5
The space $\widetilde{\mathbf{PSL}}_\mathbf{2}(\mathbb{R},\tau)$

In this chapter we focus our attention in the study of the geometry of the space $\widetilde{PSL}_2(\mathbb{R},\tau)$. We wish to study the H-surfaces, which are invariant by one-parameter group of isometries.

It will be make by understanding the isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ as well as the relationship between the space $\widetilde{PSL}_2(\mathbb{R},\tau)$ and the hyperbolic space.

The space $\widetilde{PSL}_2(\mathbb{R},\tau)$ is a simply connected homogeneous manifold whose isometry groups has dimension 4, such a manifold is a Riemannian submersion over the 2-dimensional hyperbolic space, which we will denote by $M^2 = M^2(-1)$. Thus, we only consider the case when the Gaussian curvature is equal to -1, that is, $\kappa = -1$.

Throughout this chapter, we will follow the ideas of Eric Toubiana, see (22). Recall that, we have,

$$\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \longrightarrow M^2$$

the Riemannian submersion; for $p \in M^2$, the fibers $\pi^{-1}(p)$ are geodesics and there exists a one-parameter family of translations along the fibers, generated by the unit Killing field E_3.

5.1
Isometries of $\widetilde{\mathbf{PSL}}_\mathbf{2}(\mathbb{R},\tau)$

Since there exist a Riemannian submersion $\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \longrightarrow M^2$, the isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ are strongly related with the isometries of the hyperbolic space M^2.

From now on we identify the Euclidean space $\mathbb{R}^2$ with the set of complex numbers $\mathbb{C}$, more precisely $z = x + iy \approx (x, y)$.

So, if we take $M^2 \equiv \mathbb{D}^2$, we obtain:

$$\widetilde{PSL}_2(\mathbb{R}) = \{(z,t) \in \mathbb{R}^3; x^2 + y^2 < 1, t \in \mathbb{R}\}$$

endowed with metric,

$$d\sigma^2 = \lambda^2(z)|dz|^2 + (i\tau\lambda(\overline{z}dz - zd\overline{z}) + dt)^2$$

If we take $M^2 \equiv \mathbb{H}^2$, we obtain:

$$\widetilde{PSL}_2(\mathbb{R}) = \{(z,t) \in \mathbb{R}^3; y > 0, t \in \mathbb{R}\}$$

endowed with metric,

$$d\sigma^2 = \lambda^2(z)|dz|^2 + (-\tau\lambda(dz + d\overline{z}) + dt)^2$$

Remark 5.1.1. *Let F an isometry of $\widetilde{PSL}_2(\mathbb{R},\tau)$. As*

$$\pi : \widetilde{PSL}_2(\mathbb{R},\tau) \longrightarrow M^2$$

is a Riemannian submersion, we can write F in the form $F(z,t) = (f(z), h(z,t))$, where

$$f : M^2 \longrightarrow M^2$$

is an isometry of the hyperbolic space M^2.

To see this, we take $p \in M^2$ and u, $v \in T_pM^2$. Denotes by $\overline{u}$, $\overline{v}$ the horizontal lifts of u and v respectively. Let $\overline{p}$ be the point over the fiber $\pi^{-1}(p)$ such that $\overline{u}$, $\overline{v} \in T_{\overline{p}}\widetilde{PSL}_2(\mathbb{R},\tau)$.

Denoting by $dF(\overline{u}) = \overline{a}$ and $dF(\overline{v}) = \overline{b}$, then $\overline{a}$ and $\overline{b}$ are horizontal vectors at $T_{F(\overline{p})}\widetilde{PSL}_2(\mathbb{R},\tau)$, hence $d\pi(\overline{a}) = a \in T_q\mathbb{M}^2$ and $d\pi(\overline{b}) = b \in T_q\mathbb{M}^2$, where $q = \pi(F(\overline{p}))$.

Now, consider f an isometry of M^2, such that, $f(p) = q$, $df(u) = a$, and $df(v) = b$. Then,

$$\begin{aligned}
\langle d\pi(dF(\overline{u})), d\pi(dF(\overline{v}))\rangle_{\widetilde{PSL}_2(\mathbb{R},\tau)} &= \langle d\pi(\overline{a}), d\pi(\overline{b})\rangle_{\widetilde{PSL}_2(\mathbb{R},\tau)} \\
&= \langle a, b\rangle_{M^2} \\
&= \langle u, v\rangle_{M^2}. \\
&= \langle df(a), df(b)\rangle_{M^2}
\end{aligned}$$

the least equality holds since that $f : M^2 \to M^2$, $f(p) = q$ is an isometry, so we have $F(z,t) = (f(z), h(z,t))$.

Proposition 5.1.1. *The isometries of $\widetilde{PSL}_2(\mathbb{R},\tau)$ are given by,*
In the half-plane model for the hyperbolic space M^2

$$F(z,t) = (f(z), t - 2\tau \arg f' + c)$$

or

$$G(z,t) = (-\overline{f}(z), -t + 2\tau \arg f' + c)$$

where f is a positive isometry of $\mathbb{H}^2$ and c is a real number.
In the disk model,

$$F(z,t) = (f(z), t - 2\tau \arg f' + c)$$

or

$$G(z,t) = (\overline{f}(z), -t + 2\tau \arg f' + c)$$

where f is a positive isometry of $\mathbb{D}^2$ and c is a real number.

Proof. We will consider the half-plane model. The proof for the disk model is analogous. As F is a isometry of $\widetilde{PSL}_2(\mathbb{R},\tau)$, we must have,

$$F^*\left(\lambda^2(z)|dz|^2 + (-\tau\lambda(z)(dz + d\overline{z}) + dt)^2\right) = \lambda^2(z)|dz|^2 + (-\tau\lambda(z)(dz + d\overline{z}) + dt)^2$$

since,

$$f^*(\lambda^2|dz|^2) = \lambda^2(z)|dz|^2$$

we have

$$F^*(-\tau\lambda(z)(dz + d\overline{z}) + dt) = \pm(-\tau\lambda(z)(dz + d\overline{z}) + dt).$$

We suppose first that

$$F^*(-\tau\lambda(z)(dz + d\overline{z}) + dt) = +(-\tau\lambda(z)(dz + d\overline{z}) + dt). \tag{5-1}$$

If f is a positive isometry of $\mathbb{H}^2$, then

$$(5-1) \Leftrightarrow -\tau\lambda(f(z))(df(z) + d\overline{f(z)}) + dh = -\tau\lambda(z)(dz + d\overline{z}) + dt$$

$$\Leftrightarrow -\tau\lambda(f(z))(f'dz + \overline{f}'d\overline{z}) + dh = -\tau\lambda(z)(dz + d\overline{z}) + dt$$

$$\Leftrightarrow -\tau\lambda(f(z))(f'dz + \overline{f}'d\overline{z}) + h_z dz + h_{\overline{z}} d\overline{z} + h_t dt = -\tau\lambda(z)(dz + d\overline{z}) + dt$$

$$\Leftrightarrow \begin{cases} -\tau\lambda(f(z))f' + h_z = -\tau\lambda(z); \\ -\tau\lambda(f(z))\overline{f}' + h_{\overline{z}} = -\tau\lambda(z); \\ h_t = 1; \end{cases}$$

$$\Leftrightarrow \begin{cases} -\tau\lambda(f(z))f' + h_z = -\tau\lambda(z); \\ h_t = 1; \end{cases}$$

Consequently, h is a function of the form $h(z,t)=\varphi(z)+t$, where φ is a real function that verifies,

$$\begin{aligned}\varphi_z(z) &= \tau(\lambda(f(z))f'-\lambda(z))\\ &= \tau\left[\frac{2if'}{f-\overline{f}}-\frac{2i}{z-\overline{z}}\right]\\ &= 2\tau i[\log(f-\overline{f})-\log(z-\overline{z})]_z\end{aligned}$$

$$\Leftrightarrow \varphi = 2\tau i\log\left(\frac{f-\overline{f}}{z-\overline{z}}\right)+\overline{\psi}$$

where $\overline{\psi}$ is a holomorphic function. By other hand, if f is a positive isometry of $\mathbb{H}^2$, then

$$f(z)=\frac{az+b}{cz+d},\qquad a,b,c,d\in\mathbb{R},\qquad ad-bc=1.$$

A simple computation gives,

$$\frac{f-\overline{f}}{z-\overline{z}}=\frac{1}{|cz+d|^2}=|f'(z)|$$

so, we obtain,

$$\varphi=2\tau i\arg|f'(z)|+\overline{\psi}$$

as ψ is holomorphic and φ is a real function, we must have

$$\psi=2\tau i\log(f')+c$$

where c is a constant real. So we conclude that

$$\varphi=-2\tau\arg(f'(z))+c.$$

Thus

$$h(z,t)=t-2\tau\arg(f'(z))+c.$$

Now let f be a negative isometry, that is $f=-\overline{g}$ where g is a positive isometry of $\mathbb{H}^2$. Thus, we have,

$$(5-1)\Leftrightarrow -\tau\lambda(f(z))(df(z)+d\overline{f(z)})+dh=-\tau\lambda(z)(dz+d\overline{z})+dt$$

$$\Leftrightarrow \tau\lambda(f(z))(d\overline{g}(z)+d\overline{g(z)})+dh=-\tau\lambda(z)(dz+d\overline{z})+dt$$

$$\Leftrightarrow \tau\lambda(f(z))(\overline{g}'d\overline{z}+g'd\overline{z})+h_zdz+h_{\overline{z}}d\overline{z}+h_tdt=-\tau\lambda(z)(dz+d\overline{z})+dt$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = -\tau\lambda(z); \\ \tau\lambda(f(z))\overline{g}' + h_{\overline{z}} = -\tau\lambda(z); \\ h_t = 1; \end{cases}$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = -\tau\lambda(z); \\ h_t = 1; \end{cases} .$$

Again h is of the form $h(z,t) = \varphi(z)+t$, where φ is a real function that verifies,

$$\begin{aligned} \varphi_z &= -\tau(\lambda(f(z))g' + \lambda(z)) \\ &= -\tau\left[\frac{2ig'}{g-\overline{g}} + \frac{2i}{z-\overline{z}}\right] \\ &= -2\tau i\left[\log(g-\overline{g}) + \log(z-\overline{z})\right]_z \end{aligned}$$

so,

$$\varphi = -2\tau i[\log(g-\overline{g}) + \log(z-\overline{z})] + \overline{\psi}$$

where ψ is holomorphic. Since φ is real, this implies that $[\log(g-\overline{g})+\log(z-\overline{z})]$ is harmonic, which is false. Thus f must be a positive function.

Now, we suppose that F verify,

$$F^*(-\tau\lambda(z)(dz+d\overline{z})+dt) = -(-\tau\lambda(z)(dz+d\overline{z})+dt) \tag{5-2}$$

Again, we consider a negative isometry, that is, we consider $f = -\overline{g}$ where g is a positive isometry of $\mathbb{H}^2$, so

$$(5-2) \Leftrightarrow -\tau\lambda(f(z))(df(z) + d\overline{f(z)}) + dh = -(-\tau\lambda(z)(dz+d\overline{z})+dt)$$

$$\Leftrightarrow -\tau\lambda(f(z))(-d\overline{g}(z) - d\overline{g(z)}) + dh = -(-\tau\lambda(z)(dz+d\overline{z})+dt)$$

$$\Leftrightarrow \tau\lambda(f(z))(\overline{g}'d\overline{z} + g'd\overline{z}) + dh = -(-\tau\lambda(z)(dz+d\overline{z})+dt)$$

$$\Leftrightarrow \tau\lambda(f(z))(\overline{g}'d\overline{z} + g'd\overline{z}) + h_z dz + h_{\overline{z}}d\overline{z} + h_t dt = \tau\lambda(z)(dz+d\overline{z}) - dt$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = \tau\lambda(z); \\ \tau\lambda(f(z))\overline{g}' + h_{\overline{z}} = \tau\lambda(z); \\ h_t = -1; \end{cases}$$

$$\Leftrightarrow \begin{cases} \tau\lambda(f(z))g' + h_z = \tau\lambda(z); \\ \\ h_t = -1; \end{cases}$$

Again h is of the form $h(z,t) = \varphi(z) - t$, where φ is a real function verify

$$\begin{aligned} \varphi_z &= -\tau\left[\lambda(f(z))g' - \lambda(z)\right] \\ &= -\tau\left[\frac{2ig'}{g-\overline{g}} - \frac{2i}{z-\overline{z}}\right] \\ &= -2\tau i\left[\log(g-\overline{g}) - \log(z-\overline{z})\right]_z \end{aligned}$$

so,

$$\varphi = -2\tau i \log\left(\frac{g-\overline{g}}{z-\overline{z}}\right) + \overline{\psi}$$

where $\overline{\psi}$ is a holomorphic function. On the other hand, making the same calculation as above, we obtain

$$\varphi = -2\tau i \log|g'(z)| + \overline{\psi}$$

as ψ is holomorphic and φ is a real function, we must have

$$\psi = -2\tau i \log(g') + c$$

where c is a constant real. So we conclude that

$$\varphi = 2\tau \arg(g'(z)) + c$$

Thus

$$h(z,t) = -t + 2\tau \arg(f'(z)) + c$$

Finally, in this case, if we consider an isometry positive f we get a contradiction. □

5.2
The mean curvature equation in $\widetilde{\mathbf{PSL_2}}(\mathbb{R},\tau)$

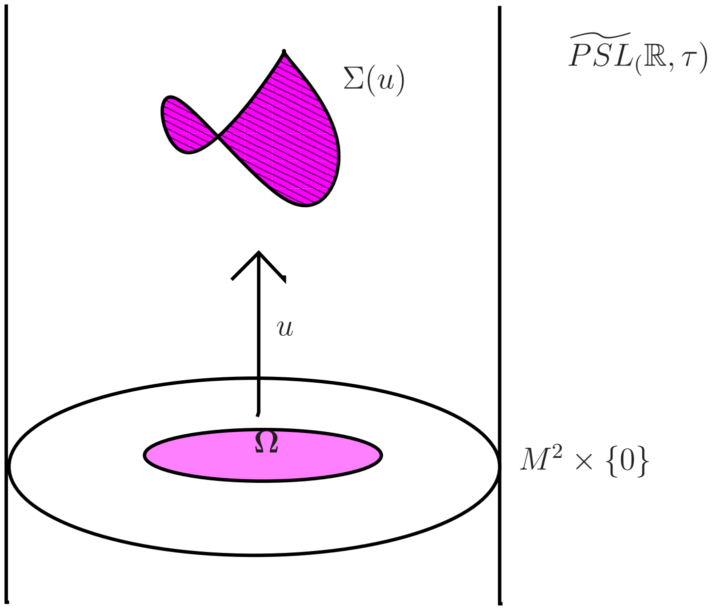
In this section we will explore the equation of the mean curvature in the divergence form.

Recall that, for a Riemannian submersion we have the notion of graph.

Definition 5.2.1. *A graph in* $\widetilde{PSL}_2(\mathbb{R},\tau)$ *over a domain* Ω *of* M^2 *is the image of a section* $s_0 : \Omega \subset M^2 \longrightarrow \widetilde{PSL}_2(\mathbb{R},\tau)$.

Given a domain $\Omega \subset M^2$ we also denote by Ω its lift to $M^2 \times \{0\}$, with this identification we have that the graph $\Sigma(u)$ of $u \in (C^0(\partial\Omega) \cap C^\infty(\Omega))$ is given by (see figure),

$$\Sigma(u) = \{(x, y, u(x, y)) \in \widetilde{PSL}_2(\mathbb{R}, \tau); (x, y) \in \Omega\}$$



Lemma 5.2.1. *Let $\Sigma(u)$ be a graph in $\widetilde{PSL}_2(\mathbb{R}, \tau)$ of the function*

$$u : \Omega \subset M^2 \longrightarrow \mathbb{R}$$

having constant mean curvature H. Then, the function u satisfies the equation

$$2H = div_{M^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right),$$

where $W = \sqrt{1 + \alpha^2 + \beta^2}$ and,

- $\alpha = \dfrac{u_x}{\lambda} + 2\tau\dfrac{\lambda_y}{\lambda^2}$,
- $\beta = \dfrac{u_y}{\lambda} - 2\tau\dfrac{\lambda_x}{\lambda^2}$.

Proof. The proof follows directly from Lema 4.1.2. □

Taking into account the notations of the Lemma 5.2.1 we obtain the next Proposition.

Proposition 5.2.1. *By expanding the equation of the mean curvature from the equation of divergence form, we obtain*

$$2H\lambda^2W^3 = \lambda\alpha_x(1+\beta^2)+\lambda\beta_y(1+\alpha^2)-\lambda\alpha\beta(\alpha_y+\beta_x)+(\lambda_x\alpha+\lambda_y\beta)W^2 \quad (5\text{-}3)$$

where, $W^2 = 1+\alpha^2+\beta^2$, *and*

- $\alpha_x = \frac{1}{\lambda^2}\left[u_{xx}\lambda - u_x\lambda_x + \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{xy} - 2\lambda\lambda_x\lambda_y)\right]$
- $\alpha_y = \frac{1}{\lambda^2}\left[u_{xy}\lambda - u_x\lambda_y + \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{yy} - 2\lambda\lambda_y^2)\right]$
- $\beta_x = \frac{1}{\lambda^2}\left[u_{xy}\lambda - u_y\lambda_x - \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{xx} - 2\lambda\lambda_x^2)\right]$
- $\beta_y = \frac{1}{\lambda^2}\left[u_{yy}\lambda - u_y\lambda_y - \frac{2\tau}{\lambda^2}(\lambda^2\lambda_{xy} - 2\lambda\lambda_x\lambda_y)\right]$

Proof. Since M^2 has conformal metric to $\mathbb{R}^2$ we can use the formula,

$$div_{M^2}\left(\frac{\alpha}{\lambda W}\partial_x + \frac{\beta}{\lambda W}\partial_y\right) = \frac{1}{\lambda^2}div_{\mathbb{R}^2}\left(\lambda^2\left(\frac{\alpha}{\lambda W}\partial_x + \frac{\beta}{\lambda W}\partial_y\right)\right)$$

we obtain,

$$\begin{aligned}
2H\lambda^2 &= div_{\mathbb{R}^2}\left(\frac{\lambda}{W}(\alpha\partial_x + \beta\partial_y)\right)\\
&= \left(\left(\frac{\lambda}{W}\right)_x\partial_x + \left(\frac{\lambda}{W}\right)_y\partial_y\right)(\alpha\partial_x+\beta\partial_y) + \\
&\quad + \frac{\lambda}{W}div_{\mathbb{R}^2}(\alpha\partial_x + \beta\partial_y)\\
&= \alpha\left(\frac{\lambda}{W}\right)_x + \beta\left(\frac{\lambda}{W}\right)_y + \frac{\lambda}{W}(\alpha_x+\beta_y)
\end{aligned}$$

Observe that:

$$\begin{aligned}
\left(\frac{\lambda}{W}\right)_x &= \frac{\lambda_x W - \lambda W_x}{W^2}\\
\left(\frac{\lambda}{W}\right)_y &= \frac{\lambda_y W - \lambda W_y}{W^2}
\end{aligned}$$

and

$$\begin{aligned}
w_x &= \frac{\alpha\alpha_x + \beta\beta_x}{W}\\
w_y &= \frac{\alpha\alpha_y + \beta\beta_y}{W}
\end{aligned}$$

by substitution, we obtain:

$$\begin{aligned}
2H\lambda^2 &= \frac{\alpha}{W^2}\left(\lambda_x W - \frac{\lambda}{W}(\alpha\alpha_x + \beta\beta_x)\right) + \frac{\beta}{W^2}\left(\lambda_y W - \frac{\lambda}{W}(\alpha\alpha_y+\beta\beta_y)\right)\\
&\quad + \frac{\lambda}{W^3}(\lambda_x W^2 + \beta_y W^2)
\end{aligned}$$

which we be write in the form:

$$\begin{aligned} 2H\lambda^2W^3 &= \alpha\lambda_x W^2 - \lambda\alpha(\alpha\alpha_x + \beta\beta_x) + \beta\lambda_y W^2 + \\ &\quad +\lambda(\alpha_x W^2 + \beta_y W^2) - \lambda\beta(\alpha\alpha_y + \beta\beta_y) \end{aligned}$$

remember that $W^2 = 1 + \alpha^2 + \beta^2$, so

$$\begin{aligned} 2H\lambda^2W^3 &= \alpha\lambda_x(1+\alpha^2+\beta^2) - \lambda\alpha^2\alpha_x - \lambda\alpha\beta\beta_x + \beta\lambda_y(1+\alpha^2+\beta^2) + \\ &\quad -\lambda\alpha\beta\alpha_y - \lambda\beta^2\beta_y + \lambda\alpha_x(1+\alpha^2+\beta^2) + \lambda\beta_y(1+\alpha^2+\beta^2) \\ &= \lambda\alpha_x(1+\beta^2) + \lambda\beta_y(1+\alpha^2) - \lambda\alpha\beta(\alpha_y+\beta_x) + \\ &\quad +\lambda_x\alpha(1+\alpha^2+\beta^2) + \lambda_y\beta(1+\alpha^2+\beta^2) \end{aligned}$$

This gives,

$$2H\lambda^2W^3 = \lambda\alpha_x(1+\beta^2) + \lambda\beta_y(1+\alpha^2) - \lambda\alpha\beta(\alpha_y+\beta_x) + (\lambda_x\alpha + \lambda_y\beta)W^2$$

A simple computation gives the other expressions.

□

An immediate consequence from Proposition 5.2.1 is the next corollary. Here we are considering the half-plane model for the hyperbolic space, that is $M^2 \equiv \mathbb{H}^2$.

Corollary 5.2.1. *By expanding the mean curvature equation of the divergence form in* $\widetilde{PSL}_2(\mathbb{R},\tau)$*, we obtain:*

$$\begin{aligned} 2H\lambda^2m^3 &= u_{xx}(\lambda^3 + \lambda u_y^2) + u_{yy}\lambda(\lambda^2 + (u_x - 2\tau\lambda)^2) - 2u_{xy}\lambda(u_x - 2\tau\lambda)u_y + \\ &\quad -u_x u_y \lambda^2(u_x - 2\tau\lambda) - \lambda^2 u_y^3 \end{aligned}$$

where $m = \sqrt{\lambda^2 + (2\tau\lambda - u_x)^2 + u_y^2}$

Proof. Since $\lambda = \dfrac{1}{y}$, so $\lambda_x \equiv 0$. The equation from the Proposition 5.2.1 becomes:

$$\begin{aligned} 2H\lambda^2W^3 &= \lambda\alpha_x(1+\beta^2) + \lambda\beta_y(1+\alpha^2) - \lambda\alpha\beta(\beta_x+\alpha_y) + \\ &\quad -\lambda^2(\beta + \alpha^2\beta + \beta^3) \end{aligned}$$

By considering:

$$\alpha = \frac{u_x}{\lambda} - 2\tau, \quad \beta = \frac{u_y}{\lambda}, \quad \lambda = \frac{1}{y}$$

$$\alpha_x = \frac{u_{xx}}{\lambda}, \qquad \lambda_y = u_x + \frac{u_{xy}}{\lambda}$$
$$\beta_x = \frac{u_{xy}}{\lambda}, \qquad \lambda_y = u_y + \frac{u_{xy}}{\lambda}$$

we obtain:

$$\begin{aligned}
2H\lambda^2W^3 &= u_{xx}\left(1+\frac{u_y^2}{\lambda^2}\right) + (\lambda u_y + u_{yy})\left(1+\frac{(u_x-2\tau\lambda)^2}{\lambda^2}\right) + \\
&\quad -\lambda\left(\frac{u_x}{\lambda}-2\tau\right)\left(\frac{u_y}{\lambda}\right)\left(u_x + 2\frac{u_{xy}}{\lambda}\right) - \lambda^2\left(\frac{u_y}{\lambda}+\frac{u_y}{\lambda}\left(\frac{u_x}{\lambda}-2\tau\right)^2+\frac{u_y^3}{\lambda^3}\right) \\
&= u_{xx}\left(\frac{\lambda^2+u_y^2}{\lambda^2}\right) + u_{yy}\left(\frac{\lambda^2+(u_x-2\tau\lambda)^2}{\lambda^2}\right) + \\
&\quad +\lambda u_y\left(\frac{\lambda^2+(u_x-2\tau\lambda)^2}{\lambda^2}\right) - 2u_{xy}\left(\frac{u_x-2\tau\lambda}{\lambda}\right)\frac{u_y}{\lambda} \\
&\quad -u_xu_y\left(\frac{u_x-2\tau\lambda}{\lambda}\right) - \lambda^2\left(\frac{u_y}{\lambda}+\frac{u_y}{\lambda}\left(\frac{(u_x-2\tau\lambda)^2}{\lambda^2}\right)+\frac{u_y^3}{\lambda^3}\right).
\end{aligned}$$

Since $\lambda^3W^3 = m^3$, where $m = \sqrt{\lambda^2+(u_x-2\tau\lambda)^2+u_y^2}$ we obtain:

$$\begin{aligned}
2H\lambda^2m^3 &= u_{xx}(\lambda^3+\lambda u_y^2) + u_{yy}\lambda(\lambda^2+(u_x-2\tau\lambda)^2) + \\
&\quad -2u_{xy}\lambda(u_x-2\tau\lambda)u_y - u_xu_y\lambda^2(u_x-2\tau\lambda) + \\
&\quad -\lambda^5\left(\frac{u_y}{\lambda}+\frac{u_y}{\lambda}\frac{(u_x-2\tau\lambda)^2}{\lambda^2}+\frac{u_y^3}{\lambda^3}\right) + \lambda^2u_y(\lambda^2+(u_x-2\tau\lambda)^2) \\
&= u_{xx}(\lambda^3+\lambda u_y^2) + u_{yy}\lambda(\lambda^2+(u_x-2\tau\lambda)^2) - 2u_{xy}\lambda(u_x-2\tau\lambda)u_y + \\
&\quad -u_xu_y\lambda^2(u_x-2\tau\lambda) - \lambda^2u_y^3
\end{aligned}$$

This complete the proof.

□

On the other hand, by considering the coefficients of the first and second fundamental form of a surface immersed into $\widetilde{PSL}_2(\mathbb{R},\tau)$ we can obtain the mean curvature equation. For example, taking the half plane model M^2 for the hyperbolic space. Taking graphs of the form $t = u(x,y)$, where u is a smooth function. Setting $S = graf(u) \in \widetilde{PSL}_2(\mathbb{R})$, and parameterizing S by

$$\varphi(x,y) = (x,y,u(x,y))$$

The coordinate global frame field to the graph is given by,

$$\begin{cases} \varphi_x = \partial_x + u_x\partial_t = \lambda E_1 + (u_x - 2\tau\lambda)E_3; \\ \\ \varphi_y = \partial_y + u_y\partial_t = \lambda E_2 + u_y E_3; \end{cases}$$

So the normal vector is given by,

$$N = \frac{1}{\sqrt{(u_x - 2\tau\lambda)^2 + \lambda^2 + u_y^2}}[-(u_x - 2\tau\lambda)E_1 - u_y E_2 + \lambda E_3]$$

Set,

$$m = m(x,t) = \sqrt{(u_x - 2\tau\lambda)^2 + \lambda^2 + u_y^2}.$$

Lemma 5.2.2. *With the notations above, and denoting by H the mean curvatura of S, then H satisfies*

$$\begin{aligned} 2H\lambda^2 m^3 &= u_{xx}(\lambda^3 + \lambda u_y^2) + u_{yy}\lambda(\lambda^2 + (u_x - 2\tau\lambda)^2) - 2u_{xy}\lambda(u_x - 2\tau\lambda)u_y + \\ &\quad -u_x u_y \lambda^2 (u_x - 2\tau\lambda) - \lambda^2 u_y^3 \end{aligned}$$

Proof. The coefficients of the second fundamental forms are given by,

$$\begin{aligned} b_{11} &= \langle \overline{\nabla}_{\varphi_x}\varphi_x, N\rangle, & g_{11} &= \langle \varphi_x, \varphi_x\rangle \\ b_{12} &= \langle \overline{\nabla}_{\varphi_x}\varphi_y, N\rangle, & g_{12} &= \langle \varphi_x, \varphi_y\rangle \\ b_{22} &= \langle \overline{\nabla}_{\varphi_y}\varphi_y, N\rangle, & g_{22} &= \langle \varphi_y, \varphi_y\rangle \end{aligned}$$

where $\langle .,.\rangle$ is the metric of $\widetilde{PSL}_2(\mathbb{R})$, then H satisfies,

$$2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2} \tag{5-4}$$

It is easily deduce that the connection is given by,

$$\begin{aligned} \overline{\nabla}_{\varphi_x}\varphi_x &= (\lambda^2 - 2\tau\lambda(u_x - 2\tau\lambda))E_2 + u_{xx}E_3 \\ \overline{\nabla}_{\varphi_x}\varphi_y &= (\lambda\tau(u_x - 2\tau\lambda) - \lambda^2)E_1 - \lambda\tau u_y E_2 + (u_{xy} + \lambda^2\tau)E_3 \\ \overline{\nabla}_{\varphi_y}\varphi_y &= 2\tau\lambda u_y E_1 - \lambda^2 E_2 + u_{yy}E_3 \end{aligned}$$

and with this,

$$\begin{aligned} b_{11} &= \lambda u_{xx} - u_y(\lambda^2 - 2\tau\lambda(u_x - 2\tau\lambda)) \\ b_{12} &= \lambda(u_{xy} + \lambda^2\tau) - (u_x - 2\tau\lambda)(\lambda\tau(u_x - 2\tau\lambda) - \lambda^2) + \lambda\tau u_y^2 \\ b_{22} &= \lambda u_{yy} - 2\tau\lambda u_y(u_x - 2\tau\lambda) + \lambda^2 u_y \end{aligned}$$

since,

$$\begin{aligned} g_{11} &= \lambda^2 + (u_x - 2\tau\lambda)^2 \\ g_{12} &= u_y(u_x - 2\tau\lambda) \\ g_{22} &= \lambda^2 + u_y^2 \end{aligned}$$

by substitution this expressions in (5-4), we obtain the lemma. □

5.3 Maximum principle in $\widetilde{\mathbf{PSL_2}}(\mathbb{R},\tau)$

An important criterium in Riemannian Geometry is the maximum principle. There are many books, which study the maximum principle. We enunciate this principle in the next form,

Theorem 5.3.1. *(17, Theorem 3.1)*[**Maximum principle**] *Let* Σ_1 *and* Σ_2 *two convex surfaces in* $\widetilde{PSL}_2(\mathbb{R},\tau)$*, such that* Σ_2 *touch* Σ_1 *at* $p \in \Sigma_1$*, that is* $p \in \Sigma_1 \cap \Sigma_2$*, and suppose that, there is a neighborhood of* p*, such that,* Σ_2 *stay in the mean convex side of* Σ_1*. Denoting by* $\overrightarrow{H}_{\Sigma_1}$*, and* $\overrightarrow{H}_{\Sigma_2}$*, the mean curvature vector field respectively, if:*

$$|\overrightarrow{H}_{\Sigma_1}| = |\overrightarrow{H}_{\Sigma_2}| = cte$$

and

$$\langle \overrightarrow{H}_{\Sigma_1}(p), \overrightarrow{H}_{\Sigma_2}(p)\rangle \geq 0.$$

Then, $\Sigma_1 = \Sigma_2$*. When the intersection point* p *belongs to the boundary of the surfaces, the result holds as well, provided further that the two boundaries are tangent and both are local graphs over a common neighborhood in* $T_p\Sigma_1 = T_p\Sigma_2$*.*

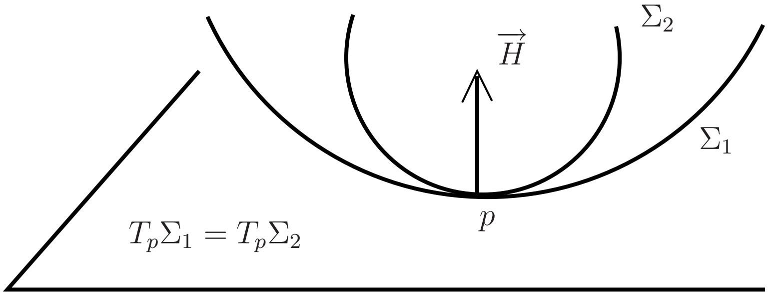


Figure 5.1: Schematic figure for the maximum principle.

The proof of the maximum principle is based on the fact that a constant mean curvature surface in $\widetilde{PSL}_2(\mathbb{R},\tau)$ locally satisfies a second order elliptic

PDE, see Lemma 5.2.2. For the proof of the maximum principle in space forms see (17, Theorem 3.1). The proof generalizes to $\widetilde{PSL}_2(\mathbb{R},\tau)$ as well.

Remark 5.3.1. *As simple application, we will show that, there is no entire* H*-graph* G *in* $\widetilde{PSL}_2(\mathbb{R},\tau)$ *having constant mean curvature* $H > \frac{1}{2}$. *From example 6.2.1, we know that, there are rotational spheres having constant mean curvature* $H > \frac{1}{2}$. *We denote this sphere by* S.

Since the vertical translations are isometries on $\widetilde{PSL}_2(\mathbb{R},\tau)$. *Moving the sphere* S, *such that* S *lies in the mean convex side of* G *and touch* G *at* $p \in S$, *then by using the maximum principle, we obtain* $G \equiv S$, *which is a contradiction.*