2 Preliminaries

In this first chapter, we will give some notions and results of Riemannian manifolds.

2.1 Riemannian manifolds

In this section we introduce the notion of a Riemannian connection on a Riemannian manifold. We will follow the presentation and notation of Manfredo do Carmo, see (2).

Let (M^n, g) be a n-dimensional Riemannian manifold with Riemannian metric g. We denote it by M for simplicity.

We will denote by $\mathfrak{X}(M)$ the set of C^∞ vector fields defined along of M.

Definition 2.1.1. *Let (M, g) be a Riemannian manifold then the map:*

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

given by:

$$\begin{aligned} 2g(\nabla_X Y, Z) &= X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + \\ &\quad g(Z,[X,Y]) + g(Y,[Z,X]) - g(X,[Y,Z]) \end{aligned}$$

is called the Levi-Civita connection on M.

Thus, we have the next theorem,

Theorem 2.1.1. *(2, Theorem 3.6,pag. 61) The Levi-Civita connection satisfies the next properties:*

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$
2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$
3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y$
4. $[X, Y] = \nabla_X Y - \nabla_Y X$

5. $Xg(Y,Z) = g(\nabla_X Y, Z) + g(X, \nabla_X Z)$

where $X, Y, Z \in \mathfrak{X}(M)$ *and* $f, g \in \mathfrak{D}(M)$*, here* $\mathfrak{D}(M)$ *is the set of* C^∞ *real functions defined on* M*.*

Let us define the notion of curvature on a Riemannian manifold M.

Definition 2.1.2. *The curvature* R *of a Riemannian manifold* M *is a map:*

$$R(X,Y) : \mathfrak{X}(M) \longrightarrow \mathfrak{X}(M)$$

given by:

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

where $X, Y, Z \in \mathfrak{X}(M)$*.*

Now, we fix a point $p \in M$, and consider the 2-dimensional subspace $\sigma \subset T_pM$, where T_pM denotes the tangent space at p of M.

Taking $X, Y \in \sigma \subset T_pM$, two linearly independent vector on σ, we have the next definition:

Definition 2.1.3. *Following the above notation we define the sectional curvature* $K_p(X,Y) = K_p(\sigma)$ *as being the real number:;*

$$K_p(X,Y) = \frac{g(R(X,Y)X,Y)}{|X \times Y|^2}$$

where $|X \times Y|$ *indicate the area of the parallelogram formed by* X *and* Y*.*

Remark 2.1.1. *It is possible to show that* $K_p(X,Y)$ *does not depend of the choice of* X *and* Y*, see (2, pag. 94).*

Definition 2.1.4. *Let* $p \in M$ *and fix* $X \in T_pM$ *an unit tangent vector at* p*, taking an orthonormal basis* $\{X_1, ..., X_{n-1}\}$ *of the hyperplane orthogonal to* X*, we define the Ricci curvature in* X *as being the real number* $Ric_p(X)$*, where:*

$$Ric_p(X) = \sum_{i=1}^{n-1} g(R(X,X_i)X,X_i).$$

Remark 2.1.2. *(2, pag. 97) It is possible to show that* $Ric_p(X)$ *does not depend of the choice of the orthonormal basis* $\{X_1, ..., X_{n-1}\}$*.*

We conclude this section given the definition of a Killing field on a Riemannian Manifold.

Definition 2.1.5. *Let* M *be a Riemannian manifold and* $X \in \mathfrak{X}(M)$*. Let* $p \in M$ *and* $U \subset M$ *a neighborhood of* p *in* M *and* $\varphi : (-\epsilon, \epsilon) \times U \to M$ *a differentiable map such that, for all* $p \in U$ *the curve* $t \to \varphi(t, q)$ *is the trajectory of* X *passing through* q *at* $t = 0$*.* X *is called a Killing field if for all* $t_0 \in (-\epsilon, \epsilon)$*, the map* $\varphi(t_0) : U \subset M \to M$ *is a isometry of* M*.*

Remark 2.1.3. *(2, pag. 82) Let* X *be a vector field along of* M*. Then* X *is a Killing field if and only if,*

$$g(\nabla_Y X, Z) + g(\nabla_Z X, Y) = 0$$

for all $Y, Z \in \mathfrak{X}(M)$*.*

2.2 Fibration

In this section we discuss briefly the notion of fibration.

Definition 2.2.1. *A fibration is given by, a submersion,*

$$\pi : E \longrightarrow B$$

where E *and* B *are differentiable manifolds. A family of neighborhoods*

$$\beta = \bigcup_{j \in J} U_j$$

covering B*, such that, in each pre-image* $\pi^{-1}(U_j)$ *we have defined a diffeomorphis,*

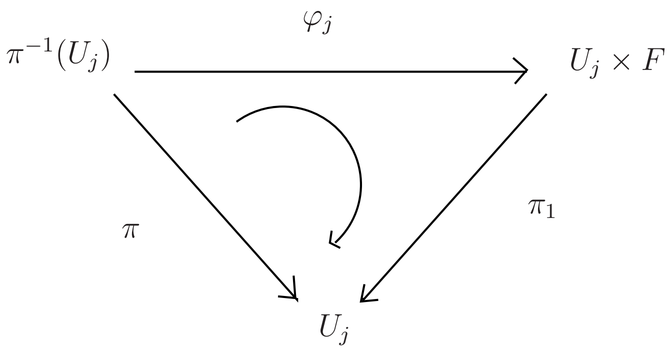
$$\varphi_j : \pi^{-1}(U_j) \longrightarrow U_j \times F$$

where F *is a fixed differentiable manifold.*

Finally, there is the compatibility condition. If

$$\pi_1 : B \times F \longrightarrow B$$

defines the projection on the first coordinate, $\pi_1(b, f) = b$*. Then, the following diagram,*



is commutative.

The manifold E is called total space, the manifold B is called the basis, F is called fiber, π is called the projection, and

$$\varphi_j : \pi^{-1}(U_j) \longrightarrow U_j \times F$$

a trivialization.

Observe that, if M^n is a differentiable manifold of dimension n, then the tangent bundle TM is a fibration, where $E = TM$, $B = M^n$ and $F = \mathbb{R}^n$.

2.3 Riemannian submersion

In this section we will follow the presentation and notation of Jeff Cheeger, see (7), and Peter Petersen, see (13). We begin this section with the definition of Riemannian submersion.

Definition 2.3.1. *A submersion is a differentiable map,*

$$\pi : M^{n+k} \longrightarrow N^n$$

such that at each point the derivative of π (which we denote by $d\pi$), has rank n.

It follows from the implicit-function theorem that $\pi^{-1}(p)$ is a smooth k-dimensional submanifold of M for all $p \in N$.

Definition 2.3.2. *We shall use the notation $\overline{p}$ and p as well as $\overline{X}$ and X for points and vector fields that are π-related, i.e., such that,*

$$\pi(\overline{p}) = p$$

and

$$d\pi(\overline{X}) = X.$$

Fixing $\overline{p} \in d\pi^{-1}(p)$. Let V denotes the tangent space to the fiber $\pi^{-1}(p)$ at $\overline{p}$. Assume that M and N have Riemannian metrics, and set $H = V^{\perp}$.

Definition 2.3.3. *We call H and V the horizontal and vertical subspaces, respectively, and we use H and V as superscripts to denote horizontal and vertical component.*

Definition 2.3.4. *The map π is called a Riemannian submersion if $d\pi|_H$ is an isometry.*

Remark 2.3.1. *If X is a vector field on N, then there is a unique vector field $\overline{X}$ on M such that $\overline{X} \in H$ and $d\pi(\overline{X}) = X$. we call $\overline{X}$ the horizontal lift of X.*

Also if $c : [0,1] \to N$ is a piecewise smooth curve, and $\overline{p} \in \pi^{-1}(c(0))$, then there is a unique curve $\overline{c} : [0,1] \to M$ such that $\overline{c}(0) = \overline{p}$, $\pi \circ \overline{c} = c$, $\overline{c}'(t) \in H$, see (7).

Proposition 2.3.1. *(7, pag. 66) Let T be a vertical vector field on M and X, Y, Z vector fields on N with horizontal lifts $\overline{X}, \overline{Y}, \overline{Z}$.*

1. $[T, \overline{X}]$ *is vertical,*
2. $\langle [\overline{X}, \overline{Y}], \overline{Z} \rangle = \langle [X, Y], Z \rangle$
3. $\langle [\overline{X}, \overline{Y}], T \rangle = 2\langle \overline{\nabla}_{\overline{X}} \overline{Y}, T \rangle$
4. $\overline{\nabla}_{\overline{X}} \overline{Y} = \overline{\nabla_X Y} + \frac{1}{2}[\overline{X}, \overline{Y}]^v$, *where* $[\overline{X}, \overline{Y}]^v$ *is the vertical component of* $[\overline{X}, \overline{Y}]$.
5. $[\overline{X}, \overline{Y}]$ *is π-related to* $[X, Y]$.
6. $K(X, Y) = \overline{K}(\overline{X}, \overline{Y}) + \frac{3}{4}\|[\overline{X}, \overline{Y}]^v\|^2$, *where $\overline{K}$ and K denotes the sectional curvatures of M and N respectively.*

Proposition 2.3.2. *(7, Proposition 3.31) If $\pi : M \to N$ is a Riemannian submersion,*

$$\gamma : [0,1] \longrightarrow N$$

and,

$$\overline{\gamma} : [0,1] \longrightarrow M$$

a horizontal lift, then γ is a geodesic if and only if $\overline{\gamma}$ is.

We conclude this section by given the definition of section of a submersion.

Definition 2.3.5. *(21, pag.161) A section of the submersion,*

$$\pi : M^{n+k} \longrightarrow N^n$$

is a left inverse to the projection π*, that is, a continuous map,*

$$s : N^n \longrightarrow M^{n+k}$$

such that $\pi \circ s = Id$*, where* Id *denotes the identity map on* N^n*.*

2.4 The Cheeger's constant

Now, we consider an important quantity the Cheeger's constant.

Definition 2.4.1. *Let* M^n *be a Riemannian manifold of dimension* n*, and* $\Omega \subset M^n$ *be an open domain in* M^n*, such that* $\overline{\Omega}$ *is compact and* $\partial\Omega$ *is of class* C^∞*. The Cheeger constant which is denoted by* $C(M^n)$ *is given by*

$$C(M^n) = \inf_{\Omega} \left\{ \frac{A(\partial\Omega)}{V(\Omega)}; \Omega \subset M^n, \overline{\Omega} \text{ compact} \right\}$$

where A *is the area function on* $\partial\Omega$ *and* V *the volume function on* Ω*.*

It will be useful to find the Cheeger's constant for the 2-dimensional space form, that is, for $\mathbb{R}^2$, $\mathbb{S}^2$ and $\mathbb{H}^2$. We start with the most simple, the Euclidean space, we follow with the Euclidean sphere $\mathbb{S}^2$ and we ended with the hyperbolic space $\mathbb{H}^2$.

Remark 2.4.1. *It is sufficient to consider* $\Omega \subset M^2$ *as a geodesic ball in the space form.*

- Let $D(r) \subset \mathbb{R}^2$ denote the geodesic disk in $\mathbb{R}^2$, with Euclidean radius r centered at the origin of $\mathbb{R}^2$. We denote by $l(\partial D(r))$ the length of $\partial D(r)$ (the boundary of $D(r)$), and by $Area(D(r))$ the area of $D(r)$. Since $l(D(r)) = 2\pi r$ and $Area(D(r)) = \pi r^2$, then,

$$f(r) = \frac{l(\partial D(r))}{Area(D(r))} = \frac{2}{r}.$$

Observe that, the function $f(r)$ is decreasing for $0 < r < +\infty$, so the infimum is attached when r goes to $+\infty$. In this case,

$$C(\mathbb{R}^2) = \lim_{r \to +\infty} f(r) = 0.$$

- In the Euclidean sphere $\mathbb{S}^3$ centered at the origin of $\mathbb{R}^3$ and of radius one. We consider the next argument, consider a curve in $\mathbb{R}^3$ (the 3-dimensional Euclidean space), given by,

$$\alpha(t) = (0, a(t), b(t)), \qquad a_0 < t < b_0,$$

a_0 and b_0 real numbers, and consider the surface of revolution, obtained by apply one-parameter group of rotations on α. So the surface obtained, which we denote by S is parametrized by,

$$\varphi(t,\theta) = (\cos(\theta)a(t), \sin(\theta)a(t), b(t)), \qquad 0 < \theta < 2\pi.$$

The first fundamental form of S is given by,

$$\begin{aligned} g_{11} &= \langle \varphi_t, \varphi_t \rangle = (a'(t))^2 + (b'(t))^2 \\ g_{12} &= \langle \varphi_t, \varphi_\theta \rangle = 0 \\ g_{22} &= \langle \varphi_\theta, \varphi_\theta \rangle = a(t). \end{aligned}$$

By considering α parametrized by arc length, we obtain that the first fundamental form take the next form,

$$ds^2 = dt^2 + a^2(t)d\theta^2.$$

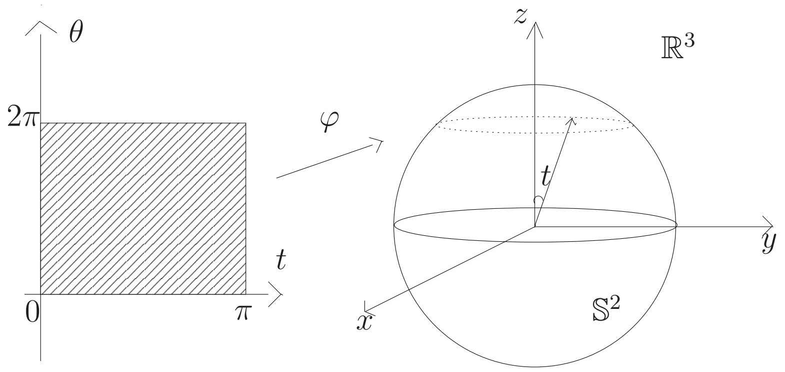
Considering $a(t) = \sin(t)$, $b(t) = \sin(t)$, we obtain a natural parametrization of the unit sphere. Thus, the sphere $\mathbb{S}^2$ parametrized by,

$$\varphi(t,\theta) = (\cos(\theta)\sin(t), \sin(\theta)\sin(t), \cos(t))$$

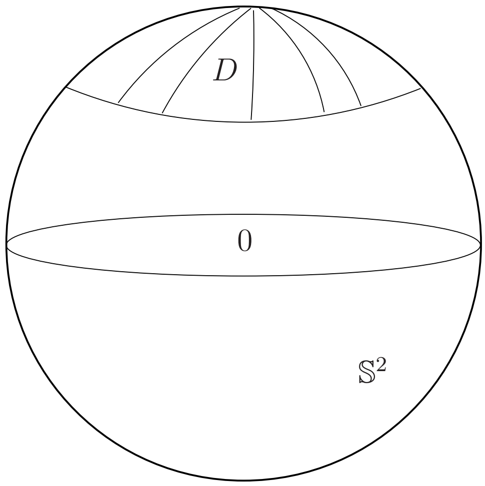
endowed with induced metric:

$$ds^2 = dt^2 + \sin^2(t)d\theta^2.$$

Now, we consider the geodesic ball D centered at the noth pole and of radius $r_0 > 0$. That is,



$$D(r_0) = \{(t,\theta); 0 \leq t \leq r_0, \theta \in (0,2\pi)\}.$$



We obtain: $l(\partial(D(r_0))) = 2\pi\sin(r_0)$ and $Area(D(r_0)) = 2\pi(1-\cos(\theta))$. Hence,

$$f(r_0) = \frac{l(\partial(D(r_0)))}{Area(D(r_0))} = \frac{\sin(r_0)}{1-\cos(r_0)}$$

since this function is decreasing for $0 < r_0 < \pi$, then the infimum is attained when r_0 goes to π. So, we obtain:

$$C(\mathbb{S}^2) = \lim_{r_0\to\pi} f(r_0) = 0.$$

- Finally we compute the Cheeger's constant for the hyperbolic space. For this purpose, we can consider the next model for the 2-dimensional hyperbolic space,

$$\mathbb{H}^2 = \{(\rho,\theta); 0 < \rho < +\infty, 0 < \theta < 2\pi\}$$

endowed with the metric:

$$ds^2 = d\rho^2 + \sinh^2(\rho)d\theta^2.$$

following the ideas as for the Euclidean space and the Euclidean sphere,we consider a geodesic ball $D(\rho) \subset \mathbb{H}^2$ with hyperbolic radius ρ, immersed in $\mathbb{H}^2$, so we obtain: $l(\partial(D(\rho))) = 2\pi\sinh(\rho)$ and $Area(D(\rho)) = 2\pi(\cosh(\rho) - 1)$. In this case:

$$f(\rho) = \frac{l(\partial(D(\rho)))}{Area(D(\rho))} = \frac{\sinh(\rho)}{\cosh(\rho) - 1}$$

since this function is decreasing for $0 < \rho < +\infty$, then the infimum is attached when ρ goes to $+\infty$. So, we obtain:

$$C(\mathbb{H}^2) = \lim_{\rho\to+\infty} f(\rho) = 1.$$

2.5
An introduction to the maximum principle

In this section we discuss briefly about an important criterium in differential geometry, the maximum principle. Our main reference is (20, Chap. 10, Add. 2).

Let $U \subset \mathbb{R}^n$ be an open set, consider the second order differential operator L defined by,

$$Lu := \sum_{i=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=n}^{n} b_i \frac{\partial u}{\partial x_i} + cu, \tag{2-1}$$

for certain functions a_{ij}, b_i, c on U, we are considering $C^0(\partial U) \cap C^\infty(U)$, where ∂U denotes the boundary of U. We assume that $a_{ij} = a_{ji}$, and the matric $A = (a_{ij})$ is positive definite, in this case, we say that L is an elliptic operator. Thus,

$$\sum_{i,j} a_{ij}\xi_i\xi_j > 0$$

for $0 \neq \xi \in \mathbb{R}^n$, or equivalently, the 1×1 matrix,

$$\xi \cdot A \cdot \xi^t > 0$$

for $0 \neq \xi \in \mathbb{R}^n$.

Suppose that B is also a positive definite, so that,

$$\xi \cdot B \cdot \xi^t > 0$$

for $0 \neq \xi \in \mathbb{R}^n$. For any non-singular matrix P we have,

$$\xi \cdot PBP^t \cdot \xi^t = (\xi P)B(\xi P)^t > 0$$

for $0 \neq \xi \in \mathbb{R}^n$, hence $PBP^t = C = (c_{ij})$ is also positive definite. Now the symmetric matrix A can be diagonalized , that is, there is an orthogonal matrix P such that,

$$PAP^{-1} = PAP^t = \begin{pmatrix} \lambda_1 & \dots & 0 \\ 0 & \lambda_i & 0 \\ 0 & \dots & \lambda_n \end{pmatrix} \qquad \lambda_i > 0.$$

Then: traceAB= trace$PABP^t$= trace$(PAP^t)(PBP^t)$= trace$(\lambda_i c_{ij}) > 0$, where $C = (PBP^t)$.

Similarly, we have trace$AB \geq 0$ if B is positive semi-definite, and trace$AB \leq 0$ if B is negative semi-definite.

Now consider the operator from equation (2-1), where we assume that,

(i) $c \leq 0$ in U.

Suppose that $u : U \to \mathbb{R}$ is a twice differentiable function with a relative maximum at $p \in U$. Assume that,

(ii) $u(p) \geq 0$.

From (i) and (ii), we have:

(iii) $\displaystyle\sum_{i,j=1}^{n} a_{ij}(p)\frac{\partial^2 u}{\partial x_i \partial x_j}(p) = (Lu)(p) - c(p)u(p) \geq L(u)(p).$

On the other hand, since u has a maximum at p, the matrix

$$B = \left(\frac{\partial^2}{\partial x_i \partial x_j} u(p)\right)$$

is negative semi-definite. Hence we have:

(iv) $\displaystyle 0 \geq \text{trace} A(p) \cdot B = \sum_{i,j=1}^{n} a_{ij}(p)\frac{\partial^2 u}{\partial x_i \partial x_j}(p).$

Since (iii) and (iv) imply that $Lu(p) \geq 0$, we obtain the next theorem.

Theorem 2.5.1. *If L is an elliptic operator on U and $c \leq 0$ on U, and the twice differentiable function u satisfies $Lu > 0$ on U, then u cannot have a non-negative relative maximum on U.*

Actually, we have the following theorem, which is due to E. Hopf.

Theorem 2.5.2. *(E. Hopf)(20, Theorem 17, pag. 183) Consider a second order differential operator*

$$Lu := \sum_{i=1}^{n} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=n}^{n} b_i \frac{\partial u}{\partial x_i} + cu, \qquad c \leq 0$$

on a connected open set $U \subset \mathbb{R}^n$. *Assume that the function* b_i *and* c *are locally bounded, and that in a neighborhood of any point of* U *there are* $a, b > 0$ *such that the matrix* (a_{ij}) *satisfies:*

$$a\sum_{i,j=1}^{n} \xi_i^2 \leq \sum_{i,j=1}^{n} a_{ij}\xi_i\xi_j \leq b\sum_{i,j=1}^{n} \xi_i^2, \qquad \xi \in \mathbb{R}^n.$$

Suppose that u *is twice differentiable function on* U *satisfying:*

$$Lu \geq 0.$$

Then u *cannot have a non-negative relative maximum on* U, *unless* u *is a constant.*

A corollary of this theorem is the following:

Corollary 2.5.1. *(E. Hopf)(20, Corollary 19, pag. 187) Consider the operator* L *with arbitrary* c. *If* u *is twice differentiable function on* U *with* $Lu \geq 0$ *and* $u \leq 0$, *then* u *cannot have the value 0 anywhere on* U *unless* u *is identically 0.*