4
The space $\mathbf{E^3}(\kappa, \tau)$

The classification of complete simply connected homogeneous manifolds of dimension 3 is well known. Such a manifold has an isometry group of dimension 3, 4 or 6. When the dimension of the isometry group is 6, we have a space form. When the dimension of the isometry group is 3, the manifold has the geometry of the Lie group $Sol3$.

In this paper we will consider the complete homogeneous manifolds $E^3(\kappa, \tau)$ whose isometry groups have dimension 4. Such a manifold is a Riemannian fibration over a 2-dimensional space form $M^2(\kappa)$ having Gauss curvature κ. That is, there is a Riemannian submersion

$$\pi : E^3(\kappa, \tau) \longrightarrow M^2(\kappa)$$

which also is a Killing submersion (see Definition 4.1.1). If $E^3(\kappa, \tau)$ is not compact, then $E^3(\kappa, \tau)$ is topologically $M^2(\kappa) \times \mathbb{R}$, each fiber is diffeomorphic to $\mathbb{R}$ (the real line), the bundle curvature of the submersion is τ. If $E^3(\kappa, \tau)$ is compact, with $\kappa > 0$ and $\tau \neq 0$, then $E^3(\kappa, \tau)$ are the Berger spheres, here each fiber is diffeomorphic to S^1 (the unit circle).

The tangent unit vector field to the fiber is a Killing vector field which we will denote by E_3. This vector field will be called the vertical vector field. These manifolds are classified, up to isometry, by the curvature κ of the base surface of the submersion and the bundle curvature τ, where κ and τ can be any real numbers satisfying $\kappa \neq 4\tau^2$. Namely, these manifolds are [see (21), (22)]:

– $E^3(\kappa, \tau) = \mathbb{H}^2(\kappa) \times \mathbb{R}$, if $\kappa < 0$ and $\tau = 0$
– $E^3(\kappa, \tau) = \mathbb{S}^2(\kappa) \times \mathbb{R}$, if $\kappa > 0$ and $\tau = 0$
– $E^3(\kappa, \tau) = Nil_3$ (Heisenberg space) if $\kappa = 0$ and $\tau \neq 0$
– $E^3(\kappa, \tau) = \widetilde{PSL}_2(\mathbb{R}, \tau)$, if $\kappa < 0$ and $\tau \neq 0$
– $E^3(\kappa, \tau) = \mathbb{S}^3_\tau$ (Spheres of Berger), if $\kappa > 0$ and $\tau \neq 0$

In this chapter we focus our attention in the study of $H\ multi-graphs$ in $E^3(\kappa, \tau)$, that is, multi-graphs of constant mean curvature H. We also focus our attention on the problem of existence of entire graphs of constant mean curvature in $E^3(\kappa,\tau)$ [see Definition 4.1.3].

4.1
Knowing the space $\mathbf{E^3}(\kappa,\tau)$

We begin this section by given the definition of Killing submersion which is due to *H. Rosenberg*, *R. Souam*, and *E. Toubiana* see (4).

Consider a Riemannian 3-manifolds (M^3,g) which fiber over a Riemannian surface (M^2,h), where g and h denote the Riemannian metrics respectively

Definition 4.1.1. *A Riemannian submersion* $\pi:(M^3,g)\longrightarrow(M^2,h)$ *such that:*

1. *each fiber is a complete geodesic,*
2. *the fibers of the fibration are the integral curves of an unit Killing vector field* ξ *on* M^3*.*

will be called a Killing submersion.

Definition 4.1.2. *Let* $\pi:(M^3,g)\longrightarrow(M^2,h)$ *be a Killing submersion.*

1. *Let* $\Omega\subset M^2$ *be a domain. An* H*-section over* Ω *is an* H*-surface which is the image of a section.*
2. *Let* $\gamma\subset M^2$ *be a smooth curve with geodesic curvature 2H. Observe that the surface* $\pi^{-1}(\gamma)\subset M^3$ *has mean curvature H [see Lemma 4.2.2]. We call such surface a vertical* H*-cylinder.*

Observe that, $\pi:E(\kappa,\tau)\longrightarrow M^2(\kappa)$ given by $\pi(x,y,t)=(x,y)$ is a Killing submersion.

4.1.1
Riemannian connection of $\mathbf{E^3}(\kappa,\tau)$

We following ideas of Eric Toubiana to see the relationship between the Riemannian metric of $E^3(\kappa,\tau)$ with the Riemannian metric of $M^2(\kappa)$, for more details see (22).

When the space $E^3(\kappa,\tau)$ is not compact, it is given by (topologically),

$$E^3(\kappa,\tau)=\{(x,y,t);(x,y)\in M^2(\kappa),t\in\mathbb{R}\},$$

Since that the projection $\pi:E^3(\kappa,\tau)\longrightarrow M^2(\kappa)$ given by $\pi(x,y,z)=(x,y)$ is the Riemannian submersion, and the translations along the fibers are isometries, the unit tangent field to the fibers is a Killing field [see Definition 2.1.5], which we denote by E_3. Let $\{E_1, E_2\}$ be the horizontal lift of $\{e_1, e_2\}$ (where, $\{e_1, e_2\}$ is an orthonormal frame of $M^2(\kappa)$ [see Chapter 4]), that is $d\pi(E_i) = e_i$, $i = 1, 2$; so $\{E_1, E_2, E_3\}$ is a orthonormal frame on $E^3(\kappa, \tau)$.

Let $X \in \mathfrak{X}(M^2(\kappa))$ be a vector field on $M^2(\kappa)$ and $\overline{X}$ its horizontal lift to $E^3(\kappa, \tau)$.

Denote by $\overline{\nabla}$ the Riemannian connection of $E^3(\kappa, \tau)$, and let $\overline{X}, \overline{Y}$ be horizontal fields, then

$$\begin{aligned}\langle[\overline{X}, \overline{Y}], E_3\rangle &= \langle\overline{\nabla}_{\overline{X}}\overline{Y}, E_3\rangle - \langle\overline{\nabla}_{\overline{Y}}\overline{X}, E_3\rangle \\ &= \overline{X}\langle\overline{Y}, E_3\rangle - \langle\overline{Y}, \overline{\nabla}_{\overline{X}}E_3\rangle - \overline{Y}\langle\overline{X}, E_3\rangle + \langle\overline{X}, \overline{\nabla}_{\overline{Y}}E_3\rangle \\ &= \langle\overline{X}, \overline{\nabla}_{\overline{Y}}E_3\rangle - \langle\overline{Y}, \overline{\nabla}_{\overline{X}}E_3\rangle,\end{aligned}$$

since $\langle\overline{X}, E_3\rangle = 0 = \langle\overline{Y E_3}\rangle$, that is:

$$\langle[\overline{X}, \overline{Y}], E_3\rangle = \langle\overline{X}, \overline{\nabla}_{\overline{Y}}E_3\rangle - \langle\overline{Y}, \overline{\nabla}_{\overline{X}}E_3\rangle \tag{4-1}$$

thus the vertical component of $[\overline{X}, \overline{Y}]$ at $p \in E^3(\kappa, \tau)$ (which we will denote by $[\overline{X}, \overline{Y}]^v$) depend only of the values $\overline{X}(p)$ and $\overline{Y}(p)$. Furthermore, there is a positive isometry f of $E^3(\kappa, \tau)$ such that, [see (22, Proposition 3)]

$$f(p) = q, \quad D_p f(E_1) = E_1(q)$$

and such that it leave invariant the field E_3. Hence, if $\langle[E_1, E_2], E_3\rangle(p) = 2\tau$, then

$$\langle[E_1, E_2], E_3\rangle(q) = \langle[df(E_1), df(E_2)], df(E_3)\rangle(p) = \langle[E_1, E_2], E_3\rangle(p) = 2\tau,$$

that is, $[E_1, E_2]^v = 2\tau E_3$.

Taking $X, Y \in \mathfrak{X}(M^2(\kappa))$, since that

$$\pi : E^3(\kappa, \tau) \longrightarrow M^2(\kappa)$$

is a Riemannian submersion, we have [see Proposition 2.3.1],

$$\langle\overline{\nabla}_{\overline{X}}\overline{Y}, \overline{Z}\rangle = \langle\nabla_X Y, Z\rangle. \tag{4-2}$$

where $\overline{\nabla}$ is the Riemannian connection of $E^3(\kappa, \tau)$ and ∇ is the Riemannian connection of $M^2(\kappa)$.

Furthermore, as E_3 is a Killing Field of $E^3(\kappa, \tau)$, then, for all $A, B \in \mathfrak{X}(E^3(\kappa, \tau))$ [see Definition 2.1.5],

$$\langle A, \overline{\nabla}_B E_3\rangle + \langle B, \overline{\nabla}_A E_3\rangle = 0. \tag{4-3}$$

By using equation (4-2) and the definition of the frame $\{e_1, e_2\}$ [see Chapter 4], we obtain:

$$\langle \overline{\nabla}_{E_1} E_1, E_2 \rangle = \langle \nabla_{e_1} e_1, e_2 \rangle = -\frac{\lambda_y}{\lambda^2} \tag{4-4}$$

Now, considering equation (4-3), we obtain:

$$\langle \overline{\nabla}_{E_1} E_1, E_3 \rangle = E_1 \langle E_1, E_3 \rangle - \langle E_1, \overline{\nabla}_{E_1} E_3 \rangle = -\langle E_1, \overline{\nabla}_{E_1} E_3 \rangle = 0$$

this implies,

$$\langle \overline{\nabla}_{E_1} E_1, E_3 \rangle = 0 \tag{4-5}$$

thus, considering equation (4-4) and equation (4-5), we obtain:

$$\overline{\nabla}_{E_1} E_1 = -\frac{\lambda_y}{\lambda^2} E_2. \tag{4-6}$$

From equation (4-2), we have:

$$\langle \overline{\nabla}_{E_2} E_1, E_2 \rangle = \langle \nabla_{e_2} e_1, e_2 \rangle = \frac{\lambda_x}{\lambda^2} \tag{4-7}$$

and by using equation (4-3), we obtain:

$$\begin{aligned} \langle \overline{\nabla}_{E_2} E_1, E_3 \rangle &= -\langle E_1, \overline{\nabla}_{E_2} E_3 \rangle \\ &= \langle E_2, \overline{\nabla}_{E_1} E_3 \rangle \\ &= -\langle \overline{\nabla}_{E_1} E_2, E_3 \rangle \end{aligned}$$

this implies:

$$\begin{aligned} \langle \overline{\nabla}_{E_2} E_1, E_3 \rangle &= -\langle [E_1, E_2] + \overline{\nabla}_{E_2} E_1, E_3 \rangle \\ &= -\langle [E_1, E_2], E_3 \rangle - \langle \overline{\nabla}_{E_2} E_1, E_3 \rangle \end{aligned}$$

that is:

$$\langle \overline{\nabla}_{E_2} E_1, E_3 \rangle = -\frac{1}{2} \langle [E_1, E_2], E_3 \rangle = -\tau. \tag{4-8}$$

From equation (4-7) and equation (4-8), we obtain:

$$\overline{\nabla}_{E_2} E_1 = \frac{\lambda_x}{\lambda^2} E_2 - \tau E_3. \tag{4-9}$$

Observe that, for all $X \in \mathfrak{X}(M^2(\kappa))$, the field $[\overline{X}, E_3]$ is vertical, that is

$\langle [E_1, E_3], E_2\rangle = 0$, hence, we obtain:

$$\begin{aligned}
\langle \overline{\nabla}_{E_3} E_1, E_2\rangle &= \langle \overline{\nabla}_{E_1} E_3 + [E_1, E_3], E_2\rangle \\
&= \langle \overline{\nabla}_{E_1} E_3, E_2\rangle \\
&= E_1\langle E_3, E_2\rangle - \langle \overline{\nabla}_{E_1} E_2, E_3\rangle \\
&= -\langle \overline{\nabla}_{E_1} E_2, E_3\rangle \\
&= -[\langle \overline{\nabla}_{E_2} E_1 + [E_1, E_2], E_3\rangle] \\
&= -[-\tau + 2\tau] \\
&= -\tau.
\end{aligned}$$

that is:

$$\overline{\nabla}_{E_3} E_1 = -\tau E_2. \tag{4-10}$$

Since the fiber are geodesic we have:

$$\overline{\nabla}_{E_3} E_3 = 0. \tag{4-11}$$

The others connections are compute analogously. Taking into account (4-6), (4-9), (4-10) and (4-11) we obtain:

$$\begin{array}{ccc}
\overline{\nabla}_{E_1} E_1 = -\dfrac{\lambda_y}{\lambda^2} E_2 & \overline{\nabla}_{E_1} E_2 = \dfrac{\lambda_y}{\lambda^2} E_1 + \tau E_3 & \overline{\nabla}_{E_1} E_3 = -\tau E_2 \\
\overline{\nabla}_{E_2} E_1 = \dfrac{\lambda_x}{\lambda^2} E_2 - \tau E_3 & \overline{\nabla}_{E_2} E_2 = -\dfrac{\lambda_x}{\lambda^2} E_1 & \overline{\nabla}_{E_2} E_3 = \tau E_1 \\
\overline{\nabla}_{E_3} E_1 = -\tau E_2 & \overline{\nabla}_{E_3} E_2 = \tau E_1 & \overline{\nabla}_{E_3} E_3 = 0 \\
[E_1, E_2] = \dfrac{\lambda_y}{\lambda^2} E_1 - \dfrac{\lambda_x}{\lambda^2} E_2 + 2\tau E_3 & [E_1, E_3] = 0 & [E_2, E_3] = 0
\end{array}$$

The natural frame field for the space form $M^2(\kappa)$ is given by $\{\partial_x, \partial_y\}$ [see Chapter 4], since $E^3(\kappa,\tau)$ is topologically $M^2 \times \mathbb{R}$, the natural frame field will be $\{\partial_x, \partial_y, \partial_t\}$, where ∂_t is tangent to the fibers, in this case $E_3 = \partial_t$.

Lemma 4.1.1. *(22, pag. 10) The fields E_1, E_2, E_3 in the frame $\{\partial_x, \partial_y, \partial_t\}$ are given by,*

$$\begin{array}{cc}
\kappa \neq 0 & \kappa = 0 \\
E_1 = \dfrac{1}{\lambda}\partial_x - 2\tau\dfrac{\lambda_y}{\lambda^2}\partial_t & E_1 = \partial_x - \tau y \partial_t \\
E_2 = \dfrac{1}{\lambda}\partial_y + 2\tau\dfrac{\lambda_x}{\lambda^2}\partial_t & E_2 = \partial_y + \tau x \partial_t \\
E_3 = \partial_t & E_3 = \partial_t
\end{array}$$

Where E_i for $i = 1, 2$ is the horizontal lift of e_i, and E_3 is the vertical vector field. Furthermore, the space $E^3(\kappa,\tau)$ is endowed with the metric:

$$g = \begin{cases} \lambda^2(dx^2 + dy^2) + \left(2\tau\left(\frac{\lambda_y}{\lambda}dx - \frac{\lambda_x}{\lambda}dy\right) + dt\right)^2; & \text{if, } \kappa \neq 0 \\ \lambda^2(dx^2 + dy^2) + (\tau(ydx - xdy) + dt)^2; & \text{if, } \kappa = 0 \end{cases}$$

4.1.2 Graphs in $\mathbf{E^3}(\kappa,\tau)$

Since $E^3(\kappa,\tau)$ is a Riemannian submersion over a 2-dimensional space form $M^2(\kappa)$, it is possible to consider graphs. For this, recall the definition of a section [see Definition 2.3.5].

Definition 4.1.3. *A graph in $E^3(\kappa,\tau)$ is the image of a section of the Killing submersion $\pi : E^3(\kappa,\tau) \longrightarrow M^2(k)$. When the section is defined over the entire $M^2(\kappa)$, we will say that the graph is entire.*

Remark 4.1.1. *There is no entire graph in $\mathbb{S}^3_\tau$, since the fibration is not trivial, that is, $\mathbb{S}^3_\tau$ is not the product $\mathbb{S}^2 \times \mathbb{S}^1$.*

We will prove that the remark 4.1.1 holds for the others spaces when the graph has constant mean curvature H and such that $2H > C(M^2(\kappa))$ where $C(M^2(\kappa))$ is the Chegeer's constant [see Definition 2.4.1].

Given a domain $\Omega \subset M^2(\kappa)$ and let

$$s : \Omega \subset M^2(\kappa) \longrightarrow \{(x, y, u(x, y)) \in E^3(\kappa,\tau)\}$$

be a section, where $u \in (C^0(\partial\Omega) \cap C^\infty(\Omega))$ is a function. We will identify Ω with its lift to $M^2 \times \{0\}$, then the graph $\Sigma(u)$ of $u \in (C^0(\partial\Omega) \cap C^\infty(\Omega))$ is given by,

$$\Sigma(u) = \{(x, y, u(x, y)) \in E^3(\kappa,\tau); (x, y) \in \Omega\}.$$

With the above notation, we have the following lemma:

Lemma 4.1.2. *Let $\Sigma(u)$ be the graph of the function $u : \Omega \subset M^2 \longrightarrow \mathbb{R}$, having constant mean curvature H. Then the function u satisfies the equation*

$$2H = div_{M^2(\kappa)}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right),$$

where $W = \sqrt{1 + \alpha^2 + \beta^2}$ and,

$\kappa \neq 0$	$\kappa = 0$
$\alpha = \frac{u_x}{\lambda} + 2\tau \frac{\lambda_y}{\lambda^2}$	$\alpha = u_x - \tau y$
$\beta = \frac{u_y}{\lambda} - 2\tau \frac{\lambda_x}{\lambda^2}$	$\beta = u_y + \tau x$

Proof. We consider the smooth function $u^* : E^3(\kappa, \tau) \longrightarrow \mathbb{R}$ defined by $u^*(x, y, t) = u(x, y)$. Set $F(x, y, t) = t - u^*(x, y, t)$, observe that with this choice of F, we are fixing the unit normal vector of $\Sigma(u)$ which pointing up in $E^3(\kappa, \tau)$, that is $g(N, E_3) > 0$, where N is the unit normal vector along $\Sigma(u)$. If the mean curvature vector points up, then $H > 0$, if it points down, then $H < 0$. Since 0 is a regular value of F and $\Sigma(u) = F^{-1}(0)$, we have that (2, pag. 156):

$$2H = -\overline{div}\left(\frac{\overline{\nabla} F}{|\overline{\nabla} F|}\right),$$

where $\overline{div}$ and $\overline{\nabla}$ denote the divergence and gradient in $E^3(\kappa, \tau)$.

We consider the case $\kappa \neq 0$, for $\kappa = 0$ the proof is analogous. We will calculate $\overline{\nabla} F$. Since $\{E_1, E_2, E_3\}$ is an orthonormal frame, we have

$$\begin{aligned}
\overline{\nabla} F &= aE_1 + bE_2 + cE_3, \qquad with, \\
a &= g(\overline{\nabla} F, E_1) = dF(E_1) = dF\left(\frac{1}{\lambda}\partial_x - 2\tau\frac{\lambda_y}{\lambda^2}\partial_t\right) = -\frac{u_x}{\lambda} - 2\tau\frac{\lambda_y}{\lambda^2} \\
b &= g(\overline{\nabla} F, E_2) = dF(E_2) = dF\left(\frac{1}{\lambda}\partial_y + 2\tau\frac{\lambda_x}{\lambda^2}\partial_t\right) = -\frac{u_y}{\lambda} + 2\tau\frac{\lambda_x}{\lambda^2} \\
c &= g(\overline{\nabla} F, E_3) = dF(E_3) = dF(\partial_t) = 1
\end{aligned}$$

hence, $\overline{\nabla} F = -\alpha E_1 - \beta E_2 + E_3$, where $\alpha = \frac{u_x}{\lambda} + 2\tau\frac{\lambda_y}{\lambda^2}$ and $\beta = \frac{u_y}{\lambda} - 2\tau\frac{\lambda_x}{\lambda^2}$.

We denote by $W = |\overline{\nabla} F| = \sqrt{1 + \alpha^2 + \beta^2}$ and by $\overline{X} = -\frac{\overline{\nabla} F}{|\overline{\nabla} F|}$.

Following the above notations, we obtain:

$$\overline{X} = \left(\frac{1}{W}(\alpha E_1 + \beta E_2)\right) - \left(\frac{1}{W}E_3\right) = A - B$$

where A is horizontal and B is vertical, hence:

$$2H = \overline{div}(X) = \overline{div}(A) - \overline{div}(B).$$

We will compute $\overline{div}(A)$ and $\overline{div}(B)$. To calculate $\overline{div}(A)$, we use that

$$\pi : E^3(\kappa,\tau) \longrightarrow M^2(\kappa)$$

is a Riemannian submersion, thus:

$$\begin{aligned}
\overline{div}(A) &= g(\overline{\nabla}_{E_1}A, E_1) + g(\overline{\nabla}_{E_2}A, E_2) + g(\overline{\nabla}_{E_3}A, E_3) \\
&= g(\overline{\nabla}_{E_1}A, E_1) + g(\overline{\nabla}_{E_2}A, E_2) \\
&= ds(\nabla_{e_1}d\pi(A), e_1) + ds(\nabla_{e_2}d\pi(A), e_2) \\
&= div_{M^2(\kappa)}d\pi(A) \\
&= div_{M^2(\kappa)}\left(\frac{1}{W}(\alpha e_1 + \beta e_2)\right).
\end{aligned}$$

Now, we will calculate $\overline{div}(B)$ by using that E_3 is a Killing field, that is,

$$\begin{aligned}
\overline{div}(B) &= \overline{div}\left(\frac{1}{W}E_3\right) \\
&= g\left(\overline{\nabla}\frac{1}{W}, E_3\right) + \frac{\overline{div}(E_3)}{W} \\
&= \partial_t\left(\frac{1}{W}\right) = 0.
\end{aligned}$$

So, we conclude that

$$2H = div_{M^2(\kappa)}\left(\frac{1}{W}(\alpha e_1 + \beta e_2)\right).$$

□

Recall that for the space form $M^2(\kappa)$, we have been denoted by $C(M^2(\kappa))$ the Cheeger's constant of $M^2(\kappa)$. As a consequence, we obtain the following proposition:

Proposition 4.1.1. *There is no entire H-graph in $E^3(\kappa,\tau)$ having constant mean curvature H, such that $2H > C(M^2(\kappa))$.*

Proof. Let us suppose that such an entire graph exists. Let

$$\Sigma(u) = \{(x,y,u(x,y)) \in E^3(\kappa,\tau); (x,y) \in M^2(\kappa)\},$$

where $u : M^2(\kappa) \longrightarrow \mathbb{R}$ is a solution of

$$2H = div_{M^2(\kappa)}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right). \tag{4-12}$$

Let $\Omega \subset M^2(\kappa)$ be an open domain with compact closure and smooth boundary $\partial\Omega$. Let us denote by η the unit outwards normal to $\partial\Omega$ (the boundary of Ω). Thus, from the equation (4-12) and the divergence theorem, we obtain:

$$\begin{aligned} 2HV(\Omega) &= \int_\Omega div_{M^2(\kappa)}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right) dV \\ &= \int_{\partial\Omega} ds\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2, \eta\right) dA \\ &\leq A(\partial\Omega). \end{aligned}$$

Taking into account the Cheeger's constant [see Definition 2.4.1], we obtain,

$$C(M^2(\kappa)) < 2H \leq C(M^2(\kappa)).$$

This contradiction completes the proof. □

Remark 4.1.2. *In the case $2H = C(M^2(\kappa))$ there are many examples of entire graphs in $E^3(\kappa,\tau)$. For instance in the case $\tau \equiv 0$ and $\kappa \equiv -1$, that is $E^3(\kappa,\tau) \equiv \mathbb{H}^2 \times \mathbb{R}$ we have the next beautiful example due to Ricardo Sa Earp (3, pag. 24):*

$$t = \frac{\sqrt{x^2+y^2}}{y}, \qquad y > 0.$$

This entire $H = \frac{1}{2}$ graph is invariant by hyperbolic translation

We have found one example of an entire graph having constant mean curvature H with $2H = C(M^2(\kappa))$ in the case $\tau \equiv -\frac{1}{2}$ and $\kappa \equiv -1$, that is when $E^3(\kappa,\tau) \equiv \widetilde{PSL}_2(\tau,\mathbb{R})$, see example 6.2.2. In this example we have an entire $H = \frac{1}{2}$ graph, which is invariant by rotational isometries, this graph is given by:

$$u(x,y) = 2\sqrt{\cosh(2\tanh^{-1}(\sqrt{x^2+y^2}))} - 2\arctan(\sqrt{\cosh(2\tanh^{-1}(\sqrt{x^2+y^2}))})$$

4.2
Surfaces in $\mathbf{E^3}(\kappa,\tau)$

Now, we state the integrability equations of surfaces in $E^3(\kappa,\tau)$, which was given by Benoit Daniel.

Let M be an oriented surfaces immersed in $E^3(\kappa,\tau)$, we denote by ∇ and S the Riemannian connection and the shape operator of M respectively. Let N the unit normal vector field of M, so we can write E_3 in the form $E_3 = T + \nu N$, where E_3 is the Killing field, T is the projection of E_3 to M, and $\nu = g(N, E_3)$. Denoting by K the Gauss curvature of M, we have,

Proposition 4.2.1. *(1, Proposition 3.3) With the notation above, for all* $X, Y \in \mathfrak{X}(M)$ *we have,*

Gauss equation

$$K = detS + \tau^2 + (\kappa - 4\tau^2)\nu^2$$

Mainardi-Codazzi's equation

$$\begin{aligned} T_S(X,Y) &= (\kappa - 4\tau^2)\nu(<Y,T>X - <X,T>Y) \\ \nabla_X T &= \nu(Sx - \tau JX) \\ 0 &= d\nu(X) + <SX - \tau JX, T> \\ 1 &= \|T\| + \nu^2 \end{aligned}$$

where $T_S(X,Y) = \nabla_X Y - \nabla_Y X - S[X,Y]$ *is the Codazzi's Tensor,* $\langle .,. \rangle$ *is the induced metric on* M *and* J *is the rotation of angle* $\pi/2$ *on* TM*.*

We seize this section to discuss briefly about sectional curvatures in $E^3(\kappa,\tau)$.

Fix a point $p \in E^3(\kappa,\tau)$ and consider a plane $P \subset T_pE^3(\kappa,\tau)$ (where $T_pE^3(\kappa,\tau)$ denotes the tangent space of $E^3(\kappa,\tau)$ at p), suppose that P is not horizontal. Since rotations about the vertical axis (the axis generated by the vertical field E_3) are isometries, we can suppose that, the intersection between P and the horizontal plane passing by $p \in E^3(\kappa,\tau)$ (that is the plane generated by $E_1(p) \in T_pE^3(\kappa,\tau)$ and $E_2(p) \in T_pE^3(\kappa,\tau)$) is the axis generated by the field E_1.

Thus, we can suppose that P is generated by the vector $E_1(p) \in T_pE^3(\kappa,\tau)$ and $aE_2(p) + bE_3(p) \in T_pE^3(\kappa,\tau)$, where $a^2 + b^2 = 1$

Denoting by $\overline{K}(P)$ the sectional curvature (see Definition 2.1.3) with respect to P, we have the next lemma:

Lemma 4.2.1. *(22, pag. 14) Following the above notations, we have:*

$$\overline{K}(P) = \overline{K}(aE_2 + bE_3, E_1) = a^2(\kappa - 3\tau^2) + b^2\tau^2.$$

In particular $\overline{K}(E_1,E_2)=\kappa-3\tau^2$ *and* $\overline{K}(E_1,E_3)=\tau^2$.

4.2.1 Cylinders in $\mathbf{E}^3(\kappa,\tau)$

Let $\gamma\subset M^2(k)$ be a curve with $k_g=2H$ (where k_g is the geodesic curvature), and we consider $Q=\pi^{-1}(\gamma)$, the vertical *H-cylinder*. Observe that topologically $Q=\gamma\times\mathbb{R}$ (not metrically).

If $X\in TM^2$, we denote by $\overline{X}$ its horizontal lift to $TE^3(\kappa,\tau)$ [see section: Riemannian submersion]. From now on we will always consider horizontal lifts unless explicitly stated otherwise.

Lemma 4.2.2. *The mean curvature of Q is H, in particular Q has constant mean curvature.*

Proof. We can suppose that γ is parameterized by arc length, let γ' the vector velocity of γ and denote by $\overrightarrow{n}$ the unit normal vector to γ. Let $\overline{\gamma}'=lift(\gamma')$ and $N=lift(\overrightarrow{n})$ be the horizontal lift of γ' and $\overrightarrow{n}$ respectively. Thus, $\overline{\gamma}'$ is tangent to Q. As E_3 is also tangent to Q, then $\{\overline{\gamma}',E_3\}$ form an orthonormal frame field on Q. It is clear that N is an unit normal field along Q.

As the fibers are geodesic, we have, $\overline{\nabla}_{E_3}E_3=0$. So the normal curvature of E_3 is 0 (i.e. $k_N(E_3=0)$).

Observe that

$$\begin{aligned} k_N(\gamma') &= g(-\overline{\nabla}_{\overline{\gamma}'}N,\overline{\gamma}') \\ &= ds(-\nabla_{\gamma'}\overrightarrow{n},\gamma') \\ &= k_g \\ &= 2H. \end{aligned}$$

Since the mean curvature can be obtained by the half sum of normal curvatures with respect to orthogonal field. Thus taking into account the above normal curvatures we obtain that, the mean curvature of Q in this case is H. $\square$

We denote by K_{ext} the extrinsic curvature of $Q\equiv\pi^{-1}(\gamma)$.

Lemma 4.2.3. *The surface H-cylinder Q is flat, and has constant extrinsic curvature K_{ext} equal to $-\tau^2$, that is $K_{ext}=-\tau^2$.*

Proof. We can suppose that the curve γ is parameterized by arc length, i.e., $\gamma(s)=(f(s),g(s))\subset M^2(\kappa)$ with $ds^2(\gamma',\gamma')=1$, denote by n the normal to

γ, then,

$$\begin{aligned}\gamma'(s) &= \lambda f'(s)e_1 + \lambda g'(s)e_2 \\ n(s) &= -\lambda g'(s)e_1 + \lambda f'(s)e_2\end{aligned}$$

Where λ, e_1, and e_2 are defined in [section: Space form].

Let $\overline{\gamma}'$ and N be the horizontal lift of γ' and n respectively. Thus $\overline{\gamma}'$ is tangent to Q and N is normal to Q. The other unit tangent vector field to Q, which is orthogonal to $\overline{\gamma}'$ is E_3. Thus we have an orthonormal frame field:

$$\begin{aligned}X_1 &= \overline{\gamma}' = \lambda f'(s)E_1 + \lambda g'(s)E_2 \\ X_2 &= E_3 \\ N &= -\lambda g'(s)E_1 + \lambda f'(s)E_2\end{aligned}$$

By using the Riemannian connection we obtain $[X_1, X_2] = 0$. Thus, we have around of $p \in Q$, a coordinate system $\Phi(s,t)$ such that

$$\begin{aligned}\Phi_s &= \lambda f'(s)E_1 + \lambda g'(s)E_2 \\ \Phi_t &= E_3 \\ N &= -\lambda g'(s)E_1 + \lambda f'(s)E_2\end{aligned}$$

We denote by $\langle .,. \rangle$ the induced metric on Q,

$$\begin{aligned}\langle \Phi_s, \Phi_s \rangle &= \lambda^2(f')^2 + \lambda^2(g')^2 = 1 \\ \langle \Phi_s, \Phi_t \rangle &= 0 \\ \langle \Phi_t, \Phi_t \rangle &= 1,\end{aligned}$$

Then, $K = 0$, i.e., the Gaussian curvature is 0, thus the surface Q is flat. From the Gauss equation,

$$K = detS + \tau^2 + (k - 4\tau^2)\nu^2$$

since $\nu = 0$, we obtain $K_{ext} = \det S = -\tau^2$. $\square$

Recall that the Ricci curvature in the direction X is denoted by $Ric(X)$ [see Definition 2.1.4].

Lemma 4.2.4. *The Ricci curvature in the direction* N, *is given by*

$$Ric(N) = \kappa - 2\tau^2$$

where N is the unit normal field along Q.

Proof. We have denoted by R the curvature tensor of $E^3(\kappa,\tau)$ [see Definition 2.1.2]. Furthermore, we have an orthonormal frame field on Q:

$$\begin{aligned}\Phi_s &= \lambda f'(s)E_1+\lambda g'(s)E_2\\ \Phi_t &= E_3\\ N &= -\lambda g'(s)E_1+\lambda f'(s)E_2.\end{aligned}$$

Let $Y \in TE$ be a vector field, by the linearity of R we have

$$\begin{aligned}R(N,Y)N &= R(-\lambda g'(s)E_1+\lambda f'(s)E_2,Y)(-\lambda g'(s)E_1+\lambda f'(s)E_2)\\ &= [-\lambda g'(s)R(E_1,Y)+\lambda f'(s)R(E_2,Y)](-\lambda g'(s)E_1+\lambda f'(s)E_2)\\ &= \lambda^2(g')^2R(E_1,Y)E_1-\lambda^2 f'g'R(E_1,Y)E_2-\lambda^2 f'g'R(E_2,Y)E_1+\\ &\quad \lambda^2(f')^2R(E_2,Y)E_2.\end{aligned}$$

We use this expression to show the following two affirmation,

Afirmation 4.2.1. *We have*

$$g_E(R(N,\Phi_s)N,\Phi_s)=k-3\tau^2$$

where g_E is the metric of E.

Proof. Putting $Y=\Phi_s=\lambda f'(s)E_1+\lambda g'(s)E_2$, by using linearity of the metric g_E of $E^3(\kappa,\tau)$ we obtain,

$$\begin{aligned}g_E(R(N,\Phi_s)N,\Phi_s) &= \overline{K}(E_1,E_2)[\lambda^2(g')^2+\lambda^2(f')^2]^2\\ &= \overline{K}(E_1,E_2)\\ &= \kappa-3\tau^2\end{aligned}$$

where $\overline{K}(E_1,E_2)$ denote the sectional curvature in the direction $\{E_1,E_2\}$ [see Lemma 4.2.1]. □

Afirmation 4.2.2. *We have*

$$g(R(N,\Phi_z)N,\Phi_z)=\tau^2$$

Proof. Putting $Y = \Phi_t = E_3$, by using linearity of the metric g_E of $E^3(\kappa,\tau)$ we have,

$$\begin{aligned} g_E(R(N,\Phi_t)N,\Phi_t) &= \overline{K}(E_1,E_3)[\lambda^2(g')^2 + \lambda^2(f')^2] \\ &= \overline{K}(E_1,E_3) \\ &= \tau^2 \end{aligned}$$

where $\overline{K}(E_1,E_3)$ denote the sectional curvature in the direction $\{E_1,E_3\}$ [see Lemma 4.2.1]. □

This affirmations implies:

$$\begin{aligned} Ric(N) &= g_E(R(N,\Phi_s)N,\Phi_s) + g_E(R(N,\Phi_z)N,\Phi_z) \\ &= (\kappa - 3\tau^2) + \tau^2 \\ &= \kappa - 2\tau^2. \end{aligned}$$

□

We ended this section by given the following theorem, which is due to José Espinar and Harold Rosenberg, see (9):

Theorem 4.2.1. *(9, Theorem 2.2) Let $\Sigma \subset E^3(\kappa,\tau)$ be a complete H surface with constant angle function. Then Σ is either a vertical H cylinder over a complete curve of curvature $2H$ on $M^2(\kappa)$ or a slice in $\mathbb{H}^2\times\mathbb{R}$ or $\mathbb{S}^2\times\mathbb{R}$.*

4.2.2 Stability in $\mathbf{E}^3(\kappa,\tau)$

Let us recall the definition of stability due to $M.\ do\ Carmo$, $L.\ Barbosa$, and $J.\ Eschenburg$ (see (12)). Consider an immersion $x : M \longrightarrow E^3(\kappa,\tau)$ of a compact manifold M of dimension 2. Denote by dM the element of area of M with the induced metric from the immersion x.

A variation of x is a differentiable map $X : (-\epsilon,\epsilon)\times M \longrightarrow E$ such that $X_t(p) = X(t,p)$ is an immersion for each $t \in (-\epsilon,\epsilon)$, and $p \in M$, $X_0 = x$ and $X_{t|\partial M} = x_{|\partial M}$.

Define the area function $A : (-\epsilon,\epsilon) \longrightarrow \mathbb{R}$ and the volume function $V : (-\epsilon.\epsilon) \longrightarrow \mathbb{R}$ by

$$A(t) = \int_M dM_t$$

$$V(t) = \int_{[0,t]\times M} X^* dv$$

where dv is the volume element of the ambient space and X^* is the standard linear map on forms induced by X; so X^*dv is the induced (algebraic) volume form.

Let N be an unit normal vector field to M. Denote by $f = g(\frac{\partial X}{dt}|_{t=0}, N)$, then we have,

$$\dot{A}(0) = -\int_M 2fHdM$$

Let M be a surface with constant mean curvature H and consider the function $S : (-\epsilon, \epsilon) \longrightarrow \mathbb{R}$, defined by

$$S(t) = A(t) + 2HV(t)$$

It is possible shows that $\dot{S}(0) = 0$. Furthermore,

$$\ddot{S}(0) = -\int_M [f\Delta f + (Ricc(N) + |B|^2 f^2)]dM$$

where $|B|^2$ is square of the norm of the second fundamental form of M and $Ricc(N)$ is the Ricci curvature of the ambient space in the direction of N (see (6)).

We call $L = \Delta + Ricc(N) + |B|^2$ the stability operator of M.

Definition 4.2.1. *Let $x : M^2 \longrightarrow E$ be an immersion with constant mean curvature H. The immersion x is weakly stable if $\ddot{A}(0) \geq 0$ for all volume-preserving variation of x. If M is noncompact, we say that x is weakly stable if for every manifold with boundary $\widetilde{M} \subset M$, the restriction $x|_{\widetilde{M}}$ is weakly stable.*

Now, we define strong stability, let $\mathfrak{F}$ the set defined by

$$\mathfrak{F} = \{h : M \longrightarrow \mathbb{R}; h|\partial M = 0\}$$

Definition 4.2.2. *$x : M^2 \longrightarrow E^3(\kappa,\tau)$ is strongly stable (or simply stable) if and only if*
$\ddot{S}(0)h \geq 0$ for all $h \in \mathfrak{F}$.

An important property of stability is the following.

Proposition 4.2.2. *Let W be a Killing vector field on $E^3(\kappa,\tau)$. Then $f = g(W,N)$ satisfies*

$$\Delta f + (Ric(N) + |B|^2)f = 0$$

Here f is said a Jacobi function.

4.2.3
Stability of H Cylinders in $\mathbf{E^3}(\kappa,\tau)$

Now, we see whenever a cylinder is unstable.

Proposition 4.2.3. *The H cylinder Q immersed in $E^3(\kappa,\tau)$ is an unstable H surface for $4H^2+\kappa>0$.*

Proof. The stability operator L of Q is

$$L = \Delta + |B|^2 + Ric(N),$$

where Δ is the Laplacian operator associated to the Riemannian metric induced on Q, $|B|^2$ is the square of the norm of the shape operator associated to Q and $Ric(N)$ is the Ricci curvature in the direction of the unit normal vector field N along Q. We have,

$$\begin{aligned} |B|^2 + Ric(N) &= 4H^2 - 2K_{ext} + \kappa - 2\tau^2 \\ &= 4H^2 + \kappa \\ &= 4H^2 + \kappa. \end{aligned}$$

So L$=\Delta + a$, where $a = 4H^2 + \kappa > 0$.

We consider the operator L on $[0,T]\times[-r,r]$ for $r>0$, where $[0,T]$ is an arc of length T on γ.

It is known that $\phi_1 = cos(\pi s/T)$ is a first eigenfunction of $\dfrac{\partial^2}{\partial^2 s}$ on $[0,T]$, with eigenvalue $\lambda_1 = \pi^2/T^2$.

Similarly, a first eigenfunction ϕ_2 of $\dfrac{\partial^2}{\partial^2 t}$ on $[-r,r]$ is $\phi_2 = cos(\pi t/2r)$ with eigenvalue $\lambda_2 = \pi^2/4r^2$.

Let $\phi = \phi_1 * \phi_2$, observe that $\phi \in \mathfrak{F}$ and

$$\Delta\phi + (\lambda_1+\lambda_2)\phi = 0, on \quad [0,T]\times[-r,r]$$

then,

$$L\phi + (\lambda_1+\lambda_2-a)\phi = 0, on \quad [0,T]\times[-r,r]$$

Hence, if r and T satisfy $\lambda_1+\lambda_2-a<0$, then the domain is unstable, that is, there is a negative eigenvalue for L.

This condition is equivalent to

$$\frac{\pi^2}{T^2} + \frac{\pi^2}{4r^2} < 4H^2 + \kappa,$$

but for T or r large enough is clear that

$$\frac{\pi^2}{T^2}+\frac{\pi^2}{4r^2}<4H^2+\kappa.$$

This complete the proof. □

Corollary 4.2.1. *The H cylinder Q immersed in $E^3(\kappa,\tau)$ is an unstable H surface for $2H > C(M^2(\kappa))$.*

Proof. Since that, $C(\mathbb{H}^2)\equiv 1$, $C(\mathbb{R}^2)\equiv 0$ and $C(\mathbb{S}^2)\equiv 0$, the corollary holds. □

4.3 Curvature estimates for stable H surfaces in 3 manifolds

In the paper: "*General Curvature Estimates for Stable H-Surfaces in 3-Manifold and Applications*" (see (4)), the authors gave an estimate for the norm of the second fundamental form of stable H surfaces in Riemannian 3-manifolds with bounded sectional curvature. They also gave an interior estimates for H sections and applications. This paper plays an important role in our study of multi-graphs.

Theorem 4.3.1. *(4, Main theorem) Let (M,g) be a complete smooth Riemannian 3-manifold of bounded sectional curvature $|\overline{K}|\leq\Lambda<+\infty$. Then there exist a universal constant C which depends neither on M nor on Λ, satisfying the following,*

For any immersed stable H surface $\Sigma\to M$ with trivial normal bundle, and for any $p\in\Sigma$ we have:

$$|A(p)|\leq\frac{C}{\min\{d(p,\partial\Sigma),\frac{\pi}{2\sqrt{\Lambda}}\}}$$

where $|A(p)|$ denotes the norm of the second fundamental form, and $d(.,.)$ denotes the intrinsic distance on Σ.

Now, we will present some applications. Consider a Killing submersion [see Definition 4.1.1],

$$\pi:(M^3,g)\longrightarrow(M^2,h)$$

where (M^3,g) is a Riemannian 3-manifold with Riemannian metric g, and (M^2,h) is Riemannian 2-manifold with Riemannian metric h.

Remark 4.3.1. *It is possible shows that when the fiber of the Killing submersion are complete geodesic of infinite length, then such fibration is (topologically) trivial (see (4, pag. 13)).*

Remark 4.3.2. *Let* $s:\overline{\Omega}\to\Sigma\subset M^3$ *be an* H *section [see Definition 4.1.2], where* $\Omega\subset M^2$ *is a relative compact domain. Then,*

- *For any interior point* $p\in\Omega$ *the* H *surface* Σ *is transversal at* $s(p)$ *to the fiber. Indeed, assume by contradiction that* Σ *is tangent to the fiber at the interior point* $s(p)$. *Let* $S\subset M^3$ *be a vertical* H *cylinder tangent to* Σ *at* $s(p)$ *with the same mean curvature vector. Then, in a neighborhood of* $s(p)$, *the intersection* $\Sigma\cap S$ *is composed of* $n\geq 2$ *smooth curves passing through* $s(p)$. *but the union of those curves cannot be a graph, contradicting the assumption that* Σ *is a graph over* $\overline{\Omega}$.

Theorem 4.3.2. *(4, Theorem 3.3) Let* $\pi:(M^3,g)\to(M^2,h)$ *be a Killing submersion and let* $s:\Omega\to\Sigma\subset M^3$ *be an* H *section over a domain* $\Omega\subset M^2$. *Let* U_0 *be a neighborhood of an arc* $\gamma\subset\partial\Omega$ *and* $s_0:U_0\to M^3$ *a section.*

Assume that for any sequence (p_n) *of* Ω *which converges to a point* $p\in\gamma$, *the height of* $s(p_n)$ *goes to* $+\infty$, *that is* $s(p_n)-s_0(p_n)\to+\infty$.

Then, γ *is a smooth curve with geodesic curvature* $2H$. *If* $H>0$ *then* γ *is convex with respect to* Ω *if, and only if, the mean curvature vector* $\overrightarrow{H}$ *of* Σ *points up, that is, if* $g(\overrightarrow{H},E_3)>0$ *along* Σ, *where* E_3 *denotes the tangent unit vector field along the fibers. Moreover,* Σ *converges to the vertical* H *cylinder* $\pi^{-1}(\gamma)$ *with respect to the* C^k*- topology for any* $k\in\mathbb{N}$.

Remark 4.3.3. *In a Riemannian product* $(M^3,g)=(M^2,h)\times\mathbb{R}$, *consider a domain* $\Omega\subset M^2$ *and a smooth surface* $\Sigma\subset M^3$ *which is the vertical graph of a function* u *on* Ω. *Let* N *be a unit normal field along* Σ *and let* $E_3=\partial_t$ *be the unit vertical field. Then we have:*

$$|g(N,E_3)|=\frac{1}{\sqrt{1+|\nabla_h u|^2}}.$$

Therefore, bounding $||\nabla_h u$ *from above is equivalent to bounding* $|g(N,E_3)|$ *from below away from 0.*

With this in mind, we have:

Theorem 4.3.3. *(4, Theorem 3.6) Let* $\pi:(M^3,g)\to(M^2,h)$ *be a Killing submersion. let* $\Omega\subset M^2$ *be a relatively compact domain and* $s_0:\overline{\Omega}\to\Sigma_0$ *a* C^0 *section over* Ω.

Then, for any $C_1,C_2>0$, *there exists a constant* $\alpha=\alpha(C_1,C_2,\Omega)$ *such that for any* $p\in\Omega$ *with* $d(p,\partial\Omega)>C_2$ *and for any* H *section* $s:\overline{\Omega}\to\Sigma\subset M^3$ *over* $\overline{\Omega}$ *with* $|s-s_0|<C_1$ *on* Ω, *we have:*

$$|g(N,E_3)(s(p))|>\alpha$$

where N is a unit normal field along Σ.

We conclude this section by given the following proposition,

Proposition 4.3.1. *(4, Proposition 4.3) Let (M^3, g) be a smooth Riemannian 3-manifold (not necessarily complete), with bounded sectional curvature $|\overline{K}| \leq \Lambda < +\infty$ and let $r > 0$. Let $\Omega \subset M$ be an open subset of M such that the injectivity radius in M at any point $x \in \Omega$ is $\geq r$.*

Then for any $C_1 > 0$ and $C_2 > 0$ there exist constants $\delta, \delta_0 > 0$ depending only on C_1, C_2, Λ and r and neither M nor on Ω satisfying the following,

For any immersed surface $S \to M^3$ whose second fundamental form A satisfies $|A| < C_1$ and for any $p \in S$ such that $d_S(p, \partial S) > C_2$ then a part S_0 of S is a Euclidean graph over the disk of T_pS centered at p with Euclidean radius δ, here T_pS denotes the tangent plane at p of S. Furthermore, the subset S_0 contains the geodesic disk of S centered at p with radius δ_0.

Remark 4.3.4. *Consider the space $M^3 \equiv E^3(\kappa,\tau)$ and let $\Sigma \to E^3(\kappa,\tau)$ be a complete H surface, immersed in $E^3(\kappa,\tau)$. Denote by $\nu := g(N, E_3)$ the angle function, where N denotes the tangent unit vector field along Σ, and E_3 is the tangent unit vector field along the fibers.*

Suppose that either $\nu \geq 0$ or $\nu \leq 0$ along Σ, then ν is a non-negative Jacobi function [see Proposition 4.2.2]. Thus, the first eigenvalue for the Jacobi operator is zero, hence there is no negative eigenvalue. This implies that, Σ is strongly stable (or simply stable).

Then by using curvature estimates, we have by Proposition 4.3.1, that for any point $p \in \Sigma$, a neighborhood of p in Σ is a Euclidean graph over a disk of Euclidean radius δ in the tangent plane $T_p\Sigma$, this δ is uniform, that is δ is independent of p.

4.4 Multi-graphs in $\mathbf{E}^3(\kappa,\tau)$

Throughout this section, we consider a surface Σ having constant mean curvature $H \neq 0$ immersed in $E^3(\kappa,\tau)$.

Let N be the unit normal vector field along Σ. Recall that we have denoted by $\nu = g(N, E_3)$ the angle function on Σ.

Definition 4.4.1. Σ *is called a multi-graph if the angle function ν does not changes sign, i.e., either $\nu \geq 0$ or $\nu \leq 0$ on Σ.*

Lemma 4.4.1. *Let Σ be a complete H multi-graph immersed in $E^3(\kappa,\tau)$. Then, either one of the following conditions hold:*

(i) $\nu > 0$ *or* $\nu < 0$ *on* Σ.

(ii) Σ *is an* H *cylinder.*

Proof. We can suppose $\nu \leq 0$. Suppose that, there is a point $p \in \Sigma$ such that $\nu(p) = 0$. In this case we will show that Σ is an H cylinder.

Since E_3 is a killing vector field, then ν is a Jacobi function on Σ, that is:

$$L\nu = \Delta\nu + (Ric(N) + |B|^2)\nu = 0$$

Setting $c = Ric(N) + |B|^2$, we have the operator $L = \Delta + c$. Let U be a neighborhood of p, so $\nu \leq 0$ on U and $\nu(p) = 0$. Working on U, we have

$$\begin{aligned}\Delta\nu + \min_U(c,0)\nu &= L\nu - c\nu + \min_U(c,0)\nu \\ &= (\min_U(c,0) - c)\nu + L\nu \\ &\geq 0\end{aligned}$$

so, $(\min_U(c,0) - c)\nu \geq 0$. Thus, $P\nu = \Delta\nu + \min_U(c,0)\nu \geq 0$ with $\min_U(c,0) \leq 0$. Applying the maximum principle of Hopf [see Theorem 2.5.2], we conclude that $\nu \equiv 0$ on U.

Now, setting $A = \{q \in \Sigma; \nu(q) = 0\}$. Observe that $A \neq \phi$ and A is closed and open, so $A = \Sigma$.

Thus, the angle function is constant on Σ, moreover $\nu \equiv 0$. By using Theorem 4.2.1, we obtain that Σ is an H cylinder. □

4.4.1
The main theorem

When $E^3(\kappa,\tau)$ in not compact, that is $E^3(\kappa,\tau) \neq \mathbb{S}^3_\tau$. The space $E^3(\kappa,\tau)$ is topologically $M^2(\kappa) \times \mathbb{R}$, which we have denoted by $E^3(\kappa,\tau) \simeq M^2(\kappa) \times \mathbb{R}$.

In this case, there is a *trivial section* $s_0 : M^2(\kappa) \longrightarrow E^3(\kappa,\tau)$ given by $s_0(p) = (p,0) \in M^2(\kappa) \times \mathbb{R}$ [see Remark 4.3.1]. We identify $M^2(\kappa)$ with its lift $s_0(M^2(\kappa))$.

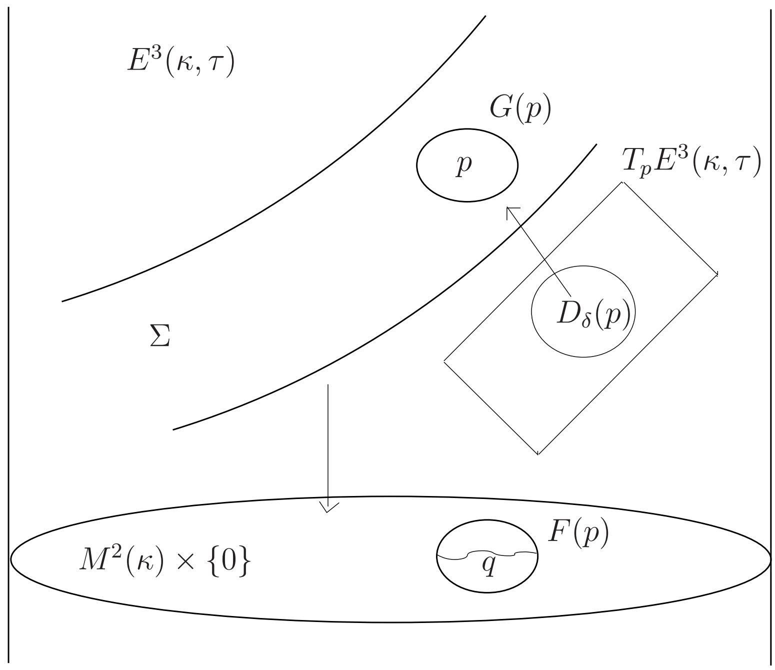
Denote by $I : E^3(\kappa,\tau) \longrightarrow s_0(M^2(\kappa))$ the vertical translation along the fiber, taking the point (p_0,t_0) to the point $(p_0,0)$, that is:

$$I(p_0,t_0) = (p_0,0) \in s_0(M^2(\kappa))$$

If Σ is a multi-graph, then ν is a non-negative Jacobi function, so Σ is stable and this implies that Σ has bounded curvature, so there is a real number

$\delta > 0$, such that for each $p \in \Sigma$, a piece of Σ is a Euclidean graph (of bounded geometry) over the disk of radius $D_\delta(p) \subset T_p\Sigma$ [see Remark 4.3.4]. We denote this graph by $G(p)$.

Applying the vertical translation I to $G(p)$, taking the point p to the point q, that is $I(p) = q$, we obtain a piece of a surface which we denote by $F(q)$, that is, $I(G(p)) = F(q)$. This surface $F(q)$ is a Euclidean graph over the disk $D_\delta(q) \subset T_qF(q)$.



Recall that there are entire graphs in $E^3(\kappa,\tau)$ having constant mean curvature H, with $2H = C(M^2(\kappa))$, where $C(M^2(\kappa))$ is the Cheeger's constant, for instance see Remark 4.1.2. But there is no entire graph in $E^3(\kappa,\tau)$ having constant mean curvature H such that $2H > C(M^2(\kappa))$ [see Proposition 4.1.1]. In a more general sense we have the following theorem: (To prove this theorem, we follow the ideas from (8, Theorem 4.1))

Theorem 4.4.1. *Let $E^3(\kappa,\tau)$ be a complete simply connected homogeneous 3-manifold. Let Σ be a complete H multi-graph immersed in $E^3(\kappa,\tau)$. If $2H > C(M^2(\kappa))$, then Σ is a vertical H cylinder.*

Proof. If there is a point $p \in \Sigma$ such that $\nu(p) = 0$, then Σ is a vertical H cylinder [see Lemma 4.4.1].

Thus, we can suppose that the angle function ν, satisfies $\nu < 0$ on Σ, and $g(N, \overrightarrow{H}) > 0$, where N is the unit normal vector field on Σ, $\overrightarrow{H}$ is the mean curvature vector field of Σ and g is the metric of $E^3(\kappa,\tau)$.

We divide the proof in two step: First we will prove the theorem for the case $E^3(\kappa,\tau) \neq \mathbb{S}^3_\tau$. After, we will prove the theorem for the case $E^3(\kappa,\tau) = \mathbb{S}^3_\tau$.

The case $\mathbf{E^3}(\kappa,\tau) \neq \mathbb{S}^3_\tau$: There is no complete H multi-graph Σ immersed into $E^3(\kappa,\tau)$, having angle function $\nu < 0$, and $2H > C(M^2(\kappa))$.

Proof. The idea of the proof is shows that such multi-graph is actually an entire graph. Hence, using Proposition 4.1.1, we conclude that such an entire graph does not exist.

Let $B_{\widetilde{R}}(0) = B_{\widetilde{R}}$ be the geodesic ball in $s_0(M^2(\kappa))$ centered at the origin of $s_0(M^2)$, with radius $\widetilde{R}$. Let $p \in \Sigma$ and assume that on a neighborhood of p in Σ, Σ is a H-section of $\pi : E^3(\kappa,\tau) \longrightarrow M^2(\kappa)$ [see Definition 4.1.2], over the ball $B_{\widetilde{R}}$. Let $f : B_{\widetilde{R}} \longrightarrow \mathbb{R}$ such that the H-section is equals to $graf(f)$.

If Σ is not an entire graph, then let R be the biggest such that f exists. Let $q \in \partial B_R$ such that f cannot extends to a H-section in any neighborhood of q.

Afirmation 4.4.1. *For any sequence $q_n \subset B_R$ such that $q_n \longrightarrow q$, we have $f(q_n) \longrightarrow +\infty$ (or $f(q_n) \longrightarrow -\infty$).*

Proof. Suppose that there exists a sequence $q_n \subset B_R$ such that $q_n \longrightarrow q$ and $f(q_n)$ is bounded. Then, there exists a subsequence $f(q_{n_i})$ that converges to a point x; since Σ is complete we have $(q,x) \in \Sigma$. This implies that Σ must have a horizontal normal at (p,x), which is false since $\nu < 0$.

Observe that, is not possible that there are two subsequences (q_n) and $(\widetilde{q}_n)$ such that: $f(q_n) \longrightarrow +\infty$ and $f(\widetilde{q}_n) \longrightarrow -\infty$. To see this, takes n big and consider the points q_n and $\widetilde{q}_n$, this points are close to q. Now consider the family of complete geodesics passing by q, this family gives rise to a family of vertical planes passing by q. Since Σ is connected and complete, there is a plane in the above family such that, the intersection of this plane with Σ is a connected curve. Supposing that there $f(q_n) \longrightarrow +\infty$ and $f(\widetilde{q}_n) \longrightarrow -\infty$, we obtain that this connected curve has a vertical tangent, this is impossible since $\nu < 0$ □

Now, consider the piece of curve $C_\delta(q)$ passing by q, tangent to ∂B_R at q, having geodesic curvature $2H$ ($k_g = 2H$) and such that B_R stays in the concave side of $C_\delta(q)$. Let $Q_\delta = \pi^{-1}(C_\delta(q))$ the "δ-cylinder" which contains $C_\delta(q)$.

Recall that, we have denoted by I the vertical translation, that is, the translation along the fibers. Using curvature estimates, we have for any $\widetilde{p} \in \Sigma$, that a neighborhood of $\widetilde{p}$ is a Euclidean graph over a disk $D_\delta(\widetilde{p}) \subset T_{\widetilde{p}}\Sigma$. We have denoted this graph by $G(\widetilde{p})$ and by $F(\widetilde{q}) = I(G(\widetilde{p}))$ the translated surface to hight $t = 0$, that is, when $I(\widetilde{p}) = \widetilde{q} \in s_0(M^2(\kappa))$. By using this notations, we rewrite the Theorem 4.3.2 in the following form:

Afirmation 4.4.2. [*Theorem 4.3.2*] *Let* $(p_n)_n$ *be a sequence of points of* Σ *such that* $I(p_n) = q_n$ *converge to* q. *Then there exist a sub-sequence of* $(F(q_n))_n$ *that converge to a* δ*-piece of* $Q_\delta = \pi^{-1}(C_\delta(q))$. *This convergence is in the* C^2*-topology.*

Let γ_0 be the arc geodesic having length 2ϵ, centered at q and orthogonal to $C_\delta(q)$, denote by γ_0^+ the part of γ_0 that stays in B_R.

Let q_n be a sequence that converge to q with $q_n \in \gamma_0^+$.

Let C be the complete curve of $M^2(\kappa)$ which has geodesic curvature $2H$ ($k_g = 2H$), and such that C contains $C_\delta(q)$. Parametrize C by arc length; denote by $q(s) \in C$ the point at distance s on C from $q(0) = q$, $s \in \mathbb{R}$. Note that C may be compact.

Denote by γ_s an arc geodesic orthogonal to C at $q(s)$, that is, $q(s)$ is the midpoint of γ_s. Assume that the length of each γ_s is 2ϵ. Thus

$$\bigcup_{s\in\mathbb{R}} \gamma_s = T_\epsilon(C)$$

is the ϵ tubular neighborhood of C.

We denote by γ_s^+ the part of γ_s on the concave side of C.

Afirmation 4.4.3. *For* n *large, each* $F(q_n)$ *is disjoint from* $C \times \mathbb{R}$. *Also, for* $|s| < \delta$, $F(q_n) \cap (\gamma_s \times \mathbb{R})$ *is a vertical graph over an interval of* γ_s.

Proof. Observe that the $F(q_n)$ are C^2 close to $C_\delta(q) \times J_\delta \subset Q_\delta$ (where J_δ is an interval of length 2δ). Choose n_0 so that for $n \geq n_0$, $\Gamma_n(s) = F(q_n) \cap (\gamma_s \times \mathbb{R})$ is one connected curve and the intersection of $F(q_n)$ with $\gamma_s \times \mathbb{R}$ is transverse, for each $s \in [-\delta, \delta]$. Thus $\Gamma_n(s)$ has no horizontal or vertical tangents and is a graph over an interval in γ_s.

Now, we show that this interval is $\gamma_s^+ - q(s)$. Suppose not, so $\Gamma_n(s)$ goes beyond $C \times \mathbb{R}$ on the convex side. Recall that $p_n = (q_n, f(q_n))$. Lift each $\Gamma_n(s)$ to $G(p_n)$ by the inverse of the vertical translation I. Still we denote by $\Gamma_n(s)$ its lift to Σ. The curve

$$\Gamma(s) = \bigcup_{n\geq n_0} \Gamma_n(s)$$

is a vertical graph over an interval in γ_s This curve has points in the convex side of $C \times \mathbb{R}$ for some $s_0 \in [-\delta, \delta]$. For $s = 0$, $\Gamma(0)$ stays on the mean concave side of $C \times \mathbb{R}$. So, for some s_1, $0 < s_1 \leq s_0$, $\Gamma(s_1)$ has a point on $C \times \mathbb{R}$ and also inside of the mean convex side of $C \times \mathbb{R}$. But the $F(q_n)$ converge uniformly to $C_\delta(q) \times J_\delta$ as $n \longrightarrow +\infty$, so the curve $\Gamma(s_1)$ converges to $q(s_1) \times \mathbb{R}$ as the height goes to $+\infty$. This ensures $\Gamma(s_1)$ to have a vertical tangent on the convex side of $C \times \mathbb{R}$, this is impossible since $\nu < 0$. $\square$

Now, we choose $\epsilon_1 < \epsilon$ (which we call ϵ as well) such that

$$\bigcup_{s\in[-\delta,\delta]} \Gamma(s)$$

is a vertical graph of a function g on

$$\bigcup_{s\in[-\delta,\delta]} (\gamma_s^+ - q(s))$$

here γ_s^+ have length ϵ_1. The graph of g converges to $C_\delta(q) \times \mathbb{R}$ as the height goes to infinity.

Observe that, at the point $q(\delta)$ we have the same situation as for the point $q(0) = q$. Thus, replacing $\Gamma(0)$ by the curve $\Gamma(\delta)$, we begin this process again. This gives an analytic continuation for the graph g to a graph over:

$$\bigcup_{s\in[-\delta,2\delta]} (\gamma_s^+ - q(s))$$

which converges uniformly to $C(q,[-\delta,2\delta])$ as the height goes to infinity. Here $C(q,[-\delta,2\delta])$ denotes the arc of C, of length 3δ, between the points $q(-\delta)$ and $q(2\delta)$. We now continue analytically, by extending the graph about $\Gamma(2\delta)$. When we refer here to analytic continuation, we mean the unique continuation of the local pieces of the surface. Continuing this argument, we obtain that a portion of Σ is a graphic of a function g definite over

$$\bigcup_{s\in\mathbb{R}}(\gamma_s^+ - q(s)).$$

Continue this process replacing $\Gamma(\delta)$ and $\Gamma(-\delta)$ by $\Gamma(2\delta)$ and $\Gamma(-2\delta)$; again, going up high enough on these curves so that the graph is within $T_\epsilon(C \times \mathbb{R}) = \epsilon$- tubular neighborhood of $Q = \pi^{-1}(C)$.

Let M denote the surface obtained by this analytic continuation, so the curve $\partial M := \beta$ is far of Q and M is asymptotic to Q. We know that $Q = C \times \mathbb{R}$ is an unstable H surface.

Let K_0 be a compact stable domain of Q. Let K_0 expand until one reaches an unstable domain K of Q, K compact. This means that, there is a smooth function $f : K \longrightarrow \mathbb{R}$, $f = 0$ on ∂K, $f > 0$ on $int(K)$, and f satisfies

$$Lf + \lambda f = 0, \qquad \lambda < 0$$

Let $K(t)$ be the variation of K given by $K(t) = exp_p(tf(p)Z(p))$, where $p \in K$ and $Z(p)$ is a unit normal to K, with Z pointing to the mean convex side of Q. $K(t)$ is a smooth surface with $\partial K(t) = \partial K \subset Q$, and for t small, $int(K(t)) \cap Q = \phi$.

Since the linearized operator L is the first variation of the mean curvature at $t = 0$, and $Lf(p) = -\lambda f(p) > 0$ for $p \in int(K)$, we conclude $H(K(t)) > H$ for $t > 0$, and $H(K(t)) < H$ for $t < 0$.

So for t small enough the surfaces $K(t)$ are disjoint from β, and they can be slid up and down Q to remain disjoint from M. But M is asymptotic to Q. Hence for small $t < 0$, the surface $K(t)$ will touch M at a first point, when $K(t)$ is slid up Q. But this contradicts the maximum principle [see Theorem 5.3.1].

Thus Σ must be an entire graph. By Proposition 4.1.1 such entire graph cannot exist. This complete the proof in this case.

□

The case $\mathbf{E^3}(\kappa,\tau) = \mathbb{S}^3_\tau$: There is no complete H multi-graph Σ immersed into $E^3(\kappa,\tau)$, having angle function $\nu < 0$, and $2H > C(M^2(\kappa))$.

Proof. Recall that, we have assumed $\nu < 0$.

We will divide the proof in two subcases. In the first subcase, we will suppose that the angle function is far away from 0, that is $\nu \ll 0$. In the second subcase, we will suppose that, there is a sequence (p_n) such that $\nu(p_n) \to 0$. In both cases, we will show that, such H multi-graph cannot exists.

First subcase: If ν is far away from 0 ($\nu \ll 0$). Then Σ cannot exists.

Let $\pi_1 = \pi_{|_\Sigma} : \Sigma \longrightarrow \mathbb{S}^2$ be the restriction of π to Σ, where π is the Killing submersion:

$$\pi : E^3(\kappa,\tau) \longrightarrow M^2(\kappa).$$

Since that $\nu \ll 0$, then for each $p \in \Sigma$, there is a neighborhood of p in Σ which we denote by $V(p) \subset \Sigma$, such that $V(p)$ is the image of a section over a neighborhood of $\pi(p) \in \mathbb{S}^2$. Hence π_1 is a local homeomorphism, this implies that ($\mathbb{S}^2$ is simply connected), π_1 is a global homeomorphism. Thus, there is a global section of $\pi : \mathbb{S}^3_\tau \longrightarrow \mathbb{S}^2$, which is impossible [see Remark 4.1.1]. Thus, in this subcase Σ cannot exists.

Second subcase.- If There is a sequence $q_n \in \Sigma$ such that $\nu(q_n) \longrightarrow 0$. Then, such Σ cannot exists.

Suppose that, there exist a sequence $(q_n) \subset \Sigma$ such that $\nu(q_n) \longrightarrow 0$. Since $\mathbb{S}^3_\tau$ is compact, there is a subsequence (which we also denote by (q_n)), such that $q_n \longrightarrow p$, $p \in \mathbb{S}^3_\tau$.

Following the same arguments and notations from the case $E^3(\kappa,\tau) \neq \mathbb{S}^3_\tau$, and by using Affirmation 4.4.2 and Affirmation 4.4.3, we have:

Σ is asymptotic to an H cylinder Q and the mean curvature vector of Σ has the same direction as that of Q at points of Σ converging to $Q = \pi^{-1}(C)$, where $C \subset \mathbb{S}^2$ is a complete curve having Gauss curvature equal to $2H$. Furthermore, a portion of Σ is a graphic of a function g definite over

$$\bigcup_{s\in\mathbb{R}}(\gamma_s^+ - q(s)).$$

This graph is within $T_\epsilon(Q) = \epsilon$- tubular neighborhood of $Q = \pi^{-1}(C)$.

We denote by $\widetilde{C}$ the universal covering of C, and by $\widetilde{Q}$ the universal covering of Q. Setting $\widetilde{Q}_\epsilon = \epsilon-$tubular neighborhood of $\widetilde{Q}$.

We give to $\widetilde{Q}_\epsilon$ the induced metric from $\mathbb{S}^3_\tau$. Again, following the ideas from the case $E^3(\kappa,\tau) \neq \mathbb{S}^3_\tau$, we denoted by M the portion of Σ, which is the graph of the function g, and by $\widetilde{M}$ the universal cover of M.

By construction $\widetilde{M} \subset \widetilde{Q}_\epsilon$, the boundary of $\widetilde{M}$, $\partial\widetilde{M}$ is far away from $\widetilde{Q}$ and $\widetilde{M}$ is asymptotic to $\widetilde{Q}$.

Now we apply the same argument as for the first case to conclude that such Σ cannot exists.

□

By considering that on the case $E^3(\kappa,\tau) \neq \mathbb{S}^3_(\tau)$ as well as on the case $E^3(\kappa,\tau) = \mathbb{S}^3_\tau$ such H multi-graph Σ cannot exists when $\nu < 0$. We conclude the proof of the theorem. □