \qquad {}^{L}\alpha, \Delta_2 \Rightarrow \Gamma_2}{\Delta_1, \Delta_2 \Rightarrow \Gamma_1, \Gamma_2}\text{ cut}$$

Figure VI.2: The System SC$_{\mathcal{ALCQI}}$: structural rules

Besides the rules inherited from SC$_{\mathcal{ALC}}$ with some extra provisos, SC$_{\mathcal{ALCQI}}$ specific rules are: (1) the rules shift-$\leq|\geq$-{l,r} that increase (decrease) labels upper (lower) bounds; (2) the rules $\leq \exists$-{l,r} and $\exists \leq$-{l,r} transform quantified number restricted labels into existential and the order way around.

Before present the soundness and completeness of $\mathcal{SALC}$ system, let us first present a simple example of its usage. The following proof draws the conclusion *everyone that have at least one child male or at least one child female have a child* in $\mathcal{ALCQI}$ terms.

Example 5 *In the proof below,* Fem *is an abbreviation for* Female *and* child *for* hasChild*.*

$$\frac{\Delta, {}^{L^{\forall\geq}}\alpha, {}^{L^{\forall\geq}}\beta \Rightarrow \Gamma}{\Delta, {}^{L^{\forall\geq}}(\alpha \sqcap \beta) \Rightarrow \Gamma}\ \sqcap\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L^{\forall\leq}}\alpha \quad \Delta \Rightarrow \Gamma, {}^{L^{\forall\leq}}\beta}{\Delta \Rightarrow \Gamma, {}^{L^{\forall\leq}}(\alpha \sqcap \beta)}\ \sqcap\text{-r}$$

$$\frac{\Delta, {}^{L^{\exists\leq}}\alpha \Rightarrow \Gamma \quad \Delta, {}^{L^{\exists\leq}}\beta \Rightarrow \Gamma}{\Delta, {}^{L^{\exists\leq}}(\alpha \sqcup \beta) \Rightarrow \Gamma}\ \sqcup\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L^{\exists\geq}}\alpha, {}^{L^{\exists\geq}}\beta}{\Delta \Rightarrow \Gamma, {}^{L^{\exists\geq}}(\alpha \sqcup \beta)}\ \sqcup\text{-r}$$

$$\frac{\Delta \Rightarrow \Gamma, {}^{\neg L^{\forall\exists}}\alpha}{\Delta, {}^{L^{\forall\exists}}\neg\alpha \Rightarrow \Gamma}\ \neg\text{-l} \qquad \frac{\Delta, {}^{\neg L^{\forall\exists}}\alpha \Rightarrow \Gamma}{\Delta \Rightarrow \Gamma, {}^{L^{\forall\exists}}\neg\alpha}\ \neg\text{-r}$$

Figure VI.3: The System $\mathrm{SC}_{\mathcal{ALCQI}}$: $\sqcap$, $\sqcup$ and $\neg$ rules

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Fem \Rightarrow Fem}{{}^{\exists child}Fem \Rightarrow {}^{\exists child}Fem}}{{}^{\geq 1 child}Fem \Rightarrow {}^{\exists child}Fem}}{{}^{\geq 1 child}Fem \Rightarrow {}^{\exists child}Male, {}^{\exists child}Fem}}{{}^{\geq 1 child}Fem \Rightarrow {}^{\exists child}(Male \sqcup Fem)}}{{}^{\geq 1 child}Fem \Rightarrow \exists child.(Male \sqcup Fem)}}{\geq 1 child.Fem \Rightarrow \exists child.(Male \sqcup Fem)} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Male \Rightarrow Male}{{}^{\exists child}Male \Rightarrow {}^{\exists child}Male}}{{}^{\geq 1 child}Male \Rightarrow {}^{\exists child}Male}}{{}^{\geq 1 child}Male \Rightarrow {}^{\exists child}Male, {}^{\exists child}Fem}}{{}^{\geq 1 child}Male \Rightarrow {}^{\exists child}(Male \sqcup Fem)}}{{}^{\geq 1 child}Male \Rightarrow \exists child.(Male \sqcup Fem)}}{\geq 1 child.Male \Rightarrow \exists child.(Male \sqcup Fem)}}{\geq 1 child.Male \sqcup \geq 1 child.Fem \Rightarrow \exists child.(Male \sqcup Fem)}$$

VI.3 $\mathrm{SC}_{\mathcal{ALCQI}}$ Soundness

Theorem 32 ($\mathcal{SALCQ}$ is sound) *Considering Ω a set of sequents, a theory presentation or a TBox, let an Ω-proof be any $\mathcal{SALCQ}$ proof in which sequents from Ω are permitted as initial sequents (in addition to the logical axioms). The soundness of $\mathcal{SALCQ}$ states that if a sequent $\Delta \Rightarrow \Gamma$ has an Ω-proof, then $\Delta \Rightarrow \Gamma$ is satisfied by every interpretation which satisfies Ω. That is,*

$$\textit{if} \qquad \Omega \vdash_{\mathrm{SC}_{\mathcal{ALCQI}}} \Delta \Rightarrow \Gamma \qquad \textit{then} \qquad \Omega \models \prod_{\delta \in \Delta} \sigma(\delta) \sqsubseteq \coprod_{\gamma \in \Gamma} \sigma(\gamma)$$

for all interpretation $\mathcal{I}$.

Proof: We proof Theorem 32 by induction on the length of the Ω-proofs. The length of a Ω-proof is the number of applications for any derivation rule of the calculus.

For the base case, proofs with length zero are proofs $\Omega \vdash \Delta \Rightarrow \Gamma$ where $\Delta \Rightarrow \Gamma$ occurs in Ω. In that case, it is easy to see that the theorem holds.

As inductive hypothesis, we will consider that for proofs of length n the theorem holds. It is now sufficient to show that each of the derivation rules preserves the truth. That is, if the premises holds, the conclusion must also

$$\frac{\Delta, {}^{L,\forall R}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}(\forall R.\alpha)L_2 \Rightarrow \Gamma}\ \forall\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\forall R}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}(\forall R.\alpha)}\ \forall\text{-r}$$

$$\frac{\Delta, {}^{L,\exists R}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}(\exists R.\alpha) \Rightarrow \Gamma}\ \exists\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\exists R}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}(\exists R.\alpha)}\ \exists\text{-r}$$

$$\frac{\Delta, {}^{L,\leq nR}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}\!\leq nR.\alpha \Rightarrow \Gamma}\ \leq\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\leq nR}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}\!\leq nR.\alpha}\ \leq\text{-r}$$

$$\frac{\Delta, {}^{L,\geq nR}\alpha \Rightarrow \Gamma}{\Delta, {}^{L}\!\geq nR.\alpha \Rightarrow \Gamma}\ \geq\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{L,\geq nR}\alpha}{\Delta \Rightarrow \Gamma, {}^{L}\!\geq nR.\alpha}\ \geq\text{-r}$$

$$n \leq m\ \frac{\Delta, {}^{\geq nR,L}\alpha \Rightarrow \Gamma}{\Delta, {}^{\geq mR,L}\alpha \Rightarrow \Gamma}\ \text{shift-}\geq\text{-l} \qquad n \geq m\ \frac{\Delta \Rightarrow {}^{\geq nR,L}\alpha, \Gamma}{\Delta \Rightarrow {}^{\geq mR,L}\alpha, \Gamma}\ \text{shift-}\geq\text{-r}$$

$$n \geq m\ \frac{\Delta, {}^{\leq nR,L}\alpha \Rightarrow \Gamma}{\Delta, {}^{\leq mR,L}\alpha \Rightarrow \Gamma}\ \text{shift-}\leq\text{-l} \qquad n \leq m\ \frac{\Delta \Rightarrow {}^{\leq nR,L}\alpha, \Gamma}{\Delta \Rightarrow {}^{\leq mR,L}\alpha, \Gamma}\ \text{shift-}\leq\text{-r}$$

$$\frac{\Delta, {}^{\geq 1R,L}\alpha \Rightarrow \Gamma}{\Delta, {}^{\exists R,L}\alpha \Rightarrow \Gamma}\ \geq\exists\text{-l} \qquad n \geq 1\ \frac{\Delta \Rightarrow \Gamma, {}^{\geq nR,L}\alpha}{\Delta \Rightarrow \Gamma, {}^{\exists R,L}\alpha}\ \geq\exists\text{-r}$$

$$n \geq 1\ \frac{\Delta, {}^{\exists R,L}\alpha \Rightarrow \Gamma}{\Delta, {}^{\geq nR,L}\alpha \Rightarrow \Gamma}\ \exists\geq\text{-l} \qquad \frac{\Delta \Rightarrow \Gamma, {}^{\exists R,L}\alpha}{\Delta \Rightarrow \Gamma, {}^{\geq 1R,L}\alpha}\ \exists\geq\text{-r}$$

$$\frac{\Delta, {}^{\exists R,L_1}\alpha \Rightarrow {}^{L_2}\beta, \Gamma}{\Delta, {}^{L_1}\alpha \Rightarrow {}^{\forall R^-,L_2}\beta, \Gamma}\ \exists\text{-inv} \qquad \frac{\Delta, {}^{L_1}\alpha \Rightarrow {}^{\forall R^-,L_2}\beta, \Gamma}{\Delta, {}^{\exists R,L_1}\alpha \Rightarrow {}^{L_2}\beta, \Gamma}\ \text{inv-}\exists$$

$$\frac{\Delta \Rightarrow \Gamma}{{}^{+\geq nR}\Delta \Rightarrow {}^{+\geq nR}\Gamma}\ \text{prom-}\geq \qquad \frac{\delta \Rightarrow \gamma}{{}^{+\leq nR}\gamma \Rightarrow {}^{+\leq nR}\delta}\ \text{prom-}\leq$$

$$\frac{\delta \Rightarrow \Gamma}{{}^{+\exists R}\delta \Rightarrow {}^{+\exists R}\Gamma}\ \text{prom-}\exists \qquad \frac{\Delta \Rightarrow \gamma}{{}^{+\forall R}\Delta \Rightarrow {}^{+\forall R}\gamma}\ \text{prom-}\forall$$

Figure VI.4: The system $\text{SC}_{\mathcal{ALCQI}}$: $\forall$, $\exists$, $\leq$, $\geq$ and *inverse* rules hold. Remembering from Section III.1 that the natural interpretation of a sequent $\Delta \Rightarrow \Gamma$ (Δ and Γ range over labelled concepts) is the $\mathcal{ALC}$ formula

$$\prod_{\delta \in \Delta} \sigma(\delta) \sqsubseteq \bigsqcup_{\gamma \in \Gamma} \sigma(\gamma)$$

For clear presentation, we will sometimes omit the translation from labelled concepts to $\mathcal{ALCQ}$ concepts and directly take Δ as the conjunction of $\mathcal{ALCQ}$ concepts and Γ as the disjunction of $\mathcal{ALCQ}$ concepts and assume that $\Delta \Rightarrow \Gamma$ has $\Delta \sqsubseteq \Gamma$ as a natural interpretation.

For rules on Figure VI.2, we can apply standard set theory. The proof of their soundness are the same presented in Section III.2 for $\mathcal{SALC}$. For instance, let us consider A, B, C, D and X sets. Rules weak-l and weak-r following from $(A \cap B) \subseteq A$ and $A \subseteq (A \cup B)$. Rules contraction-l and contraction-r follows from $A \cap A = A$ and $A \cup A = A$. In rules perm-l and perm-r, the premises and conclusions have the same semantics. The cut rule is also easily justified by set theory: if $A \subseteq (B \cup X)$ and $(X \cap C) \subseteq D$, we must have $(A \cap C) \subseteq (B \cup D)$.

In Figure **??**, rules $\forall$-l, $\forall$-r, $\exists$-l, $\exists$-r, $\leq$-l, $\leq$-r, $\geq$-l and $\geq$-r represent steps in the translation of labelled concepts to $\mathcal{ALCQ}$ concepts (reading top-bottom), so that, premises and conclusion have the same semantics, if the former subsumption holds, the later will also hold.

Rule $\exists \geq$-l is sound regarding the $\mathcal{SALCQ}$ semantic fact that $\geq nR.A \sqsubseteq \exists R.A$ if $n \geq 1$. If we take $A = \Delta^{\mathcal{I}}$, $B = \Gamma^{\mathcal{I}}$, $C = ({}^{\geq 1 R,L}\alpha)^{\mathcal{I}}$ and $D = ({}^{\exists R,L}\alpha)^{\mathcal{I}}$ for any given $\mathcal{I}$. Then we can conclude that if $A \cap C \subset B$ (premise) and $C \subset D$ (fact) then $A \cap D \subset B$ (conclusion).

The argument to show rule $\exists \geq$-r soundness is similar, Considering now the fact that $\exists R.A \equiv \geq 1R.A$ follows from the $\mathcal{ALCQ}$ semantics, we can show that: if we take $A = \Delta^{\mathcal{I}}$, $B = \Gamma^{\mathcal{I}}$, $C = ({}^{\exists R,L}\alpha)^{\mathcal{I}}$ and $D = ({}^{\geq 1 R,L}\alpha)^{\mathcal{I}}$ for any given $\mathcal{I}$, then if $A \subset B \cup C$ (premise) and $C \equiv D$ (fact) then $A \subset B \cup D$ (conclusion).

Rules $\neg$-l and $\neg$-r do not deal with quantified labeled concepts, their soundess were provided in Section III.2.

From the $\mathcal{ALCQ}$ semantics, we know that if $n \leq m$: (1) $\geq mR.C \sqsubseteq \geq nR.C$; and (2) $\leq nR.C \sqsubseteq \leq mR.C$ for any concept C. Taking $A = \Delta^{\mathcal{I}}$ and $B = \Gamma^{\mathcal{I}}$ for any $\mathcal{I}$, rules shift-$\geq$-l and shift-$\leq$-r are sound:

- if $A \cap ({}^{\geq nR,L}\alpha)^{\mathcal{I}} \subseteq B$ (premise), and ${}^{\geq mR,L}\alpha \subseteq {}^{\geq nR,L}\alpha$ (by 1 if $n \leq m$), then $A \cap ({}^{\geq mR,L}\alpha)^{\mathcal{I}} \subseteq B$ (conclusion);
- if $A \subseteq ({}^{\leq nR,L}\alpha)^{\mathcal{I}} \cup B$ (premise) and ${}^{\leq nR,L}\alpha \subseteq {}^{\leq mR,L}\alpha$ (by 2 if $n \leq m$), then $A \subseteq ({}^{\leq mR,L}\alpha)^{\mathcal{I}} \cup B$ (conclusion);

Rules shift-$\leq$-l and shift-$\geq$-r are similar, using the same semantics facts 1 and 2 above.

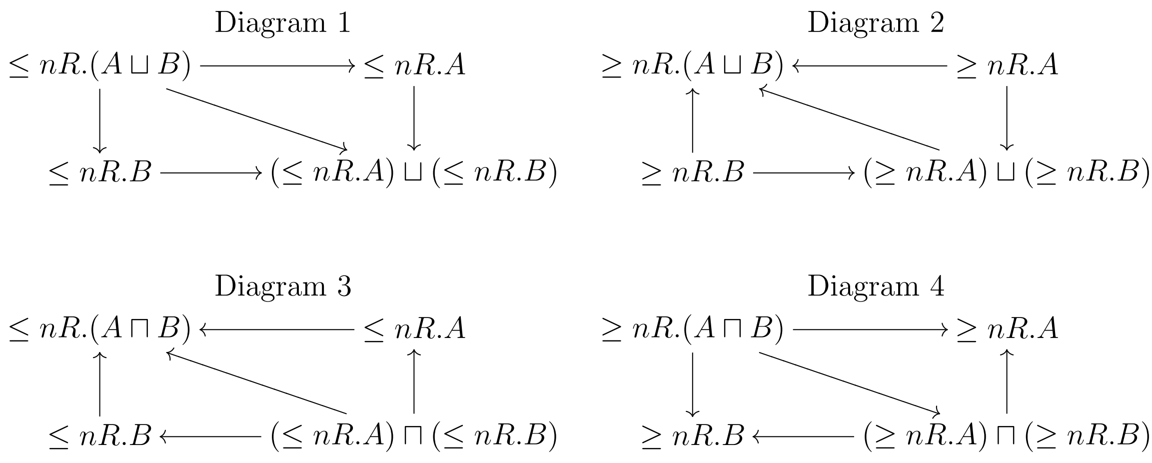

Figure VI.5: The inclusion diagrams for $\leq$ and $\geq$ over $\sqcup$ and $\sqcap$. The arrow $A \to B$ means $A \sqsubseteq B$.

For rules $\sqcup$-l, $\sqcup$-r, prom-$\exists$, prom-$\forall$, $\sqcap$-l and $\sqcap$-r we use the inclusion relations expressed in the diagrams of Figure VI.5. The arrows in the Figure indicate the inclusion direction, that is, if $A \to B$, than $A \sqsubseteq B$. Following the traditional proof theory terminology for sequent calculi, we call the principal formula, the formula occurring in the lower sequent of the inference which is not in the designated sets (Δ and Γ) and the auxiliary formulas are the formulas from the premises, subformulas of the principal formula in the conclusion.

Rule $\sqcup$-l with the proviso that the lists labels in auxiliary formulas can only contain $\exists R$ or $\leq nR$ labels for any role R and integer n is sound. This follows from: (1) the diagram 1 in the figure that shows that the union of the interpretation of auxiliary formulas is subset of the interpretation of the principal formula; and (2) the set theory fact that if $A \subseteq C$, $B \subseteq C$ and $X \subseteq A \cup B$ then $X \subseteq C$.

Rule $\sqcup$-r with the proviso that the list of labels in auxiliary formulas does not contain labels rather than $\exists R$ and $\geq nR$ for any role R and integer n is also sound. This follows from: (1) diagram 2 which shows that the interpretation of the principal formula contains the union of the interpretation of the auxiliary formulas; and (2) the set theory fact that if $A \subseteq B \cup C$ and $B \cup C \subseteq X$ then $A \subseteq X$.

Rule $\sqcap$-l providing that labels of auxiliary formulas does not contain labels rather than $\forall R$ and $\geq nR$ is sound given that: (1) diagram 4 shows that the intersection of the (interpretation of) the premises contains the interpretation of the conclusion, for any interpretation function; and (2) the set theory transitive property of the inclusion relation, that is, if $A \cap B \subseteq C$ and $X \subseteq A \cap B$ then $X \subseteq C$.

The soundness of rule $\sqcap$-r, providing that the list of labels of auxiliary formulas contain only $\forall$ and $\leq nR$ labels is proved with: (1) diagram 3 that shows that the intersection of the interpretation of the auxiliary formulas is included in the principal formula; (2) the fact that if $A \subseteq B$, $A \subseteq C$ and $B \cap C \subseteq X$ then $A \subseteq X$.

The proof of rules *inv*-$\exists$ and $\exists$-*inv* soudness derives from the fact that $A \sqsubseteq \forall R^{-}.B$ if and only if $\exists R.A \sqsubseteq B$. For clear presentation, we can state this fact as a rule in a natural deduction style:

$$\frac{(2) \quad \exists R.A \sqsubseteq B}{(1) \quad A \sqsubseteq \forall R^{-}.B}\, inv*$$

Now we have only to prove the double soundess of the above rule and consider $A \equiv {}^{L_1}\alpha$ and $B \equiv {}^{L_2}\beta$.

Case 1 $\rightarrow$ 2. Let $v \in \exists R.A^{\mathcal{I}} = \{v \mid (v,u) \in R^{\mathcal{I}} \wedge u \in A^{\mathcal{I}}\}$ thus $\exists u \in A^{\mathcal{I}}$ such that $(v,u) \in R^{\mathcal{I}}$ and hence $(u,v) \in (R^{-})^{\mathcal{I}}$. But from (1) we have that $u \in \forall R^{-}.B^{\mathcal{I}}$, thus $\forall v((u,v) \in (R^{-})^{\mathcal{I}} \rightarrow v \in B^{\mathcal{I}})$, hence $v \in B^{\mathcal{I}}$ we conclude (2). Note also that this conclusion also holds if $R^{\mathcal{I}} = \emptyset$.

Case 2 $\rightarrow$ 1. Let us assume that there is a $(v,u) \in R^{\mathcal{I}}$, so, $v \in \exists R.A^{\mathcal{I}}$ and hence $v \in B^{\mathcal{I}}$, by (2). We have $(u,v) \in (R^{-})^{\mathcal{I}}$ so $\forall v((u,v) \in (R^{-})^{\mathcal{I}} \rightarrow v \in B^{\mathcal{I}})$ and hence $u \in \forall R^{-}.B^{\mathcal{I}}$. If for some $u \in A^{\mathcal{I}}$ there is no v such that $(v,u) \in R^{\mathcal{I}}$ then $u \in \forall R^{-}.B^{\mathcal{I}}$, vacously.

■

VI.4 On SC$_{\mathcal{ALCQI}}$ Completeness

The proof of SC$_{\mathcal{ALCQI}}$ completeness should be obtained following the same strategy used for SC$_{\mathcal{ALC}}$. A deterministic version of SC$_{\mathcal{ALCQI}}$ can be designed along the same basic idea used on SC$^{\square}_{\mathcal{ALC}}$. Afterwards, provision of counter-example from fully expanded trees that are not proofs must be obtained.

Next, we show briefly how to provide a counter-example for a top-sequent that is not an axiom (initial sequent) in a fully expanded tree. Let us consider the full expanded tree in the sequel.

Example 6 *The bottom sequent represents an unsatisfiable subsumption. Clearly, it is not true that all people with at least two children necessarily have one child male and the other female. In the proof,* F *stands for* Female, M *for* Male *and* child *for* hasChild.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{M \Rightarrow M}{{}^{\exists child}M \Rightarrow {}^{\exists child}M} \quad \dfrac{F \Rightarrow M}{{}^{\exists child}F \Rightarrow {}^{\exists child}M}}{{}^{\exists child}(M \sqcup F) \Rightarrow {}^{\exists child}M}}{{}^{\geq 1 child}(M \sqcup F) \Rightarrow {}^{\exists child}M}}{{}^{\geq 2 child}(M \sqcup F) \Rightarrow {}^{\exists child}M}}{{}^{\geq 2 child}(M \sqcup F) \Rightarrow \exists child.M} \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{M \Rightarrow F}{{}^{\exists child}M \Rightarrow {}^{\exists child}F} \quad \dfrac{F \Rightarrow F}{{}^{\exists child}F \Rightarrow {}^{\exists child}F}}{{}^{\exists child}(M \sqcup F) \Rightarrow {}^{\exists child}F}}{{}^{\geq 1 child}(M \sqcup F) \Rightarrow {}^{\exists child}F}}{{}^{\geq 2 child}(M \sqcup F) \Rightarrow {}^{\exists child}F}}{{}^{\geq 2 child}(M \sqcup F) \Rightarrow \exists child.F}}{\dfrac{{}^{\geq 2 child}(M \sqcup F) \Rightarrow \exists child.M \sqcap \exists child.F}{\geq 2 child.(M \sqcup F) \Rightarrow \exists child.M \sqcap \exists child.F}}$$

Starting from any top-sequent that are not initial, one can easily construct an interpretation $\mathcal{I}$ such that

$$\mathcal{I} \not\models\ \geq 2hasChild.(Male \sqcup Female) \sqsubseteq \exists hasChild.Male \sqcap \exists hasChild.Female$$

Following [1, section 2.3.2.1] style, we use ABox assertions to represent the restrictions about the interpretation $\mathcal{I} = (\Delta, \mathcal{I})$ that we intend to construct. We started from the top-sequent $Female \Rightarrow Male$ and constructed $\mathcal{A}_1$ that falsifies it. The ABox $\mathcal{A}_2$, an extension of $\mathcal{A}_1$, is than constructed to falsify ${}^{\exists hasChild}Female \Rightarrow {}^{\exists hasChild}Male$. $\mathcal{A}_2$ falsifies all subsequent sequents until

$${}^{\geq n\ hasChild}(Male \sqcup Female) \Rightarrow {}^{\exists hasChild}Male$$

is reached. In order to falsify it we constructed $\mathcal{A}_3$ from $\mathcal{A}_2$. The main idea is that for each rule application, given a interpretation that falsifies its premise, one can provide an interpretation that falsifies its conclusion. From the natural interpretation of a sequent, Section III.1, we know that in order to falsify a sequent $\Delta \Rightarrow \Gamma$, an interpretation must contain an element c such that $c \in \Delta^{\mathcal{I}}$ and $c \notin \Gamma^{\mathcal{I}}$.

$$\begin{aligned} \mathcal{A}_1 &= \{Female(f_1)\} \\ \mathcal{A}_2 &= \mathcal{A}_1 \cup \{hasChild(a, f_1)\} \\ \mathcal{A}_3 &= \mathcal{A}_2 \cup \{hasChild(a, f_2), Female(f_2)\} \end{aligned} \tag{1}$$

The desired interpretation $\mathcal{I}$ can than be extracted from $\mathcal{A}_3$:

$$\Delta^{\mathcal{I}} = \{a, f_1, f_2\},\ Female^{\mathcal{I}} = \{f_1, f_2\},\ hasChild^{\mathcal{I}} = \{(a, f_1), (a, f_2)\} \tag{2}$$

VI.5 A Natural Deduction for $\mathcal{ALCQI}$

The Natural Deduction for $\mathcal{ALCQI}$, named $\text{ND}_{\mathcal{ALCQI}}$, is presented in Figure VI.6. $\text{ND}_{\mathcal{ALCQI}}$ is an extension of the system $\text{ND}_{\mathcal{ALC}}$ presented in

Chapter V.

When dealing with theories, sometimes is more convenient to have the following rule, since theories must be closed under generalizations.

$$\frac{\alpha \sqsubseteq \beta}{{}^{\forall R}\alpha \sqsubseteq {}^{\forall R}\alpha}$$

ND$_{\mathcal{ALCQI}}$ normalization and completeness is not presented in this thesis. A completeness proof for ND$_{\mathcal{ALCQI}}$ should follow from a (technically heavy) mapping from a complete Sequent Calculus for $\mathcal{ALCQI}$ to ND$_{\mathcal{ALCQI}}$.

Assuming that normalization holds for ND$_{\mathcal{ALCQI}}$, one can define a proof procedure for ND$_{\mathcal{ALCQI}}$. Initially decompose the (candidate) conclusion ($\alpha \sqsubseteq \beta$) by means of introduction rules applied bottom-up, until atomic labeled concepts. For each atomic concept, one chooses an hypothesis from Δ and by decomposing it, by means of elimination rules, tries to achieve this very atomic (labeled) concept. This allows us to derive a (complete) proof procedure for the logic, decomposing the conclusions and the hypothesis until atomic levels an proving one set using the other. In our case we are interested in applying this proof procedure on top of theories. In the sequel we show ND$_{\mathcal{ALCQI}}$ soundness.

VI.6 ND$_{\mathcal{ALCQI}}$ Soundness

This section extends the results of Section V.2 to prove that ND$_{\mathcal{ALCQI}}$ rules are sound. We adopted here the same notations used in Section V.2. Moreover, most part of the proof use results from Section VI.3.

Theorem 33 ND$_{\mathcal{ALCQI}}$ *is sound regarding the standard semantics of* $\mathcal{ALCQI}$*. That is,*

$$\textit{if} \quad \Omega \vdash \gamma \quad \textit{then} \quad \Omega \models \gamma$$

Proof: It follows direct from Lemma 34. ■

Lemma 34 *Let* Π *be a deduction in* ND$_{\mathcal{ALCQI}}$ *of* F *with all hypothesis in* $\Omega = (\mathcal{C}, \mathcal{S})$*, then if* F *is a concept:*

$$\mathcal{S} \models \bigsqcap_{A \in \mathcal{C}} A \sqsubseteq F$$

and if F *is a subsumption* $A_1 \sqsubseteq A_2$*:*

$$\mathcal{S} \models \bigsqcap_{A \in \mathcal{C}} A \sqcap A_1 \sqsubseteq A_2$$

Proof: The proof of Lemma 25 is done by induction on the height of a proof tree Π represented by $\mid \Pi \mid$. The proof of ND$_{\mathcal{ALCQI}}$ rules soundness is similar from the proof of soundness of their counterparts in ND$_{\mathcal{ALC}}$.

$$\frac{{}^{L^{\forall\geq}}(\alpha \sqcap \beta)}{{}^{L^{\forall\geq}}\alpha}\ \sqcap\text{-e} \qquad \frac{{}^{L^{\forall\geq}}(\alpha \sqcap \beta)}{{}^{L^{\forall\geq}}\beta}\ \sqcap\text{-e} \qquad \frac{{}^{L^{\forall\leq}}\alpha \quad {}^{L^{\forall\leq}}\beta}{{}^{L^{\forall\leq}}(\alpha \sqcap \beta)}\ \sqcap\text{-i}$$

$$\frac{{}^{L^{\exists\leq}}(\alpha \sqcup \beta) \quad \begin{matrix}[{}^{L^{\exists\leq}}\alpha] \\ \vdots \\ \gamma\end{matrix} \quad \begin{matrix}[{}^{L^{\exists\leq}}\beta] \\ \vdots \\ \gamma\end{matrix}}{\gamma}\ \sqcup\text{-e} \qquad \frac{{}^{L^{\exists\geq}}\alpha}{{}^{L^{\exists\geq}}(\alpha \sqcup \beta)}\ \sqcup\text{-i} \qquad \frac{{}^{L^{\exists\geq}}\beta}{{}^{L^{\exists\geq}}(\alpha \sqcup \beta)}\ \sqcup\text{-i}$$

$$\frac{{}^{L^{\forall\exists}}\alpha \quad {}^{\neg L^{\forall\exists}}\neg\alpha}{\bot}\ \neg\text{-e} \qquad \frac{\begin{matrix}[{}^{L^{\forall\exists}}\alpha] \\ \vdots \\ \bot\end{matrix}}{{}^{\neg L^{\forall\exists}}\neg\alpha}\ \neg\text{-i} \qquad \frac{\begin{matrix}[{}^{\neg L^{\forall\exists}}\neg\alpha] \\ \vdots \\ \bot\end{matrix}}{{}^{L^{\forall\exists}}\alpha}\ \bot_c$$

$$\frac{{}^{L}\exists R.\alpha}{{}^{L,\exists R}\alpha}\ \exists\text{-e} \qquad \frac{{}^{L,\exists R}\alpha}{{}^{L}\exists R.\alpha}\ \exists\text{-i} \qquad \frac{{}^{L}\forall R.\alpha}{{}^{L,\forall R}\alpha}\ \forall\text{-e}$$

$$\frac{{}^{L,\forall R}\alpha}{{}^{L}\forall R.\alpha}\ \forall\text{-i} \qquad \frac{{}^{L}\leq nR.\alpha}{{}^{L,\leq nR}\alpha}\ \leq\text{-e} \qquad \frac{{}^{L,\leq nR}\alpha}{{}^{L}\leq nR.\alpha}\ \leq\text{-i}$$

$$\frac{{}^{L}\geq nR.\alpha}{{}^{L,\geq nR}\alpha}\ \geq\text{-e} \qquad \frac{{}^{L,\geq nR}\alpha}{{}^{L}\geq nR.\alpha}\ \geq\text{-i}$$

$$\frac{{}^{\exists R,L}\alpha}{{}^{\geq 1R,L}\alpha}\ \geq\exists \qquad \frac{{}^{\geq nR,L}\alpha}{{}^{\exists R,L}\alpha}\ \exists\geq (n \geq 1)$$

$$\frac{{}^{\geq mR,L}\alpha}{{}^{\geq nR,L}\alpha}\ -\geq (m \geq n) \qquad \frac{{}^{\leq mR,L}\alpha}{{}^{\leq nR,L}\alpha}\ +\geq (m \leq n) \qquad \frac{{}^{L}\alpha}{{}^{\forall R,L}\alpha}\ Gen$$

$$\frac{{}^{L_1}\alpha \quad {}^{L_1}\alpha \sqsubseteq {}^{L_2}\beta}{{}^{L_2}\beta}\ \sqsubseteq\text{-e} \qquad \frac{\begin{matrix}[{}^{L_1}\alpha] \\ \vdots \\ {}^{L_2}\beta\end{matrix}}{{}^{L_1}\alpha \sqsubseteq {}^{L_2}\beta}\ \sqsubseteq\text{-i} \qquad \frac{{}^{\exists R,L_1}\alpha \sqsubseteq {}^{L_2}\beta}{{}^{L_1}\alpha \sqsubseteq {}^{\forall R^-,L_2}\beta}\ inv$$

Figure VI.6: The Natural Deduction system for $\mathcal{ALCQI}$

Base case This case is similar from the proof of Lemma 34. If $\mid \Pi \mid = 1$ then $\Omega \vdash {}^{L}\alpha$ is such that ${}^{L}\alpha$ is in Ω. In that case, is easy to see that Lemma 34 holds since by basic set theory $(A \cap B) \subseteq A$ for all A and B.

Rule $\sqcap$***-e*** this rule has one additional proviso that must be taken into account, namely, besides $\forall R$ roles, the label of the premise may only contain $\geq nR$ roles. By induction hypothesis, if $\begin{matrix}\Pi_1\\ {}^{L}(\alpha \sqcap \beta)\end{matrix}$ is a derivation with all hypothesis in $\{\mathcal{C}, \mathcal{S}\}$ then $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}(\alpha \sqcap \beta)$. From Diagram 4 on Figure VI.5 and Axiom 1 we know that ${}^{L}(\alpha \sqcap \beta) \sqsubseteq {}^{L}\alpha \sqcap {}^{L}\beta$ and from basic set theory ${}^{L}\alpha \sqcap {}^{L}\beta \sqsubseteq {}^{L}\alpha$ so $\mathcal{S} \models \mathcal{C} \sqsubseteq {}^{L}\alpha$ as desired.

Rule $\sqcap$***-e*** let us take the proof of soudness of its counterpart in Section V.2 and consider the additional proviso that L may only contain $\forall R$ and $\leq nR$ labels. Given $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq {}^{L}\alpha \sqcap {}^{L}\beta$ (by arguments of Section V.2) and ${}^{L}\alpha \sqcap {}^{L}\beta \sqsubseteq {}^{L}(\alpha \sqcap \beta)$ by Diagram 3 on Figure VI.5 and Axiom 1, we can write $\mathcal{S}_1 \cup \mathcal{S}_2 \models (\mathcal{C}_1 \sqcap \mathcal{C}_2) \sqsubseteq {}^{L}(\alpha \sqcap \beta)$.

$\sqcup$***-e and*** $\sqcup$***-i*** As in the cases above, the proof is similar of their counterparts in Section V.2. We have also to consider diagrams 1 and 2 on Figure VI.5 to prove that ${}^{L^{\exists\geq}}\alpha \sqcup {}^{L^{\exists\geq}}\beta \sqsubseteq {}^{L^{\exists\geq}}(\alpha \sqcup \beta)$ and ${}^{L^{\exists\leq}}(\alpha \sqcup \beta) \sqsubseteq {}^{L^{\exists\leq}}\alpha \sqcup {}^{L^{\exists\leq}}\beta$.

Rules $\neg$-{i,e} and $\bot$-c are the same of $\text{ND}_{\mathcal{ALC}}$ since they do not handle number restrictions and inverse. Rules $\forall$-{i,e}, $\exists$-{i,e}, $\leq$-{i,e} and $\geq$-{i,e} have the same semantics of their premise and conclusion, thus they are sound.

The soundness of $- \geq$ and $+ \geq$ are direct consequence of the $\mathcal{ALCQI}$ semantics and they are actually used to prove the soundness of $\text{SC}_{\mathcal{ALCQI}}$ shift rules in Section VI.3.

Rule *inv* is not only sound but also double sound, once more, we point to the proof of soundness in Section VI.3.

The soundness of the remain rules *Gen* and $\sqsubseteq$-{i,e} are consequence of the soundness of their counterparts in $\text{ND}_{\mathcal{ALC}}$, see Section V.2. ■