VII
Proofs and Explanations

VII.1 Introduction

From a logical point of view, the conceptual modeling tasks can be summarized by the following steps:

1. Observe the "world";
2. Determine what is relevant;
3. Choose or define your terminology by means of non-logical linguistic terms;
4. Write down the main laws, the axioms, governing your "world";
5. Verify the correctness (sometimes completeness too) of your set of laws, that is, the theory constructed.

Steps 1, 2 and 3 may be facilitated by the use of an informal notation (UML, ER, Flow-Charts, etc) and their respective methodology, but it is essentially "Black Art" [42]. Step 4 demands quite a lot of knowledge of the "world" begin specified (the model). Step 5 essentially provides finitely many tests as support for the correctness of an infinite quantified property.

A deduction of a proposition α from a set of hypothesis Γ is essentially a mean of convincing that Γ entails α. When validating a theory, represented by a set of logical formulas, we mainly test entailments, possibly using a theorem prover. Considering a model M specified by the set of axioms $Spec(M)$, given a property ϕ about M, from the entailment tests results one can rise the following questions:

1. If $M \models \phi$ and $Spec(M) \vdash \phi$, why ϕ is truth? One must provide a proof of ϕ;
2. If $M \models \phi$, but $Spec(M) \nvdash \phi$ from the attempt to construct the proof of ϕ one may obtain a counter-model and from that counter-model an

explanation for the failed entailment. Model-checking based reasoning can be used in such situation;

3. If $M \not\models \phi$, but $Spec(M) \vdash \phi$, why does this false proposition holds? In this case, one must provide a proof of ϕ.

Here we are interested in the last case, tests providing a false positive answer, that is, the prover shows a deduction/proof for an assertion that must be invalid in the theory considered. This is one of the main reasons to explain a theorem when validating a theory. We need to provide explanation on why a false positive is entailed. Another reason to provide explanations of theorem has to do with providing explanation on why some assertion is a true positive, which is the first case. This latter use is concerned with certification; in this case the proof/deduction itself serves as a certification document. This section does not take into account educational uses of theorem provers, and their resulting theorems, since explanations in these cases are more demanding.

For the tasks of providing proofs and explanations, we compare three deduction systems, Analytic Tableaux (**AT**) [64], Sequent Calculus (**SC**) [66] and Natural Deduction (**ND**) [49] as presented in the respective references. In this section we consider the propositional logic (Minimal, Intuitionistic and Classical, as defined in [49]). Let us consider a theory (presented by a knowledge base $\mathcal{KB}$) containing the single axiom

$$\mathcal{KB} \equiv (Quad \wedge PissOnFireHydrant) \rightarrow Dog$$

which classifies a *dog* as a *quadruped* which pisses on a fire hydrant. This $\mathcal{KB}$ draws the following proposition

$$(Quad \rightarrow Dog) \vee (PissOnFireHydrant \rightarrow Dog)$$

Figure VII.1 presents three from many more possible proofs of this entailment in Propositional Tableaux system. Figure VII.2 presents three possible proofs in Sequent Calculus, they are also sorted out from many others possible proofs in Sequent Calculus. Figure VII.3 present the only two possible normal proofs for this entailment.

Consider the derivations from Figure VII.1 and VII.2. They all correspond to the Natural Deduction derivations that is showed in Figure VII.3. The Tableaux and Sequent Calculus variants only differ in the order of rule applications. In **ND** there is no such distinction. In this example, the order of application is irrelevant in terms of explanation, although it is not for the prover's implementation. The pattern represented by the **ND** deduction is

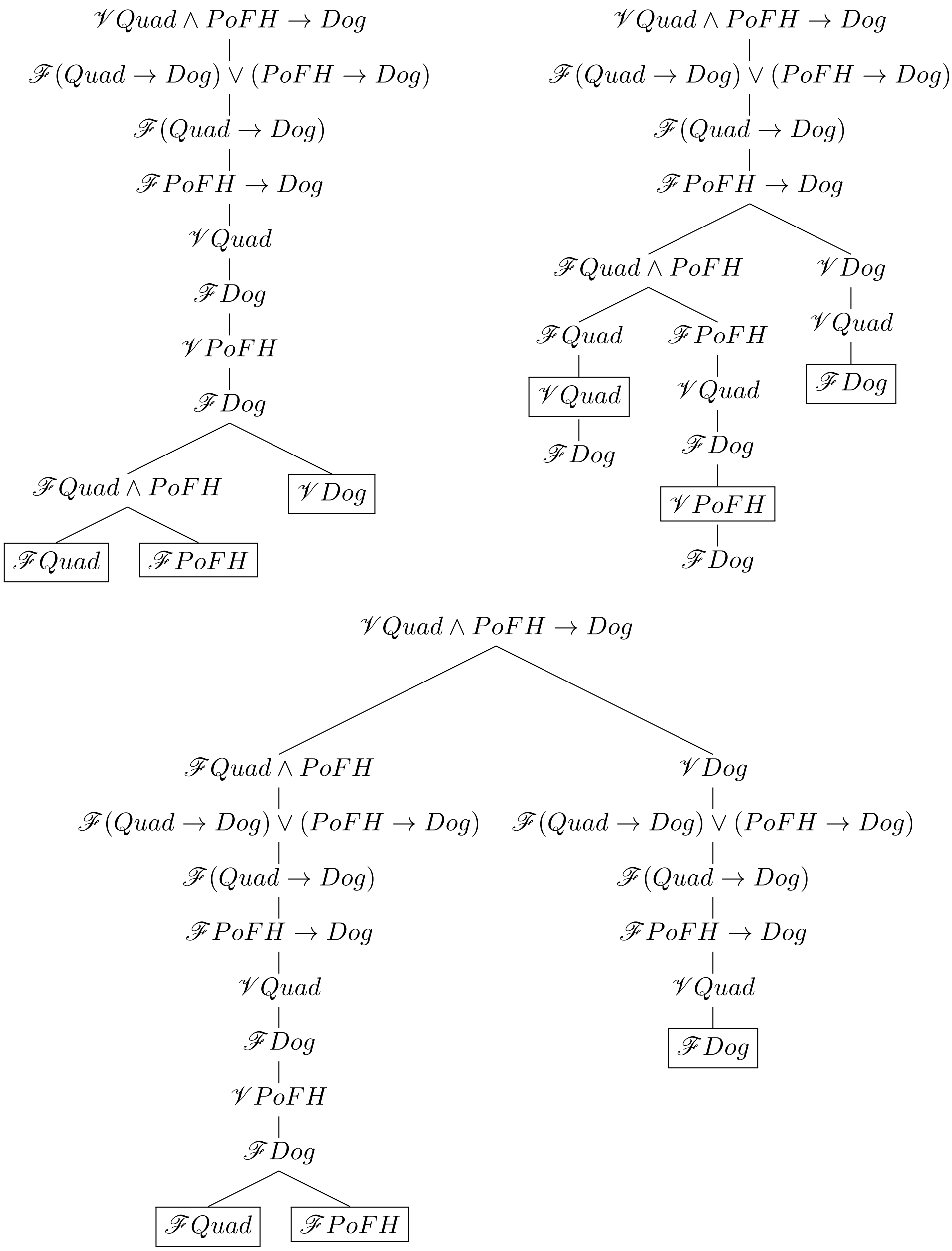

Figure VII.1: Tableaux proofs

$$\dfrac{\dfrac{\dfrac{\mathcal{KB} \Rightarrow PoFH \wedge Quad \rightarrow Dog \qquad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{Quad \Rightarrow Quad}{Quad, PoFH \Rightarrow Quad} \quad \dfrac{PoFH \Rightarrow PoFH}{Quad, PoFH \Rightarrow PoFH}}{Quad, PoFH \Rightarrow Quad \wedge PoFH} \qquad Dog \Rightarrow Dog}{Quad, PoFH, PoFH \wedge Quad \rightarrow Dog \Rightarrow Dog}}{Quad, PoFH, PoFH \wedge Quad \rightarrow Dog \Rightarrow Dog, Dog}}{PoFH, PoFH \wedge Quad \rightarrow Dog \Rightarrow (Quad \rightarrow Dog), Dog}}{PoFH, \mathcal{KB} \Rightarrow (Quad \rightarrow Dog), Dog}}{\mathcal{KB} \Rightarrow (Quad \rightarrow Dog), (PoFH \rightarrow Dog)}}{\mathcal{KB} \Rightarrow (Quad \rightarrow Dog) \vee (PoFH \rightarrow Dog)}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{KB} \Rightarrow PoFH \wedge Quad \rightarrow Dog \qquad \dfrac{\dfrac{\dfrac{Quad \Rightarrow Quad}{Quad, PoFH \Rightarrow Quad} \quad \dfrac{PoFH \Rightarrow PoFH}{Quad, PoFH \Rightarrow PoFH}}{Quad, PoFH \Rightarrow Quad \wedge PoFH} \qquad Dog \Rightarrow Dog}{Quad, PoFH, PoFH \wedge Quad \rightarrow Dog \Rightarrow Dog}}{\mathcal{KB}, Quad, PoFH \Rightarrow Dog}}{\mathcal{KB}, Quad, PoFH \Rightarrow Dog, Dog}}{\mathcal{KB}, PoFH \Rightarrow (Quad \rightarrow Dog), Dog}}{\dfrac{\mathcal{KB} \Rightarrow (Quad \rightarrow Dog), (PoFH \rightarrow Dog)}{\mathcal{KB} \Rightarrow (Quad \rightarrow Dog) \vee (PoFH \rightarrow Dog)}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{KB} \Rightarrow PoFH \wedge Quad \rightarrow Dog \qquad \dfrac{\dfrac{\dfrac{Quad \Rightarrow Quad}{Quad, PoFH \Rightarrow Quad} \quad \dfrac{PoFH \Rightarrow PoFH}{Quad, PoFH \Rightarrow PoFH}}{Quad, PoFH \Rightarrow Quad \wedge PoFH} \qquad \dfrac{Dog \Rightarrow Dog}{Dog \Rightarrow Dog, Dog}}{Quad, PoFH, PoFH \wedge Quad \rightarrow Dog \Rightarrow Dog, Dog}}{\mathcal{KB}, Quad, PoFH \Rightarrow Dog, Dog}}{\mathcal{KB}, PoFH \Rightarrow (Quad \rightarrow Dog), Dog}}{\dfrac{\mathcal{KB}, PoFH \Rightarrow (Quad \rightarrow Dog), Dog}{\dfrac{\mathcal{KB} \Rightarrow (Quad \rightarrow Dog), (PoFH \rightarrow Dog)}{\mathcal{KB} \Rightarrow (Quad \rightarrow Dog) \vee (PoFH \rightarrow Dog)}}}$$

Figure VII.2: Sequent Calculus proofs

close to what one expects from an argument drawing a conclusion from any conjunction that it contains. This example shows how **SC** proofs carry more information than that needed for a meaningful explanation. Concerning the **AT** system, Smullyan [64] noted that **AT** proofs correspond to **SC** proofs by considering sequents formed by positively signed formulas ($T\alpha$) at the antecedent and negatively signed ones ($F\alpha$) appearing at the succedent. A Block **AT** is defined then by considering **AT** expansion rules in the form of inference rules. In this way, our example in **SC** would carry the same content useful for explanation carried by the **AT** proofs. We must note that different **SC** proofs and its corresponding **AT** proofs, as the ones shown, are represented, all of them, by only two possible variations of normal derivations in **ND**.

Sequent Calculus seems to be the oldest among the three systems here considered. Gentzen decided to move from **ND** to **SC** in order to detour from technical problems faced by him in his syntactical proof of the consistency of Arithmetic in 1936. As mentioned by Prawitz [49], **SC** can be understood as a meta-calculus for the deducibility relation in **ND**. A consequence of this is that **ND** can represent in only one deduction of α from $\gamma_1, \dots, \gamma_n$ many **SC** proofs of the sequent $\gamma_1, \dots, \gamma_n \Rightarrow \alpha$. Gentzen made **SC** formally state rules that were implicit in **ND**, such as the structural rules. We advice the reader that the **SC** used here (see [66]) is a variation of Gentzen's calculus designed with the goal of having, in each inference rule, any formula occurring in a

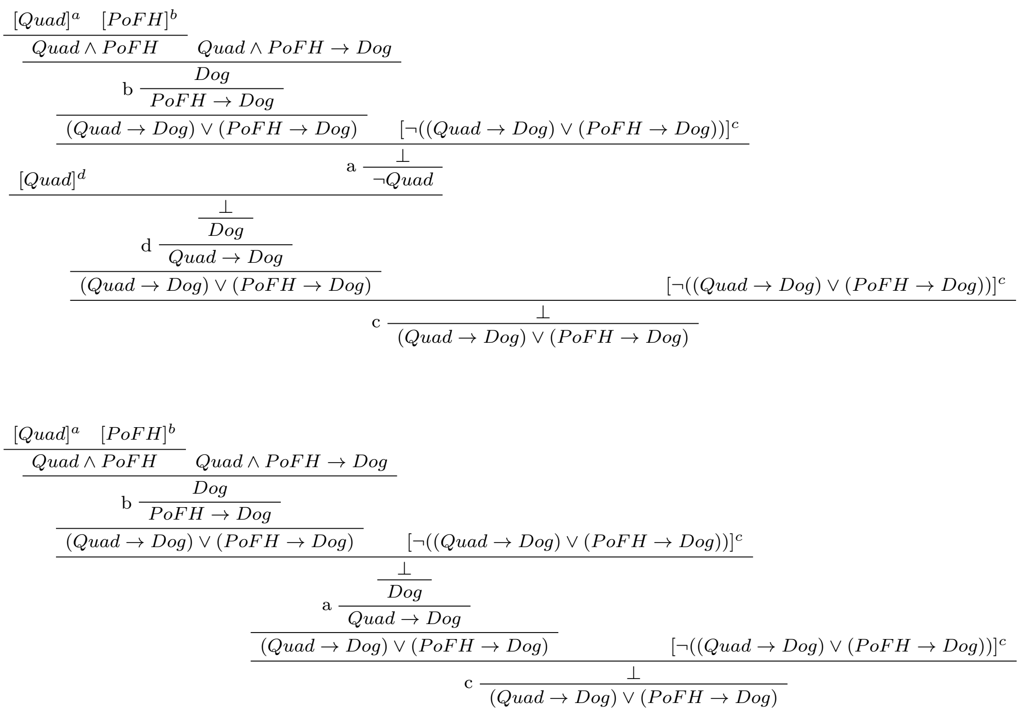

Figure VII.3: Natural Deduction proofs

premise as a sub-formula of some formula occurring in the conclusion. This sub-formula property facilitates the implementation of a prover based on this very system.

Consider a normal **ND** deduction Π_1 of α from $\gamma_1, \ldots, \gamma_k$, and, a deduction Π_2 of γ_i (for some $i = 1, k$) from $\delta_1, \ldots, \delta_n$. Using latter Π_1 in the former Π_2 deduction yields a (possibly non-normal) deduction of α from $\gamma_1, \delta_1, \ldots, \gamma_k, \delta_n$. This can be done in **SC** by applying a cut rule between the proofs of the corresponding sequents $\delta_1, \ldots, \delta_n \Rightarrow \gamma_i$ and $\gamma_1, \ldots, \gamma_k \Rightarrow \alpha$ yielding a proof of the sequent $\gamma_1, \delta_1, \ldots, \gamma_k, \delta_n \Rightarrow \alpha$. The new **ND** deduction can be normalized, in the former case, and the cut introduced in the latter case can be eliminated. In the case of **AT**, the fact that they are closed by *modus ponens* implies that closed **AT** for $\delta \rightarrow \gamma$ and $\gamma \rightarrow \alpha$ entails the existence of a closed **AT** for $\delta \rightarrow \alpha$. The use of cuts, or equivalently, lemmas may reduce the size of a derivation. However, the relevant information conveyed by a deduction or proof in any of these systems has to firstly consider normal deductions, cut-free proofs and analytic Tableaux. They are the most representative formal objects in each of these systems as a consequence of the sub-formula property, holding in **ND** too. Besides that they are computationally easier to build than their non-normal counterparts.

These examples are carried out in Minimal Logic. For Classical reasoning, an inherent feature of most DLs, including $\mathcal{ALC}$, the above scenario changes.

Any classical proof of the sequent $\gamma_1, \gamma_2 \Rightarrow \alpha_1, \alpha_2$ corresponds a **ND** deduction of $\alpha_1 \vee \alpha_2$ from γ_1, γ_2, or, of α_1 from $\gamma_1, \gamma_2, \neg\alpha_2$, or, of α_2 from $\gamma_1, \gamma_2, \neg\alpha_1$, or, of $\neg\gamma_1$ from $\neg\alpha_1, \gamma_2, \neg\alpha_2$, and so on. In Classical logic [1], each **SC** may represent more than one deduction, since we have to choose which formula will be the conclusion in the **ND** side. We recall that it still holds that to each **ND** deduction there is more than one **SC** proof. In order to serve as a good basis for explanations of classical theorems we choose **ND** as the most adequate. Note that we are not advocating that the prover has to produce **ND** proofs directly. An effective translation to a **ND** might be provided. Of course there must be a **ND** for the logic involved. If, besides that, a normalization is provided for a system, we know that it is possible to always deal with canonical proofs satisfying the sub-formula principle.

VII.2 An example of Explanations from Proofs in $\mathrm{SC}_{\mathcal{ALC}}$

Let us briefly introduce the idea of providing explanations of proofs in $\mathrm{SC}_{\mathcal{ALC}}$. Consider the proof:

$$
\begin{array}{cl}
Doctor \Rightarrow Doctor & \\ \hline
Doctor \Rightarrow Rich, Doctor & \text{weak-r} \\ \hline
Doctor \Rightarrow (Rich \sqcup Doctor) & \sqcup\text{-r} \\ \hline
{}^{\forall child}Doctor \Rightarrow {}^{\forall child}(Rich \sqcup Doctor) & \text{prom-}\forall \\ \hline
\top, {}^{\forall child}Doctor \Rightarrow {}^{\forall child}(Rich \sqcup Doctor) & \text{weak-l} \\ \hline
\top \Rightarrow {}^{\exists child}\neg Doctor, {}^{\forall child}(Rich \sqcup Doctor) & \neg\text{-r} \\ \hline
\top \Rightarrow {}^{\exists child}\neg Doctor, {}^{\exists child}Lawyer, {}^{\forall child}(Rich \sqcup Doctor) & \text{weak-r} \\ \hline
\top \Rightarrow {}^{\exists child}\neg Doctor, \exists child.Lawyer, {}^{\forall child}(Rich \sqcup Doctor) & \exists\text{-r} \\ \hline
\top \Rightarrow \exists child.\neg Doctor, \exists child.Lawyer, {}^{\forall child}(Rich \sqcup Doctor) & \exists\text{-r} \\ \hline
\top \Rightarrow (\exists child.\neg Doctor) \sqcup (\exists child.Lawyer), {}^{\forall child}(Rich \sqcup Doctor) & \sqcup\text{-r} \\ \hline
{}^{\exists child}\top \Rightarrow {}^{\exists child}((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)), {}^{\exists child,\forall child}(Rich \sqcup Doctor) & \text{prom-}\exists \\ \hline
{}^{\exists child}\top, {}^{\forall child}\neg((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)) \Rightarrow {}^{\exists child,\forall child}(Rich \sqcup Doctor) & \neg\text{-l} \\ \hline
{}^{\exists child}\top, {}^{\forall child}\neg((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)) \Rightarrow {}^{\exists child}\forall child.(Rich \sqcup Doctor) & \forall\text{-r} \\ \hline
{}^{\exists child}\top, \forall child.\neg((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)) \Rightarrow {}^{\exists child}\forall child.(Rich \sqcup Doctor) & \forall\text{-l} \\ \hline
{}^{\exists child}\top, \forall child.\neg((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)) \Rightarrow \exists child.\forall child.(Rich \sqcup Doctor) & \exists\text{-r} \\ \hline
\exists child.\top, \forall child.\neg((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)) \Rightarrow \exists child.\forall child.(Rich \sqcup Doctor) & \exists\text{-l} \\ \hline
\exists child.\top \sqcap \forall child.\neg((\exists child.\neg Doctor) \sqcup (\exists child.Lawyer)) \Rightarrow \exists child.\forall child.(Rich \sqcup Doctor) & \sqcap\text{-l}
\end{array}
$$

This proof tree could be explained by the following text:

(1) Doctors are Doctors or Rich (2) So, Everyone having all children Doctors has all children Doctors or Rich. (3) Hence, everyone either has at least a child that is not a doctor or every children is a doctor or rich. (4) Moreover, everyone is of the kind above , or,

[1]Intuitionistic Logic and Minimal Logic have similar behavior concerning the relationship between their respective systems of ND and SC.

alternatively, have at least one child that is a lawyer. (5) In other words, if everyone has at least one child, then it has one child that has at least one child that is a lawyer, or at least one child that is not a doctor, or have all children doctors or rich. (6) Thus, whoever has all children not having at least one child not a doctor or at least one child lawyer has at least one child having every children doctors or rich.

The above explanation was build from top to bottom (toward the conclusion of the proof), by a procedure that tries not to repeat conjunctive particles (if - then, thus, hence, henceforth , moreover etc) to put together phrases derived from each subproof. In this case, phrase (1) come from weak-r, $\sqcup$-r; phrase (2) come from prom-2; (3) is associated to weak-l, neg-r; (4) corresponds to weak-r, the two following $\exists$-r and the $\sqcap$; (5) is associated to prom-1 and finally (6) corresponds to the remaining of the proof. The reader can note the large possibility of using endophoras in the construction of texts from structured proofs as the ones obtained by either $\mathrm{SC}_{\mathcal{ALC}}$ or $\mathrm{SC}^{\Box}{}_{\mathcal{ALC}}$.

In Section VII.3 an example illustrating the use of theoremhood to explain reasoning on UML models is accomplished by proofs in **ND**, **SC** and **AT**.

VII.3 Explaining UML in $\mathrm{ND}_{\mathcal{ALCQI}}$

In [4], *DL*s are used to formalize UML diagrams. It uses two DL languages: $\mathcal{DLR}_{ifd}$ or $\mathcal{ALCQI}$. The diagram on Figure VII.4 and its formalization on Figure VII.5, are from [4].

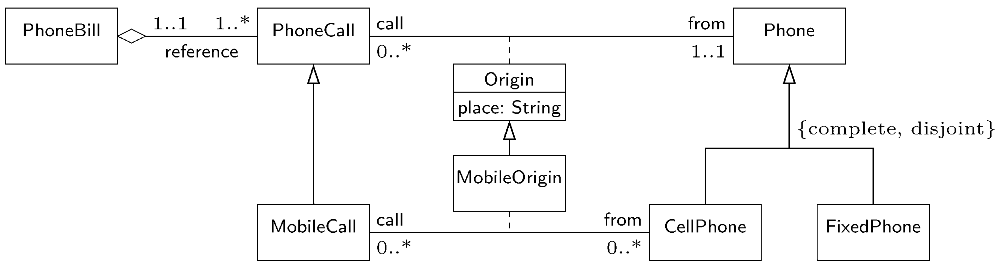

Figure VII.4: UML class diagram

We use examples of *DL* deductions from [4, page 84], using $\mathrm{ND}_{\mathcal{ALCQI}}$ to reason on the $\mathcal{ALCQI}$ KB. The idea is to exemplify how one can obtain from $\mathrm{ND}_{\mathcal{ALCQI}}$ proofs, a more precise and direct explanation.

The first example concerns a refinement of a multiplicity. That is, from reasoning on the diagram, one can deduce that the class `MobileCall`

$$
\begin{aligned}
\mathtt{Origin} &\sqsubseteq \forall\mathtt{place.String}\\
\mathtt{Origin} &\sqsubseteq \exists\mathtt{place}.\top \sqcap (\leq 1\ \mathtt{place})\\
\mathtt{Origin} &\sqsubseteq \exists\mathtt{call.PhoneCall} \sqcap (\leq 1\ \mathtt{call}) \sqcap \exists\mathtt{from.Phone} \sqcap (\leq 1\ \mathtt{from})\\
\mathtt{MobileOrigin} &\sqsubseteq \exists\mathtt{call.MobileCall} \sqcap (\leq 1\mathtt{call}) \sqcap \exists\mathtt{from.CellPhone} \sqcap (\leq 1\ \mathtt{from})\\
\mathtt{PhoneCall} &\sqsubseteq (\geq 1\ \mathtt{call}^{-}.\mathtt{Origin}) \sqcap (\leq 1\ \mathtt{call}^{-}.\mathtt{Origin})\\
\top &\sqsubseteq \forall\mathtt{reference}^{-}.\mathtt{PhoneBill} \sqcap \forall\mathtt{reference.PhoneCall}\\
\mathtt{PhoneBill} &\sqsubseteq (\geq 1\ \mathtt{reference}^{-})\\
\mathtt{PhoneCall} &\sqsubseteq (\geq 1\ \mathtt{reference}) \sqcap (\leq 1\ \mathtt{reference})\\
\mathtt{MobileCall} &\sqsubseteq \mathtt{PhoneCall}\\
\mathtt{MobileOrigin} &\sqsubseteq \mathtt{Origin}\\
\mathtt{CellPhone} &\sqsubseteq \mathtt{Phone}\\
\mathtt{FixedPhone} &\sqsubseteq \mathtt{Phone}\\
\mathtt{CellPhone} &\sqsubseteq \neg\mathtt{FixedPhone}\\
\mathtt{Phone} &\sqsubseteq \mathtt{CellPhone} \sqcup \mathtt{FixedPhone}
\end{aligned}
$$

Figure VII.5: The $\mathcal{ALCQI}$ theory obtained from the UML diagram on Figure VII.4

participates on the association `MobileOrigin` with multiplicity $0\ldots1$, instead of the $0\ldots*$ presented in the diagram. The proof on $\mathrm{ND}_{\mathcal{ALCQI}}$ is as follows, where we abbreviate the class names for their first letters, for instance, `Origin` (`O`), `MobileCall` (`MC`), `call` (`c`) and so on. Note that $\neg \geq 2\mathtt{c}^{-}.\mathtt{MO}$ is actually an abbreviation for $\leq 1\mathtt{c}^{-}.\mathtt{MO}$.

$$
\dfrac{
  \dfrac{
    \dfrac{
      \dfrac{}{\dfrac{
        [\geq 2\ \mathtt{c}^{-}.\mathtt{MO}]^2 \quad \dfrac{\mathtt{MO} \sqsubseteq \mathtt{O}}{\geq 2\ \mathtt{c}^{-}.\mathtt{MO} \sqsubseteq\ \geq 2\ \mathtt{c}^{-}.\mathtt{O}}
      }{\geq 2\ \mathtt{c}^{-}.\mathtt{O}}}
      \quad
      \dfrac{\dfrac{\dfrac{[\mathtt{MC}]^1 \quad \mathtt{MC} \sqsubseteq \mathtt{PC}}{\mathtt{PC}} \quad \mathtt{PC} \sqsubseteq\ \geq 1\ \mathtt{c}^{-}.\mathtt{O} \sqcap \leq 1\ \mathtt{c}^{-}.\mathtt{O}}{\geq 1\ \mathtt{c}^{-}.\mathtt{O} \sqcap \leq 1\ \mathtt{c}^{-}.\mathtt{O}}}{\leq 1\ \mathtt{c}^{-}.\mathtt{O}}
    }{\bot}
  }{\neg \geq 2\ \mathtt{c}^{-}.\mathtt{MO}}\ 2
}{\mathtt{MC} \sqsubseteq \neg \geq 2\ \mathtt{c}^{-}.\mathtt{MO}}\ 1
$$

To exemplify deductions on diagrams, an incorrect generalization between two classes was introduced. The generalization asserts that each `CellPhone` is a `FixedPhone`, which means the introduction of the new axiom `CellPhone` $\sqsubseteq$ `FixedPhone` in the KB. From that improper generalization, several undesirable properties could be drawn.

The first conclusion about the modified diagram is that `Cellphone` is now inconsistent. The $\mathrm{ND}_{\mathcal{ALCQI}}$ proof below explicits that from the newly introduced axiom and from the axiom $\mathtt{CellPhone} \sqsubseteq \neg\mathtt{FixedPhone}$ in the KB, one can conclude that `CellPhone` is now inconsistent.

$$\dfrac{\dfrac{\dfrac{\mathtt{Cell} \sqsubseteq \neg\mathtt{Fixed} \quad [\mathtt{Cell}]^1}{\neg\mathtt{Fixed}} \qquad \dfrac{\mathtt{Cell} \sqsubseteq \mathtt{Fixed} \quad [\mathtt{Cell}]^1}{\mathtt{Fixed}}}{\bot}}{\mathtt{Cell} \sqsubseteq \bot}\,1$$

The second conclusion is that in the modified diagram, $\mathtt{Phone} \equiv \mathtt{FixedPhone}$. Note that we have only to show that $\mathtt{Phone} \sqsubseteq \mathtt{FixedPhone}$ since $\mathtt{FixedPhone} \sqsubseteq \mathtt{Phone}$ is an axiom already in the original KB. We can conclude from the proof below that $\mathtt{Phone} \sqsubseteq \mathtt{FixedPhone}$ is not a direct consequence of `CellPhone` being inconsistent, as stated in [4], but mainly as a direct consequence of the newly introduced axiom and a case analysis over the possible subtypes of `Phone`.

$$\dfrac{\dfrac{\dfrac{[\mathtt{Phone}]^1 \quad \mathtt{Phone} \sqsubseteq \mathtt{Cell} \sqcup \mathtt{Fixed}}{\mathtt{Cell} \sqcup \mathtt{Fixed}} \qquad \dfrac{[\mathtt{Cell}] \quad \mathtt{Cell} \sqsubseteq \mathtt{Fixed}}{\mathtt{Fixed}} \qquad [\mathtt{Fixed}]}{\mathtt{Fixed}}}{\mathtt{Phone} \sqsubseteq \mathtt{Fixed}}\,1$$

Below it is shown the above discussed subsumption proved in **SC** (Sequent Calculus).

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\mathtt{MO} \Rightarrow \mathtt{O}}}{\geq 2\ \mathtt{call}^{-}.\mathtt{MO} \Rightarrow \geq 2\ \mathtt{call}^{-}.\mathtt{O}}}{\mathtt{MC}, \geq 2\ \mathtt{call}^{-}.\mathtt{MO} \Rightarrow \geq 2\ \mathtt{call}^{-}.\mathtt{O}} \qquad \dfrac{\dfrac{\mathtt{MC} \Rightarrow \mathtt{PC} \quad \mathtt{PC} \Rightarrow \geq 1\ \mathtt{call}^{-}.\mathtt{O} \sqcap \leq 1\ \mathtt{call}^{-}.\mathtt{O}}{\mathtt{MC} \Rightarrow \geq 1\ \mathtt{call}^{-}.\mathtt{O} \sqcap \leq 1\ \mathtt{call}^{-}.\mathtt{O}}}{\mathtt{MC}, \geq 2\ \mathtt{call}^{-}.\mathtt{MO} \Rightarrow \geq 1\ \mathtt{call}^{-}.\mathtt{O} \sqcap \leq 1\mathtt{call}^{-}.\mathtt{O}}}{\mathtt{MC}, \geq 2\ \mathtt{call}^{-}.\mathtt{MO} \Rightarrow \geq 1\ \mathtt{call}^{-}.\mathtt{O} \sqcap \leq 1\mathtt{call}^{-}.\mathtt{O} \sqcap \geq 2\mathtt{call}^{-}.\mathtt{O}}}{\mathtt{MC}, \geq 2\ \mathtt{call}^{-}.\mathtt{MO} \Rightarrow \bot}}{\mathtt{MC} \Rightarrow \neg \geq 2\ \mathtt{call}^{-}.\mathtt{MO}}$$

In order to the reader concretely see that it is harder explaining on Tableaux basis than on Natural Deduction basis, we prove the same $\mathtt{MC} \sqsubseteq \neg \geq 2\ \mathtt{call}^{-}.\mathtt{MO}$ subsumption in Tableaux. We follow [1, Section 2.3.2.1] and represent the Tableaux constraints as **ABox** assertions without unique name assumption. [2] The constraint"a belongs to (the interpretation of) C" is represented by $C(a)$ and "b is an R-filler of a" by $R(a,b)$. A complete presentation of the Tableaux procedure for $\mathcal{ALCQI}$ can be found at [1].

The Tableaux procedure starts translating the subsumption problem to a satisfiability problem. The subsumption $C \sqsubseteq D$ holds iff $C \sqcap \neg D$ is unsatisfiable. In our case, $C_0 \equiv \mathtt{MC} \sqcap \geq 2\ \mathtt{call}^{-}.\mathtt{MO}$ should be unsatisfiable.

[2]Instead, we allow explicit inequality assertions of the form $x \neq y$. Those assertions are assumed symmetric.

Since C_0 is already in the NNF (negation normal form), we are ready to the Tableaux algorithm, otherwise we would have to first transform it to obtain a NNF equivalent concept description. Tableaux procedure starts with the ABox $A_0 = \{C_0(x_0)\}$ and applies consistency-preserving transformation rules to the ABox until no more rules apply. If the completed expanded ABox obtained does not contain clashes (contradictory assertions), then A_0 is consistent and thus C_0 is satisfiable, and incosistent (unsatisfiable) otherwise.

$\mathcal{A}_0$ is the initial ABox. By $\sqcap$-rule, we get $\mathcal{A}_1$. Than, by $\geq$-rule we get $\mathcal{A}_2$. $\mathcal{A}_3$ is obtained by using the theory axioms $\texttt{MO} \sqsubseteq \texttt{O}$ and $\texttt{MC} \sqsubseteq \texttt{PC}$. The ABox $\mathcal{A}_4$ is obtained by using the theory axiom $\texttt{PC} \sqsubseteq \geq 1\ \texttt{call}^-.\texttt{O} \sqcap \leq 1\ \texttt{call}^-.\texttt{O}$. Next, $\mathcal{A}_5$ by $\sqcap$-rule. ABox $\mathcal{A}_5$ now contains a contradiction, the individual a is required to have at most one successor of type $\texttt{O}$ in the role $\texttt{call}^-$. Nevertheless, b and c are also required to be of type $\texttt{O}$ and successors of a in role $\texttt{call}^-$, vide $\mathcal{A}_3$ and $\mathcal{A}_2$. This shows that C_0 is unsatisfiable, and thus $\texttt{MC} \sqsubseteq \neg \geq 2\ \texttt{call}^-.\texttt{MO}$.

$$\{(\texttt{MC} \sqcap \geq 2\ \texttt{call}^-.\texttt{MO})(a)\} \qquad (\mathcal{A}_0)$$

$$\mathcal{A}_0 \cup \{\texttt{MC}(a), (\geq 2\ \texttt{call}^-.\texttt{MO})(a)\} \qquad (\mathcal{A}_1)$$

$$\mathcal{A}_1 \cup \{\texttt{call}^-(a,b), \texttt{call}^-(a,c), \texttt{MO}(b), \texttt{MO}(c), a \neq b, b \neq c, a \neq c\} \qquad (\mathcal{A}_2)$$

$$\mathcal{A}_2 \cup \{\texttt{O}(b), \texttt{O}(c), \texttt{PC}(a)\} \qquad (\mathcal{A}_3)$$

$$\mathcal{A}_3 \cup \{(\geq 1\ \texttt{call}^-.\texttt{O} \sqcap \leq 1\ \texttt{call}^-.\texttt{O})(a)\} \qquad (\mathcal{A}_4)$$

$$\mathcal{A}_4 \cup \{(\geq 1\ \texttt{call}^-.\texttt{O})(a), (\leq 1\ \texttt{call}^-.\texttt{O})(a)\} \qquad (\mathcal{A}_5)$$