6
Rotational surfaces having constant mean curvature in $\widetilde{\mathbf{PSL}}_{\mathbf{2}}(\mathbb{R}, \tau)$

On (16) Ricardo Sa Earp and Eric Toubiana gave explicit formulas for screw motions surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$. There, they gave several examples and on (18) Barbara Nelli, Ricardo Sa Earp, Walcy Santos and Eric Toubiana have studied the geometric behavior of rotational surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$.

We follow these ideas to study the geometric behavior of rotational surfaces having constant mean curvature immersed in $\widetilde{PSL}_2(\mathbb{R}, \tau)$. In this chapter we also give explicit formulas for rotational surfaces having constant mean curvature immersed in $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

In this chapter we focus our attention on rotational surfaces. To study this kind of surfaces, we will take M^2 as being the hyperbolic disk, that is $M^2 = \mathbb{D}^2$. Thus, the metric of M^2 is given by:

$$d\sigma^2 = \lambda^2 |dz|^2, \quad \lambda = \frac{2}{1 - |z|^2}, \quad z = x + iy$$

From Proposition 5.1.1, we know that to obtain a rotational motion on $\widetilde{PSL}_2(\mathbb{R}, \tau)$, it is necessary consider a rotational motion (elliptic isometry) on $\mathbb{D}^2$ together with vertical translations. Thus, our one-parameter group of rotational isometries is given by the composition of elliptic isometries from the hyperbolic disk together with vertical translations.

6.1
Rotational surfaces main lemma

The idea to obtain rotational surfaces is simple, we will take a curve in the xt plane and we will apply one-parameter group Γ of rotational isometries to this curve. We denote by $\alpha(x) = (x, 0, u(x))$ the curve in the xt plane and by $S = \Gamma(\alpha)$, the rotational surface generate by α.

Since the most simple rotational isometry on $\mathbb{D}^2$ is the rotation around the origin, we re-parameterized the hyperbolic disk whit coordinates ρ, θ.

Thus:

$$\begin{aligned} x &= \tanh(\frac{\rho}{2})\cos(\theta) \\ y &= \tanh(\frac{\rho}{2})\sin(\theta) \end{aligned}$$

where ρ is the hyperbolic distance measure from the origin of $\mathbb{D}^2$. By abuse of notation, we still call u the graph of the function u.

Hence, the surface S is parameterized by,

$$\varphi(\rho,\theta) = \left(\tanh\left(\frac{\rho}{2}\right)\cos(\theta), \tanh\left(\frac{\rho}{2}\right)\sin(\theta), u(\rho)\right)$$

Now, we rewrite all the important expressions in this news coordinates.

Lemma 6.1.1. *By considering the above re-parametrization to the hyperbolic disk, we may rewrite all terms in the form:*

(1) $\partial_x = 2\cosh^2(\frac{\rho}{2})\cos(\theta)\partial_\rho - \coth(\frac{\rho}{2})\sin\theta\partial_\theta$

(2) $\partial_y = 2\cosh^2(\frac{\rho}{2})\sin(\theta)\partial_\rho + \coth(\frac{\rho}{2})\cos\theta\partial_\theta$

(3) $\rho_x = 2\cosh^2(\frac{\rho}{2})\cos(\theta)$

(4) $\rho_y = 2\cosh^2(\frac{\rho}{2})\sin(\theta)$

(5) $\theta_x = -\coth(\frac{\rho}{2})\sin(\theta)$

(6) $\theta_y = \coth(\frac{\rho}{2})\cos(\theta)$

(7) $\lambda = 2\cosh^2(\frac{\rho}{2})$

(8) $d\sigma^2 = d\rho^2 + \sinh^2(\rho)d\theta^2$

Proof. From:

$$\begin{cases} x = \tanh(\rho/2)\cos(\theta), \\ y = \tanh(\rho/2)\sin(\theta), \end{cases} \tag{6-1}$$

we obtain:

$$\begin{cases} dx = \frac{1}{2}sech^2(\rho/2)\cos(\theta)d\rho - \tanh(\rho/2)\sin(\theta)d\theta, \\ dy = \frac{1}{2}sech^2(\rho/2)\sin(\theta)d\rho + \tanh(\rho/2)\cos(\theta)d\theta, \end{cases} \tag{6-2}$$

this implies:

$$\begin{aligned} dx^2 + dy^2 &= \frac{1}{4}sech^4(\rho/2)d\rho^2 + \tanh^2(\rho/2)d\theta^2, \\ \lambda &= \frac{2}{1-\tanh^2(\rho/2)} \\ \lambda &= \frac{2}{sech^2(\rho/2)}. \end{aligned}$$

Thus:

$$\begin{aligned} d\sigma^2 &= \lambda^2(dx^2+dy^2) \\ &= \frac{4}{sech^4(\rho/2)}\left(\frac{1}{4}sech^4(\rho/2)d\rho^2+\tanh^2(\rho/2)d\theta^2\right) \\ &= d\rho^2+(2\sinh(\rho/2)\cosh(\rho/2))^2d\theta^2 \\ &= d\rho^2+\sinh^2(\rho)d\theta^2 \end{aligned}$$

So, the equation (8) holds.

Setting

$$\partial_x = a\partial_\rho + b\partial_\theta$$

evaluating this expressions in equation (6-2), we obtain:

$$\begin{cases} 1 = \frac{1}{2}sech^2(\rho/2)\cos(\theta)a - \tanh(\rho/2)\sin(\theta)b, \\ 0 = \frac{1}{2}sech^2(\rho/2)\sin(\theta)a + \tanh(\rho/2)\cos(\theta)b, \end{cases}$$

by solving this system we obtain:

$$a = 2\cosh^2(\rho/2)\cos(\theta), \qquad b = -\coth(\rho/2)\sin(\theta).$$

Thus:

$$\partial_x = 2\cosh^2(\rho/2)\cos(\theta)\partial_\rho - \coth(\rho/2)\sin(\theta)\partial_\theta.$$

Now, by considering:

$$\partial_y = a\partial_\rho + b\partial_\theta$$

evaluating this expression in equation (6-2), we obtain:

$$\begin{cases} 0 = \frac{1}{2}sech^2(\rho/2)\cos(\theta)a - \tanh(\rho/2)\sin(\theta)b, \\ 1 = \frac{1}{2}sech^2(\rho/2)\sin(\theta)a + \tanh(\rho/2)\cos(\theta)b, \end{cases}$$

again, by solving this system, we obtain:

$$a = 2\cosh^2(\rho/2)\sin(\theta), \qquad b = \coth(\rho/2)\cos(\theta).$$

Thus:

$$\partial_y = 2\cosh^2(\rho/2)\sin(\theta)\partial_\rho + \coth(\rho/2)\cos(\theta)\partial_\theta.$$

Hence, the equations (1) and (2)holds.

To obtain equations (3) and (4), observe that, from $x^2+y^2=\tanh^2(\rho/2)$

we obtain:

$$\begin{aligned} 2x &= \tanh(\rho/2)sech^2(\rho/2)\rho_x \\ 2y &= \tanh(\rho/2)sech^2(\rho/2)\rho_y \end{aligned}$$

by considering the equation (6-1), we conclude that:

$$\begin{aligned} \rho_x &= 2\cosh(\rho/2)\cos(\theta) \\ \rho_y &= 2\cosh(\rho/2)\sin(\theta) \end{aligned}$$

Finally, by derivation of equation (6-1) with respect to x and y respectively we have:

$$1 = \frac{1}{2}sech^2(\rho/2)\rho_x\cos(\theta) - \tanh(\rho/2)\sin(\theta)\theta_x$$
$$1 = \frac{1}{2}sech^2(\rho/2)\rho_y\sin(\theta) + \tanh(\rho/2)\cos(\theta)\theta_y$$

by using ρ_x and ρ_y, we obtain:

$$\begin{aligned} \theta_x &= -\coth(\rho/2)\sin(\theta) \\ \theta_y &= \coth(\rho/2)\cos(\theta) \end{aligned}$$

This completes the proof of the lemma.

□

The next Lemma is crucial for our study. Here, we follow the ideas presented by Laurent. Mazet, Magdalena Rodríguez and Harold Rosenberg, on (11, Appendix A).

As we are considering rotational surfaces, our function u dependes only of ρ. Thus, consider the graph $t = u(\rho)$ in the plane xt, and denote by $S = grap(\Gamma u)$, where by abuse of notation, we call u the graph of the function u and Γu simply the rotational surface generated by the one-parameter group of rotational isometries Γ.

Lemma 6.1.2. *(Main lemma) Supposing that S has constant mean curvature H. Then, the function u satisfies*

$$u(\rho) = \int \frac{(2H\cosh(\rho)+d)\sqrt{1+4\tau^2\tanh^2\left(\frac{\rho}{2}\right)}}{\sqrt{\sinh^2(\rho)-(2H\cosh(\rho)+d)^2}}d\rho \qquad (6\text{-}3)$$

where d is a real number.

Observe that, when $\tau \equiv 0$ we are in the space $\mathbb{H}^2 \times \mathbb{R}$. In this case, Ricardo Sa Earp and Eric Toubiana found explicit formulas for rotational surfaces, see (15).

Proof. Since S has mean curvature H, then by lemma 5.2.1 the function u satisfies the equation:

$$2H = div_{\mathbb{D}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right), \tag{6-4}$$

where $W = \sqrt{1+\alpha^2+\beta^2}$, $\alpha = \dfrac{u_x}{\lambda} + 2\tau y$, and $\beta = \dfrac{u_y}{\lambda} - 2\tau x$.

By abuse of notation we consider $u(x,y) = u(\rho(x,y), \theta(x,y))$, since S is a rotational surfaces, then the function u does not depend of θ, so:

$$\begin{cases} u_x = u_\rho \rho_x + u_\theta \theta_x = u_\rho \rho_x, \\ u_y = u_\rho \rho_y + u_\theta \theta_y = u_\rho \rho_y, \end{cases}$$

now, by considering the lemma (6.1.1), we have:

$$\begin{aligned} \frac{u_x}{\lambda} &= u_\rho \frac{2\cosh^2(\rho/2)}{2\cosh^2(\rho/2)}\cos(\theta), \\ \frac{u_y}{\lambda} &= u_\rho \frac{2\cosh^2(\rho/2)}{2\cosh^2(\rho/2)}\sin(\theta). \end{aligned}$$

Thus:

$$\begin{aligned} \alpha &= u_\rho \cos\theta + 2\tau(\tanh(\rho/2)\sin\theta), \\ \beta &= u_\rho \sin(\theta) - 2\tau(\tanh(\rho/2)\cos\theta), \end{aligned}$$

where:

$$\begin{aligned} \alpha^2+\beta^2 &= u_\rho^2 + 4\tau^2(\tanh(\rho/2))^2, \\ W^2 &= 1 + 4\tau^2(\tanh(\rho/2))^2 + u_\rho^2. \end{aligned}$$

Setting $X_u = \dfrac{\alpha}{W}e_1 + \dfrac{\beta}{W}e_2 = \dfrac{1}{W}\left(\alpha\dfrac{\partial_x}{\lambda} + \beta\dfrac{\partial_y}{\lambda}\right)$.

We need express X_u in coordinates ρ and θ. Observe that:

$$\begin{aligned} \frac{\partial_x}{\lambda} &= \frac{1}{2\cosh^2(\rho/2)}(2\cosh^2(\rho/2)\cos\theta\partial_\rho - \coth(\rho/2)\sin\theta\partial_\theta) \\ &= \cos\theta\partial_\rho - \left(\frac{\coth(\rho/2)}{2\cosh^2(\rho/2)}\right)\sin\theta\partial_\theta \end{aligned}$$

and

$$\begin{aligned}\frac{\partial_y}{\lambda} &= \frac{1}{2\cosh^2(\rho/2)}(2\cosh^2(\rho/2)\sin\theta\partial_\rho + \coth(\rho/2)\cos\theta\partial_\theta)\\ &= \sin\theta\partial_\rho + \left(\frac{\coth(\rho/2)}{2\cosh^2(\rho/2)}\right)\cos\theta\partial_\theta.\end{aligned}$$

Setting $A = \dfrac{\coth(\rho/2)}{2\cosh^2(\rho/2)}$, we obtain:

$$\begin{aligned}\alpha\frac{\partial_x}{\lambda} &= [u_\rho\cos\theta + 2\tau(\tanh(\rho/2))\sin\theta](\cos\theta\partial_\rho - A\sin\theta\partial_\theta)\\ &= [u_\rho\cos^2\theta + 2\tau(\tanh(\rho/2))\sin\theta\cos\theta]\partial_\rho +\\ &\quad -A[u_\rho\sin\theta\cos\theta + 2\tau(\tanh(\rho/2))\sin^2\theta]\partial_\theta\end{aligned}$$

and

$$\begin{aligned}\beta\frac{\partial_y}{\lambda} &= [u_\rho\sin\theta - 2\tau(\tanh(\rho/2))\cos\theta](\sin\theta\partial_\rho + A\cos\theta\partial_\theta)\\ &= [u_\rho\sin^2\theta - 2\tau(\tanh(\rho/2))\sin\theta\cos\theta]\partial_\rho +\\ &\quad +A[u_\rho\sin\theta\cos\theta - 2\tau(\tanh(\rho/2))\cos^2\theta]\partial_\theta.\end{aligned}$$

Hence:

$$\begin{aligned}\alpha\frac{\partial_x}{\lambda} + \beta\frac{\partial_y}{\lambda} &= u_\rho\partial_\rho - 2\tau A\tanh(\rho/2)\partial_\theta\\ &= u_\rho\partial_\rho - 2\tau\frac{1}{2\cosh^2(\rho/2)}\partial_\theta\end{aligned}$$

So, we conclude that :

$$X_u = \frac{1}{W}\left[u_\rho\partial_\rho - 2\tau\frac{\tanh(\rho/2)}{\sinh(\rho)}\partial_\theta\right]$$

and

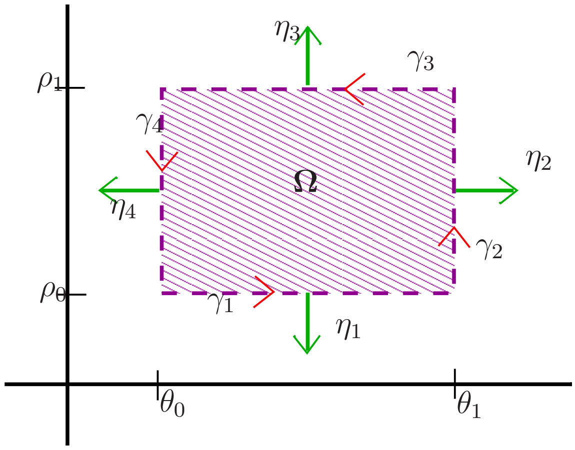
$$W = \sqrt{1 + 4\tau^2\tanh^2(\rho/2) + u_\rho^2}$$

Let $\theta_0, \theta_1 \in (0, 2\pi)$ with $\theta_0 < \theta_1$ and $\rho_0, \rho_1 \in \mathbb{R}_+$, with $\rho_0 < \rho_1$ and consider the domain $\Omega = [\theta_0, \theta_1] \times [\rho_0, \rho_1]$ in the plane $\rho\theta$.

By integrating the equation (6-4), we obtain

$$\int_{\partial(\Omega)} \langle X_u, \eta\rangle = 2H Area([\theta_0, \theta_1] \times [\rho_0, \rho_1])$$

where η is the outer co-normal.



Since

$$\int_{\partial\Omega}\langle X_u,\eta\rangle = \int_{\gamma_1}\langle X_u,\eta_1\rangle + \int_{\gamma_2}\langle X_u,\eta_2\rangle + \int_{\gamma_3}\langle X_u,\eta_3\rangle + \int_{\gamma_4}\langle X_u,\eta_4\rangle$$

Now, we compute each integral:

For the fist integral: Observe that:

$$\gamma_1(s) = (s,\rho_0), \quad \theta_0 \leq s \leq \theta_1.$$

This implies,

$$\gamma_1' = \partial_\theta,$$

so

$$|\gamma_1'| = \sinh\rho,$$

and

$$\eta_1 = -\partial_\rho.$$

Furthermore:

$$\langle X_u,\eta_1\rangle = \frac{-u_\rho(\rho_0)}{W},$$

hence:

$$\int_{\gamma_1}\langle X_u,\eta_1\rangle = \int_{\theta_0}^{\theta_1} -\frac{u_\rho}{W}(\rho_0)\sinh(\rho_0)d\theta$$

For the third integral: Observe that:

$$\gamma_3(s) = (\theta_1 - s,\rho_1), \quad 0 \leq s \leq \theta_1 - \theta_0.$$

This implies:

$$\gamma_3' = -\partial_\theta,$$

so

$$|\gamma_3'| = \sinh\rho,$$

and

$$\eta_3 = \partial_\rho.$$

Furthermore:

$$\langle X_u, \eta_3\rangle = \frac{u_\rho(\rho_1)}{W},$$

thus:

$$\int_{\gamma_3} \langle X_u, \eta_3\rangle = \int_0^{\theta_1-\theta_0} \frac{u_\rho}{W}(\rho_1)\sinh(\rho_1)ds = \int_{\theta_0}^{\theta_1} \frac{u_\rho}{W}(\rho_1)\sinh(\rho_1)d\theta$$

For the second integral: Observe that:

$$\gamma_2(s) = (\theta_1, s), \quad \rho_0 \le s \le \rho_1.$$

This implies:

$$\gamma_2' = \partial_\rho,$$

so:

$$|\gamma_2'| = 1,$$

and

$$\eta_2 = \frac{1}{\sinh(\rho)}\partial_\rho.$$

Furthermore $\langle X_u, \eta_2\rangle = -\dfrac{2\tau\tanh(\rho/2)}{W}$, hence:

$$\int_{\gamma_2} \langle X_u, \eta_2\rangle = \int_{\rho_0}^{\rho_1} -\frac{2\tau\tanh(\rho/2)}{W}d\rho$$

For the four integral: Observe that:

$$\gamma_4(s) = (\theta_0, \rho_1 - s), \quad 0 \le s \le \rho_1 - \rho_0.$$

This implies:

$$\gamma_4' = -\partial_\rho,$$

so:

$$|\gamma_4'| = 1,$$

and

$$\eta_4 = -\frac{1}{\sinh(\rho)}\partial_\rho.$$

Furthermore:

$$\langle X_u, \eta_4\rangle = \frac{2\tau\tanh(\rho/2)}{W},$$

hence:

$$\int_{\gamma_4}\langle X_u, \eta_4\rangle = \int_0^{\rho_1-\rho_0}\frac{2\tau\tanh((\rho_1-s)/2)}{W}ds = \int_{\rho_0}^{\rho_1}\frac{2\tau\tanh(\rho/2)}{W}d\rho.$$

Taking into account this four integrals, we obtain:

$$\int_{\partial\Omega}\langle X_u, \eta\rangle = \int_{\theta_0}^{\theta_1}\frac{u_\rho}{W}(\rho_1)\sinh(\rho_1)d\theta - \int_{\theta_0}^{\theta_1}\frac{u_\rho}{W}(\rho_0)\sinh(\rho_0)d\theta$$

Observe that,

$$\begin{aligned} Area(\Omega) &= \int_{\theta_0}^{\theta_1}\int_{\rho_0}^{\rho_1}\sqrt{\det(g_{ij})}d\rho d\theta \\ &= \int_{\theta_0}^{\theta_1}\int_{\rho_0}^{\rho_1}\sinh(\rho)d\rho d\theta. \end{aligned}$$

Hence, we conclude that:

$$\int_{\theta_0}^{\theta_1}\frac{u_\rho}{W}(\rho_1)\sinh(\rho_1)d\theta - \int_{\theta_0}^{\theta_1}\frac{u_\rho}{W}(\rho_0)\sinh(\rho_0)d\theta = 2H\int_{\theta_0}^{\theta_1}\int_{\rho_0}^{\rho_1}\sinh(\rho)d\rho d\theta.$$

This equality can writes in the form:

$$\int_{\rho_0}^{\rho_1}\partial_\rho\left(\frac{u_\rho}{W}(\rho)\sinh(\rho)\right)d\rho = 2H\int_{\rho_0}^{\rho_1}\sinh(\rho)d\rho.$$

Observe that that Ω is any domain in the plane $\rho\theta$. Taking the derivative with respect to ρ we obtain:

$$\partial_\rho\left(\frac{u_\rho\sinh(\rho)}{W}\right) = 2H\sinh(\rho)$$

by integrating this expression, we obtain:

$$\frac{u_\rho \sinh(\rho)}{\sqrt{1+4\tau^2(\tanh(\rho/2))^2+u_\rho^2}} = 2H\cosh(\rho)+d$$

where $d \in \mathbb{R}$. This implies:

$$u_\rho^2[\sinh^2\rho - (2H\cosh(\rho+d))^2] = (2H\cosh(\rho)+d)^2[1+4\tau^2\tanh^2(\rho/2)]$$

so the function u satisfies:

$$u(\rho) = \int \frac{(2H\cosh(\rho)+d)\sqrt{1+4\tau^2\tanh^2\left(\frac{\rho}{2}\right)}}{\sqrt{\sinh^2(\rho)-(2H\cosh(\rho)+d)^2}}$$

□

Observe that the Lemma 6.1.2 gives the first integral for the mean curvature equation of rotational surfaces. The proof use the fact that the mean curvature is constant.

Remark 6.1.1. *Observe that, the space $PSL_2(\mathbb{R})$ which is the isometry group of the hyperbolic disk can be identified with the unit tangent bundle $T_1\mathbb{D}$ of the hyperbolic disk $\mathbb{D}$. Thus, by using the Sasaki metric we can constructed the space $\widetilde{PSL}_2(\mathbb{R})$. With this natural parametrization, the natural value of τ is $\tau = -\frac{1}{2}$. see (10, Section 1.3)*

On the other hand, it is possible to obtain the first integral for rotational surfaces from the mean curvature equation when H is non-constant. Let $S = grap(\Gamma u)$ be the rotational surfaces like in Lemma 6.1.2, denoting by H the mean curvature of S. Considering $\tau \equiv -\frac{1}{2}$, we have,

Lemma 6.1.3. *The function H satisfies*

$$\left(\frac{u'(\rho)\sinh(\rho)}{\sqrt{1+\tanh^2(\rho/2)+(u')^2}}\right)' = 2H\sinh(\rho)$$

where $\frac{d}{d\rho} = {}'$ denote the derivative with respect to ρ.

Proof. Again, consider the parametrization,

$$\varphi(\rho,\theta) = (\tanh(\rho/2)\cos(\theta), \tanh(\rho/2)\sin(\theta), u(\rho))$$

that is,

$$\begin{cases} x = \tanh(\frac{\rho}{2})\cos(\theta), \\ y = \tanh(\frac{\rho}{2})\sin(\theta), \end{cases}$$

where ρ is the hyperbolic distance measure from the origin of $\mathbb{D}^2$.

Setting $f(\rho) = \tanh(\rho/2)$, thus $f'(\rho) = \dfrac{sech^2(\rho/2)}{2}$, as $\lambda = \dfrac{2}{sech^2(\rho/2)}$, so $\lambda f' = 1$. Observe that

$$\begin{cases} \partial_x = \lambda E_1 - \lambda f \sin\theta E_3, \\ \partial_y = \lambda E_2 + \lambda f \cos\theta E_3, \end{cases}$$

Let φ_ρ, φ_θ, and N, be the adapted tangent vectors and the normal unit to S respectively. Then

$$\begin{cases} \varphi_\rho = \cos\theta E_1 + \sin\theta E_2 + u' E_3, \\ \varphi_\theta = \lambda f[-\sin\theta E_1 + \cos\theta E_2 + f E_3], \\ N = \dfrac{1}{m(\rho)}[(-u'\cos\theta + f\sin\theta)E_1 - (u'\sin\theta + f\cos\theta)E_2 + E_3], \end{cases}$$

where $m(\rho) = \sqrt{1 + f^2 + (u')^2}$.

The coefficients of the second fundamental forms are given by,

$$\begin{aligned} b_{11} &=< \overline{\nabla}_{\varphi_\rho}\varphi_\rho, N >, & g_{11} &=< \varphi_\rho, \varphi_\rho > \\ b_{12} &=< \overline{\nabla}_{\varphi_\rho}\varphi_\theta, N >, & g_{12} &=< \varphi_\rho, \varphi_\theta > \\ b_{22} &=< \overline{\nabla}_{\varphi_\theta}\varphi_\theta, N >, & g_{22} &=< \varphi_\theta, \varphi_\theta > \end{aligned}$$

where $< .,. >$ is the metric of $\widetilde{PSL}_2(\mathbb{R})$, then H satisfies,

$$2H = \frac{b_{11}g_{22} + b_{22}g_{11} - 2b_{12}g_{12}}{g_{11}g_{22} - g_{12}^2} \tag{6-5}$$

By using properties from the connection, we have

$$\begin{aligned} \overline{\nabla}_{\varphi_\rho}\varphi_\rho &= \overline{\nabla}_{\varphi_\rho}[\cos\theta E_1 + \sin\theta E_2 + u' E_3] \\ &= u'' E_3 + \cos\theta[\overline{\nabla}_{\cos\theta E_1 + \sin\theta E_2 + u' E_3} E_1] + \\ &\quad + \sin\theta[\overline{\nabla}_{\cos\theta E_1 + \sin\theta E_2 + u' E_3} E_2] + \\ &\quad + u'[\overline{\nabla}_{\cos\theta E_1 + \sin\theta E_2 + u' E_3} E_3] \end{aligned}$$

$$\begin{aligned} \overline{\nabla}_{\varphi_\rho}\varphi_\rho = {} & u'' E_3 + \cos\theta[\cos\theta\overline{\nabla}_{E_1}E_1 + \sin\theta\overline{\nabla}_{E_2}E_1 + u'\overline{\nabla}_{E_3}E_1] + \\ & + \sin\theta[\cos\theta\overline{\nabla}_{E_1}E_2 + \sin\theta\overline{\nabla}_{E_2}E_2 + u'\overline{\nabla}_{E_3}E_2] + \\ & + u'[\cos\theta\overline{\nabla}_{E_1}E_3 + \sin\theta\overline{\nabla}_{E_2}E_3 + u'\overline{\nabla}_{E_3}E_3] \end{aligned}$$

now, by using the Riemannian connection of the vector fields E_1, E_2 and E_3, we obtain

$$\begin{aligned}\overline{\nabla}_{\varphi_\rho}\varphi_\rho &= u''E_3 + \\ &+\cos\theta\left[\cos\theta(-f\sin\theta E_2)+\sin\theta(f\cos\theta E_2+\frac{1}{2}E_3)+u'\left(\frac{1}{2}E_3\right)\right]+ \\ &+\sin\theta\left[\cos\theta\left(f\sin\theta E_1-\frac{1}{2}E_3\right)+\sin\theta(-f\cos\theta E_1)+u'(-\frac{1}{2}E_3)\right]+ \\ &+u'\left[\cos\theta\left(\frac{1}{2}E_2\right)+\sin\theta\left(-\frac{1}{2}E_1\right)\right]\end{aligned}$$

that is,

$$\overline{\nabla}_{\varphi_\rho}\varphi_\rho = -u'\sin\theta E_1 + u'\cos\theta E_2 + u''E_3$$

Now,

$$\begin{aligned}\overline{\nabla}_{\varphi_\theta}\varphi_\theta &= \overline{\nabla}_{\varphi_\theta}\lambda f[-\sin\theta E_1+\cos\theta E_2+fE_3] \\ &= \lambda f(\overline{\nabla}_{\varphi_\theta}[-\sin\theta E_1+\cos\theta E_2+fE_3])\end{aligned}$$

set $\alpha = \overline{\nabla}_{\varphi_\theta}[-\sin\theta E_1+\cos\theta E_2+fE_3]$, observe that,

$$\begin{aligned}\alpha &= -\cos\theta E_1-\sin\theta E_2+ \\ &\quad -\sin\theta\overline{\nabla}_{\varphi_\theta}E_1+\cos\theta\overline{\nabla}_{\varphi_\theta}E_2+f\overline{\nabla}_{\varphi_\theta}E_3 \\ &= -\cos\theta E_1-\sin\theta E_2+ \\ &\quad +\lambda f[-\sin\theta\left[-\sin\theta\overline{\nabla}_{E_1}E_1+\cos\theta\overline{\nabla}_{E_2}E_1+f\overline{\nabla}_{E_3}E_1\right]+ \\ &\quad +\cos\theta\left[-\sin\theta\overline{\nabla}_{E_1}E_2+\cos\theta\overline{\nabla}_{E_2}E_2+f\overline{\nabla}_{E_3}E_2\right]+ \\ &\quad +\cos\theta\left[-\sin\theta\overline{\nabla}_{E_1}E_2+\cos\theta\overline{\nabla}_{E_2}E_2+f\overline{\nabla}_{E_3}E_2\right]+ \\ &\quad +f\left[-\sin\theta\overline{\nabla}_{E_1}E_3+\cos\theta\overline{\nabla}_{E_2}E_3+f\overline{\nabla}_{E_3}E_3\right]]\end{aligned}$$

by using the Riemannian connection of the vector fields E_1, E_2 and E_3, we obtain

$$\alpha = -(1+2\lambda f^2)(\cos\theta E_1+\sin\theta E_2)$$

that is,

$$\overline{\nabla}_{\varphi_\theta}\varphi_\theta = -\lambda f(1+2\lambda f^2)(\cos\theta E_1+\sin\theta E_2)$$

Now,

$$\begin{aligned}\overline{\nabla}_{\varphi_\theta}\varphi_\rho &= \overline{\nabla}_{\varphi_\theta}(\cos\theta E_1+\sin\theta E_2+u'E_3)\\ &= -\sin\theta E_1+\cos\theta E_2+\lambda f[\\ &\quad+\cos\theta[-\sin\theta\overline{\nabla}_{E_1}E_1+\cos\theta\overline{\nabla}_{E_2}E_1+f\overline{\nabla}_{E_3}E_1]+\\ &\quad+\sin\theta[-\sin\theta\overline{\nabla}_{E_1}E_2+\cos\theta\overline{\nabla}_{E_2}E_2+f\overline{\nabla}_{E_3}E_2]+\\ &\quad+u'[-\sin\theta\overline{\nabla}_{E_1}E_3+\cos\theta\overline{\nabla}_{E_2}E_3+f\overline{\nabla}_{E_3}E_3]]\end{aligned}$$

by using the Riemannian connection of the vector fields E_1, E_2 and E_3, we obtain

$$\begin{aligned}\overline{\nabla}_{\varphi_\theta}\varphi_\rho &= \left[-\sin\theta-\frac{3}{2}\lambda f^2\sin\theta-\frac{u'}{2}\lambda f\cos\theta\right]E_1+\\ &\quad+\left[\cos\theta+\frac{3}{2}\lambda f^2\cos\theta-\frac{u'}{2}\lambda f\sin\theta\right]E_2+\frac{\lambda f}{2}E_3\end{aligned}$$

Analogy, we obtain

$$\begin{aligned}\overline{\nabla}_{\varphi_\rho}\varphi_\theta &= \left[-\sin\theta-\frac{3}{2}\lambda f^2\sin\theta-\frac{u'}{2}\lambda f\cos\theta\right]E_1+\\ &\quad+\left[\cos\theta+\frac{3}{2}\lambda f^2\cos\theta-\frac{u'}{2}\lambda f\sin\theta\right]E_2+\left[\lambda f^3+2f-\frac{\lambda f}{2}\right]E_3\end{aligned}$$

Thus, we obtain the important relation,

$$\lambda f^3+2f-\lambda f=0$$

Observe that,

$$\begin{aligned}m(\rho)b_{11} &= \langle\overline{\nabla}_{\varphi_\rho}\varphi_\rho,\widetilde{N}\rangle\\ &= -u'(-u'\cos(\theta)+f\sin(\theta))+\\ &\quad+u'\cos(\theta)(-u'\sin(\theta)-f\cos(\theta))+u''\\ &= u''-fu'\end{aligned}$$

$$\begin{aligned}m(\rho)b_{22} &= \langle\overline{\nabla}_{\varphi_\theta}\varphi_\theta,\widetilde{N}\rangle\\ &= -\lambda f(1+2f^2\lambda)[\cos(\theta)(-u'\cos(\theta)+f\sin(\theta))+\\ &\quad+\sin(\theta)(-u'\sin(\theta)-f\cos(\theta))]\\ &= u'\lambda f(1+2\lambda f^2)\end{aligned}$$

$$\begin{aligned} m(\rho)b_{12} &= \langle \overline{\nabla}_{\varphi_\rho}\varphi_\theta, \widetilde{N}\rangle \\ &= \frac{\lambda f}{2}(u')^2 - f - \frac{3}{2}\lambda f^3 + \frac{\lambda f}{2} \\ -2m(\rho)b_{12} &= -\lambda f(u')^2 + 2f + 3\lambda f^3 - \lambda f \\ &= 2\lambda f^3 - \lambda f(u')^2 \end{aligned}$$

Since,

$$\begin{aligned} g_{11} &= 1 + (u')^2 \\ g_{22} &= \lambda^2 f^2(1+f^2) \\ g_{12} &= \lambda f^2 u' \end{aligned}$$

we obtain,

$$\begin{aligned} m(\rho)b_{11}g_{22} &= \lambda^2 f^2(1+f^2)u'' + u'(-\lambda^2 f^3 - \lambda^2 f^5) \\ m(\rho)b_{22}g_{11} &= (u')^3(\lambda f + 2\lambda^2 f^3) + u'(\lambda f + 2\lambda^2 f^3) \\ -2m(\rho)b_{12}g_{12} &= -\lambda^2 f^3(u')^3 + 2\lambda^2 f^5 u' \end{aligned}$$

and

$$g_{11}g_{22} - g_{12}^2 = \lambda^2 f^2 m^2$$

so, the equation of the mean curvature is given by,

$$2H = \frac{\lambda f(1+f^2)u'' + u'(\lambda f^4 + \lambda f^2 + 1) + (u')^3(\lambda f^2 + 1)}{\lambda f m^3}$$

that is,

$$2H\lambda f = \left(\frac{\lambda f u'}{\sqrt{1 + f^2 + (u')^2}}\right)'$$

by substitution of f and λ, we obtain the result.

□

6.2 Examples of rotational surfaces in $\widetilde{\mathbf{PSL_2}}(\mathbb{R},\tau)$

Now, we will explore the Lemma 6.1.2. By considering $\tau = -1/2$ we obtain the next consequences:

Lemma 6.2.1. *Setting $d = -2H$, then the integral*

$$u(\rho) = \int \frac{(2H\cosh(\rho) - 2H)\sqrt{1+\tanh^2(\frac{\rho}{2})}}{\sqrt{\sinh^2(\rho) - (2H\cosh(\rho) - 2H)^2}} d\rho$$

has the following solution,

– *If $4H^2 - 1 > 0$ then,*

$$\begin{aligned} u(\rho) &= \frac{4\sqrt{2}H}{\sqrt{4H^2-1}} \left[\arctan\left(\frac{\sqrt{\cosh(\rho)}}{\frac{4H^2+1}{4H^2-1} - \cosh(\rho)} \right) \right] \\ &\quad -2\arctan\left(\frac{\sqrt{\frac{8H^2}{4H^2-1}}\sqrt{\cosh(\rho)}}{\sqrt{\frac{4H^2+1}{4H^2-1} - \cosh(\rho)}} \right) \end{aligned}$$

– *If $1 - 4H^2 > 0$ then,*

$$\begin{aligned} u(\rho) &= \frac{4\sqrt{2}H}{\sqrt{1-4H^2}} \ln\left(\sqrt{\cosh(\rho)} + \sqrt{\frac{1+4H^2}{1-4H^2} + \cosh(\rho)} \right) + \\ &+ 2\arctan\left(-\sqrt{\frac{8H^2}{1-4H^2}} \frac{\sqrt{\cosh(\rho)}}{\sqrt{\frac{1+4H^2}{1-4H^2} + \cosh(\rho)}} \right) \end{aligned}$$

Proof. First, observe that

$$\sinh^2(\rho) - (2H\cosh(\rho) - 2H)^2 = (\cosh(\rho) - 1)((1-4H^2)(\cosh(\rho) - 1) + 2)$$

so, the integral take the next form

$$u(\rho) = 2H \int \frac{(\sqrt{\cosh(\rho) - 1})\sqrt{1+\tanh^2(\frac{\rho}{2})}}{\sqrt{(1-4H^2)(\cosh(\rho)-1)+2}} d\rho$$

Now, we study this integral by considering two cases, either $1 - 4H^2 < 0$ or $1 - 4H^2 < 0$

– If $a = 4H^2 - 1 > 0$, then the integral is writing as

$$u(\rho) = 2H \int \frac{\left(\sqrt{\cosh(\rho) - 1}\right)\sqrt{\frac{2\cosh(\rho)}{\cosh(\rho)+1}}}{\sqrt{(a+2) - a\cosh(\rho)}} d\rho$$

observe that, $1+\tanh^2(\frac{\rho}{2}) = \dfrac{2\cosh(\rho)}{\cosh(\rho)+1}$.
We make change of variable, $v=\cosh(\rho)$, so $dv = \sinh(\rho)d\rho = \sqrt{v^2-1}$, then

$$u = 2\sqrt{2}H\int \frac{\sqrt{v}}{\sqrt{(a+2)-av}}\frac{dv}{(v+1)}.$$

Now, we make $v=w^2$, then $dv = 2wdw$, thus

$$u = 4\sqrt{2}H\int \frac{w^2}{\sqrt{(a+2)-aw^2}}\frac{dw}{(w^2+1)}.$$

Hence:

$$\begin{aligned} u &= 4\sqrt{2}H\int \frac{w^2}{\sqrt{(a+2)-aw^2}}\frac{dw}{(w^2+1)} \\ &= 4\sqrt{2}H\int \frac{w^2+1-1}{\sqrt{(a+2)-aw^2}}\frac{dw}{(w^2+1)} \\ &= 4\sqrt{2}H\left[\int \frac{1}{\sqrt{(a+2)-aw^2}}dw - \int \frac{1}{\sqrt{(a+2)-aw^2}}\frac{dw}{(w^2+1)}\right] \\ &= \frac{4\sqrt{2}H}{\sqrt{a}}\left[\int \frac{1}{\sqrt{b-w^2}}dw - \int \frac{1}{\sqrt{b-w^2}}\frac{dw}{(w^2+1)}\right] \end{aligned}$$

where $b = \dfrac{a+2}{a}$. So the integral is given by,

$$u = \frac{4\sqrt{2}H}{\sqrt{a}}\left[\arctan\left(\frac{w}{\sqrt{b-w^2}}\right) - \frac{\arctan\left(\frac{w\sqrt{1+b}}{\sqrt{b-w^2}}\right)}{\sqrt{1+b}}\right]$$

by substitution of the expressions in terms of ρ, we obtain the result.

– If $a = 4H^2-1>0$, following the same ideas as above, we obtain that the function u satisfies

$$u = \frac{4\sqrt{2}H}{\sqrt{a}}\left[\int \frac{1}{\sqrt{b+w^2}}dw - \int \frac{1}{\sqrt{b+w^2}}\frac{dw}{(w^2+1)}\right]$$

where $b = \dfrac{2-a}{a}$. So the integral is given by,

$$u = \frac{4\sqrt{2}H}{\sqrt{a}}\left[\ln(w+\sqrt{b+w^2}) + \frac{\arctan\left(\frac{(1-b)w}{\sqrt{b-1}\sqrt{b+w^2}}\right)}{\sqrt{b-1}}\right]$$

by substitution of the expressions in terms of ρ, we obtain the result.

□

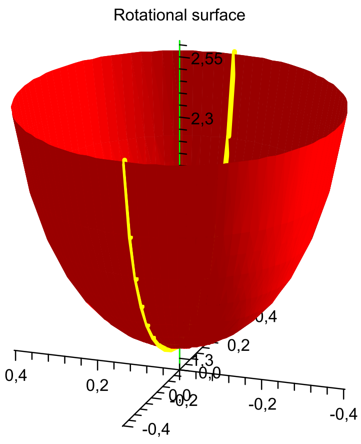
Example 6.2.1. *Making* $H = \sqrt{3}/2$ *in the Lemma 6.2.1, we obtain a rotational surface, which is a graph over a domain in* $\mathbb{D}^2$. *Since the rotation by* π *around the* x *axis is an isometry of* $\widetilde{PSL}_2(\mathbb{R},\tau)$, *the surface is actually a complete embedded rotational surface (called sphere).*

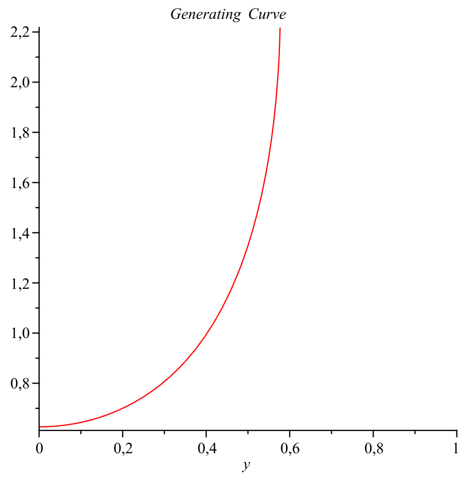
$$u(\rho) = 2\sqrt{3}\arcsin\left(\frac{\sqrt{\cosh(\rho)}}{\sqrt{2}}\right) + 2\arctan\left(\frac{-\sqrt{3}\sqrt{\cosh(\rho)}}{\sqrt{2-\cosh(\rho)}}\right)$$

Observe that, in Euclidean coordinate, we have

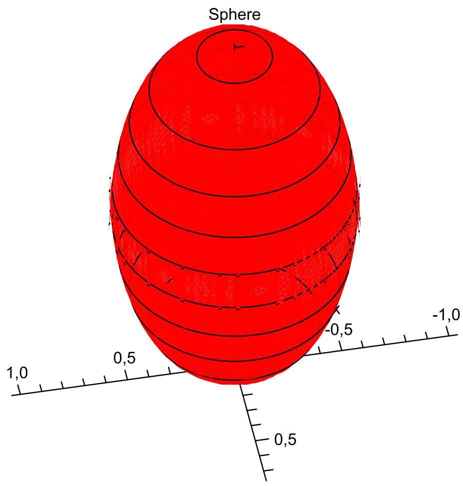
$$\begin{aligned} u(x,y) &= 2\sqrt{3}\arcsin\left(\frac{\sqrt{1+x^2+y^2}}{\sqrt{2(1-(x^2+y^2))}}\right) \\ &\quad +2\arctan\left(\frac{-\sqrt{3}\sqrt{1+x^2+y^2}}{\sqrt{1-(x^2+y^2)}\sqrt{2-\dfrac{1+x^2+y^2}{1-(x^2+y^2)}}}\right) \end{aligned}$$

Observe that, we can verify this formula by using vertical graph. By using Maple the graph is given by:





By considering the rotation around of x, *we obtain a complete sphere.*



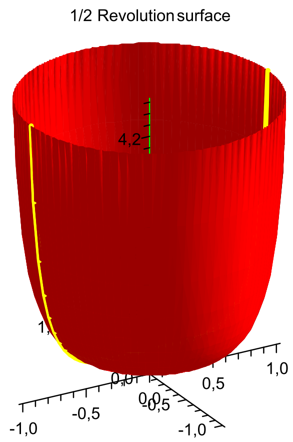
Example 6.2.2. *Making $H = 1/2$ in the Lemma 6.2.1, we obtain an $H = 1/2$ surfaces invariant by rotations in $\widetilde{PSL}_2(\mathbb{R})$ which is an entire graph. More specifically*

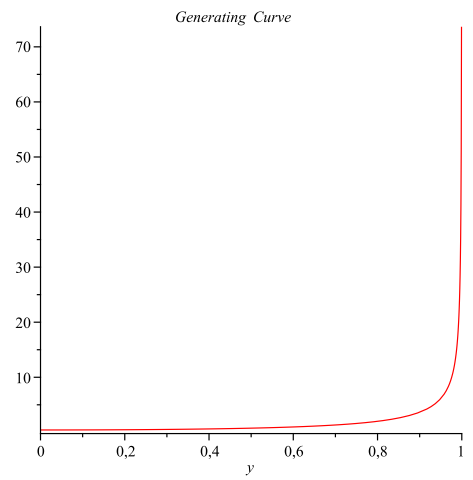
$$u(\rho) = 2\sqrt{\cosh(\rho)} - 2\arctan(\sqrt{\cosh(\rho)})$$

which expressed in Euclidian coordinates gives

$$u(x,y) = 2\sqrt{\cosh(2\tanh^{-1}(\sqrt{x^2+y^2}))} - 2\arctan(\sqrt{\cosh(2\tanh^{-1}(\sqrt{x^2+y^2}))})$$

Maple gives:





6.3 Minimal surfaces invariant by rotations in $\widetilde{\mathbf{PSL_2}}(\mathbb{R},\tau)$

In this section we discuss briefly the behavior of rotational minimal surface, that is when $H \equiv 0$.

Rami Younes gave a first integral for rotational minimal surfaces in $\widetilde{PSL}_2(\mathbb{R})$ (here $\tau \equiv -1/2$, see (23)).

Actually, he gave examples of rotational minimal surfaces as well as minimal surfaces invariant by one-parameter group of parabolic and hyperbolic isometries.

By considering $H \equiv 0$ in the Lemma 6.1.2 we obtain the next proposition.

Proposition 6.3.1. *(Minimal Rotational Surfaces) For each* $d \geq 0$ *there exist a complete minimal rotational surface* $\mathcal{M}_d$*. The surface* $\mathcal{M}_0$ *is the slice* $t = 0$*. For* $d > 0$ *the rotational surface* $\mathcal{M}_d$ *(called catenoid) is embedded and homeomorphic to an annulus.*

Proof. Observe that, the Lemma 6.1.2 gives,

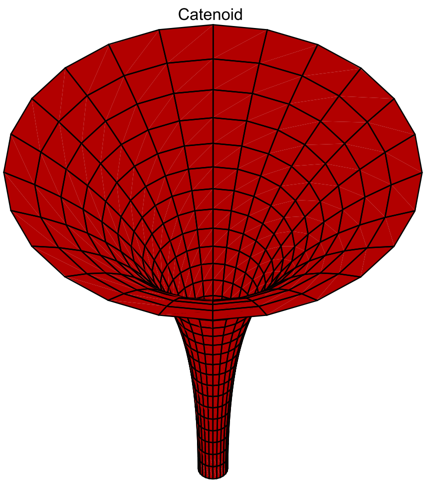
$$u(\rho) = \int_{arcsinh(d)}^{\rho} \frac{d\sqrt{1 + 4\tau^2 \tanh^2\left(\frac{r}{2}\right)}}{\sqrt{\sinh^2(r) - d^2}} dr$$

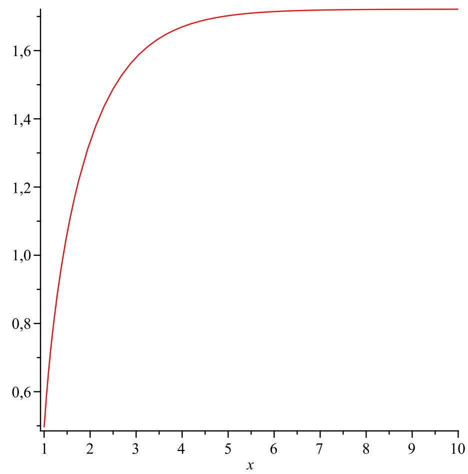
A simple computation gives $u' = \dfrac{d\sqrt{1 + 4\tau^2 \tanh^2\left(\frac{r}{2}\right)}}{\sqrt{\sinh^2(r) - d^2}} > 0$ and

$$u'' = 8\tau^2 \tanh(\rho/2)\sinh(\rho)(1 - \cosh(\rho)) - \frac{8\tau d^2 \tanh^2(\rho/2)}{\sinh(\rho)} - \sinh(2\rho) < 0$$

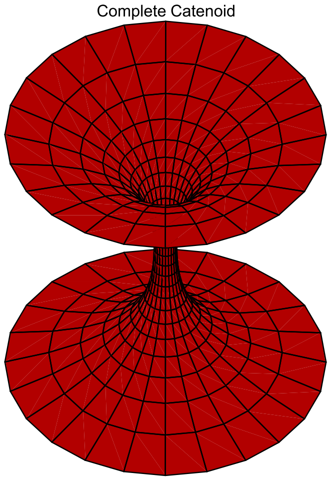
□

Example 6.3.1. *With Maple's help, we plot the catenoid* $\mathcal{M}_1$*. Observe that by considering the rotation by* π *around the* x *axis, we obtain a complete embedded surface.*





Now the complete surface is with Maple's help:



6.4 Height of catenoid

In this section, we follow the ideas presented by Barbara Nelli, Ricardo Sa Earp, Walcy Santos, and Eric Toubiana on (18, Proposition 5.1), to describe the geometric behavior of catenoids in $\widetilde{PSL}_2(\mathbb{R},\tau)$.

Remark 6.4.1. *In* $\mathbb{H}^2\times\mathbb{R}$, *that is, when* $\tau\equiv 0$, *the height of the catenoids is less than* π *and tends to* π, *when* d *tends to* $+\infty$, *see Proposition 6.3.1 for notation.*

More precisely. Let $\widetilde{C}$ *be the complete catenoid in* $\mathbb{H}^2\times\mathbb{R}$, *generated by the function, see (18, Proposition 5.1),*

$$\widetilde{u}(\rho)=\int_{arcsinh(d)}^{\rho}\frac{d}{\sqrt{\sinh^2(r)-d^2}}dr,\qquad r\geq arcsinh(d).$$

By abuse of notation, we still call $\widetilde{u}$, *the graph of* $\widetilde{u}$. *Denote by* $\widetilde{s}$ *the arc-length of* $\widetilde{C}$.

Then, $\widetilde{u}(\rho(\widetilde{s}))$ *is the curve parametrized by arc-length. Thus,*

$$\frac{d}{d\widetilde{s}}\widetilde{u}(\rho(\widetilde{s}))=\widetilde{u}_\rho\rho'(\widetilde{s}) \tag{6-6}$$

This implies,

$$\widetilde{u}(\rho(\widetilde{s}))=\int\widetilde{u}_\rho\rho'(\widetilde{s})d\widetilde{s},$$

making $t=\rho(\widetilde{s})$, *then* $dt=\rho' d\widetilde{s}$. *Hence, we can write,*

$$\widetilde{u}(\rho)=\int_{arcsinh(d)}^{\rho}\frac{d}{\sqrt{\sinh^2(t)-d^2}}dt,\qquad t\geq arcsinh(d).$$

Now, consider the function,

$$\widetilde{h}(d) = 2\int_{arcsinh(d)}^{+\infty} \frac{d}{\sqrt{\sinh^2(t) - d^2}}dt, \quad t \geq arcsinh(d).$$

This function has the following properties. Consider the function,

$$\widetilde{f}(t,d) = \frac{d}{\sqrt{\sinh^2(t) - d^2}}$$

for $t, d \geq 0$. *Observe that,*

- *If* $t_1 < t_2$, *then*
$$\frac{d}{\sqrt{\sinh^2(t_2) - d^2}} < \frac{d}{\sqrt{\sinh^2(t_1) - d^2}}$$

- *Let* $a_n = \dfrac{d}{\sqrt{\sinh^2(n) - d^2}}$, *then*
$$\lim_{n\to+\infty} \mid \frac{a_{n+1}}{a_n} \mid = \frac{1}{\sinh(1) + \cosh(1)} < 1$$

So, the integral $\widetilde{h}(d)$ *is convergent for any* $d \in [d_0, +\infty)$, $d_0 > 0$ *fixed.*

Moreover,

$$\frac{\partial}{\partial d}\widetilde{f}(t,d) = \frac{\sinh^2(t)}{(\sinh^2(t) - d^2)^{3/2}}.$$

We deduce that the integral,

$$\int_{arcsinh(d)}^{+\infty} \frac{\partial}{\partial d}\widetilde{f}(t,d)dt$$

is uniformly convergent for $d \geq d_0 > 0$. *Consequently, the function* $\widetilde{h}(d)$ *is differentiable on* $[d_0, +\infty)$ *and*

$$\widetilde{h}'(d) = 2\int_{arcsinh(d)}^{+\infty} \frac{\sinh^2(t)}{(\sinh^2(t) - d^2)^{3/2}}dt > 0.$$

That is, on (18, Proposition 5.1), the authors showed that, the function $\widetilde{h}(d)$ *satisfies,*

$$\lim_{d\to 0}\widetilde{h}(d) = 0 \quad \textit{and} \quad \lim_{d\to+\infty}\widetilde{h}(d) = \pi$$

This observation, let us obtain an important proposition. Following the notations from Proposition 6.3.1 we have.

Proposition 6.4.1. *The asymptotic boundary of* $\mathcal{M}_d \subset \widetilde{PSL}_2(\mathbb{R},\tau)$ *is two horizontal circles in* $\partial_\infty(\mathbb{D}) \times \mathbb{R}$ *and the vertical distance between them is a nondecreasing function* $h(d)$ *satisfying,*

$$\lim_{d\to 0} h(d) = 0 \quad \textit{and} \quad \lim_{d\to+\infty} h(d) = \sqrt{1+4\tau^2}\pi$$

Proof. Following the ideas of the Remark 6.4.1. We denote by C the complete catenoid in $\widetilde{PSL}_2(\mathbb{R},\tau)$, generated by the function,

$$u(\rho) = \int_{arcsinh(d)}^{\rho} \frac{d\sqrt{1+4\tau^2\tanh^2\left(\frac{r}{2}\right)}}{\sqrt{\sinh^2(r)-d^2}}dr$$

Thus, we consider the function,

$$h(d) = 2\int_{arcsinh(d)}^{+\infty} \frac{d\sqrt{1+4\tau^2\tanh^2\left(\frac{r}{2}\right)}}{\sqrt{\sinh^2(t)-d^2}}dt, \qquad t \geq arcsinh(d).$$

By using the same argument as in the Remark 6.4.1, is possible to show that, $h(d)$ is convergent for any $d \in [d_0, +\infty)$ and $h'(d) > 0$.

Furthermore, observe that,

$$\widetilde{h}(d) < h(d) < \sqrt{1+4\tau^2}\widetilde{h}(d)$$

this implies,

$$\lim_{d\to 0} h(d) = 0 \quad \textit{and} \quad \lim_{d\to+\infty} h(d) \leq \sqrt{1+4\tau^2}\pi \tag{6-7}$$

On the other hand, let $D > 0$ fix, big sufficiently. Let $d > 0$ such that

$$arcsinh(d) > 2arcsinh(D)$$

observe that, $d \to +\infty$ if $D \to +\infty$. This implies,

$$\tanh^2\left(\frac{t}{2}\right) \geq \frac{D^2}{1+D^2} = 1 - \frac{1}{1+D^2}.$$

where, $t \geq arcsinh(d)$. Thus,

$$\sqrt{1+4\tau^2\tanh^2\left(\frac{t}{2}\right)} \geq \sqrt{1+4\tau^2\left(1-\frac{1}{1+D^2}\right)}$$

Hence,

$$\sqrt{1+4\tau^2\left(1-\frac{1}{1+D^2}\right)}\widetilde{h}(d) \leq h(d).$$

Making $D \to +\infty$, we obtain,

$$\sqrt{1+4\tau^2}\pi \leq \lim_{d\to+\infty} h(d) \tag{6-8}$$

From equations (6-7) and (6-8) we obtain,

$$\lim_{d\to+\infty} h(d) = \sqrt{1+4\tau^2}\pi$$

this completes the proof.

□

6.5
Rotational surfaces in $\widetilde{\mathbf{PSL}_2}(\mathbb{R},\tau)$ having constant mean curvature $\mathbf{H} \neq \mathbf{0}$

In this section, we follow the ideas presented by Barbara Nelli, Ricardo Sa Earp, Walcy Santos, and Eric Toubiana on (18, Proposition 5.2,Proposition 5.3), to describe the behavior of rotational H-surfaces. For later use we define the functions $g(\rho)$ and $f(\rho)$:

$$\begin{aligned} g(\rho) &= d + 2H\cosh(\rho) \\ f(\rho) &= \sinh^2(\rho) - (d + 2H\cosh(\rho))^2 \\ &= (1-4H^2)\cosh^2(\rho) - 1 - d^2 \end{aligned}$$

where $d \in \mathbb{R}$ and $H > 0$.

Lemma 6.5.1. *Assume* $0 < H < 1/2$. *We have* $f(\rho) \geq 0$ *if and only if*

$$\cosh\rho \geq \frac{2dH + \sqrt{1-4H^2+d^2}}{1-4H^2}$$

Let $\rho_1 \geq 0$ *such that* $\cosh\rho_1 = \dfrac{2dH + \sqrt{1-4H^2+d^2}}{1-4H^2}$, *then* $f(\rho_1) = 0$ *and* $\rho_1 = 0$ *if and only if* $d = -2H$.

1. *If* $d > -2H$, *then* $\dfrac{-d}{2H} < \cosh\rho_1$. *Consequently the function* u *is strictly increasing for* $\rho \geq \rho_1 > 0$ *and has a nonfinite derivative at* ρ_1.
2. *If* $d = -2H$, *then* $u'(\rho) = \dfrac{2H\sqrt{\cosh\rho - 1}\sqrt{1+4\tau^2\tanh^2(\rho/2)}}{\sqrt{(1-4H^2)\cosh\rho + 4H^2 + 1}}$. *Therefore the function* u *is defined for* $\rho \geq 0$, *it has a zero derivative at* 0 *and is strictly increasing for* $\rho > 0$.

3. *If $d < -2H$, then there exist $\rho_0 > \rho_1 > 0$ such that* $\dfrac{-d}{2H} = \cosh\rho_0$. *Consequently the function u is defined for $\rho \geq \rho_1 > 0$ with a nonfinite derivative at ρ_1, it is strictly decreasing for $\rho_1 < \rho < \rho_0$, has a zero derivative at ρ_0 and it is strictly increasing for $\rho > \rho_0$.*

4. *For any d we have* $\lim_{\rho\to+\infty} u(\rho) = +\infty$.

Proof. Observe that,

$$u(\rho) = \int_*^\rho \frac{(d+2H\cosh r)\sqrt{1+4\tau^2\tanh^2 r/2}}{\sqrt{\sinh^2 r-(d+2H\cosh r)^2}}dr$$

$$u'(\rho) = \frac{g(\rho)\sqrt{1+4\tau^2\tanh^2\rho/2}}{\sqrt{f(\rho)}}$$

and, $f(\rho) \geq 0$ when $\cosh(\rho) \geq \dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}$. The last inequality is possible since

$$\begin{aligned}
\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} &\geq 1 \Longleftrightarrow\\
\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}\times\frac{2dH-\sqrt{1-4H^2+d^2}}{2dH-\sqrt{1-4H^2+d^2}} &\geq 1 \Longleftrightarrow\\
\frac{1+d^2}{\sqrt{1-4H^2+d^2}-2dH} &\geq 1 \Longleftrightarrow
\end{aligned}$$

$$1+d^2+2dH \geq \sqrt{1+d^2-4H^2}$$

which is equivalent to

$$(1+d^2+2dH)^2 \geq (\sqrt{1+d^2-4H^2})^2 \Longleftrightarrow$$

$$(1+d^2)^2+4dH(1+d^2)+4H^2d^2 \geq 1+d^2-4H^2 \Longleftrightarrow$$

$$d^2+4dH+4H^2 \geq 0 \Longleftrightarrow$$

$$(d+2H)^2 \geq 0$$

So, $\cosh\rho \geq \dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}$ always is possible.

Let ρ_1 such that $\cosh\rho_1 = \dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}$. Then $\rho_1 = 0 \Leftrightarrow$

$$\begin{aligned} 1-4H^2 &= 2dH+\sqrt{1-4H^2+d^2} \Leftrightarrow \\ (d+2H)^2 &= 0 \Leftrightarrow \\ d &= -2H \end{aligned}$$

Now, we prove each statement.

1. If $d > -2H$ then $d + 2H\cosh_1\rho > -2H + 2H = 0$ this is $\cosh\rho_1 > \frac{-d}{2H}$, so $u'(\rho) > 0$, this is the function u is strictly increasing for $\rho \geq \rho_1 > 0$ and has a nonfinite derivative at ρ_1.

2. If $d = -2H$, we set $u' = 2H\sqrt{\dfrac{(\cosh\rho-1)(1+4\tau^2\tanh^2(\rho/2))}{a\cosh\rho+(2-a)}}$, where $a = 1-4H^2 > 0$, then

$$\begin{aligned} u' &= 2H\sqrt{p(\rho)} > 0 \\ u'' &= \frac{Hp'}{\sqrt{p}} \end{aligned}$$

since

$$p' > 0$$

we have, $u'' > 0$, this is, the function u is defined for $\rho \geq 0$, it has a zero derivative at 0, is strictly increasing for $\rho > 0$, and it is up concave.

3. If $d < -2H$, then there exist $\rho_0 > \rho_1$ such that $\cosh\rho_0 = \frac{-d}{2H}$. With effect, we must have

$$\begin{aligned} \frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} &< -\frac{d}{2H} \Leftrightarrow \\ 4dH^2+2H\sqrt{1-4H^2+d^2} &< -d+4dH^2 \Leftrightarrow \\ 2H\sqrt{1-4H^2+d^2} &< -d \Leftrightarrow \\ 4H^2(1-4H^2+d^2) &< d^2 \Leftrightarrow \\ 4H^2(1-4H^2) &< d^2(1-4H^2) \Leftrightarrow \\ 2H &< |d| \end{aligned}$$

this is, either $d > 2H$ or $d < -2H$. Consequently the function u is defined for $\rho \geq \rho_1 > 0$ with a nonfinite derivative at ρ_1, it is strictly decreasing for $\rho_1 < \rho < \rho_0$, has a zero derivative at ρ_0 and it is strictly increasing for $\rho > \rho_0$.

4. Observe that,
$$\lim_{\rho\to+\infty} u' = \frac{2\sqrt{1+4\tau^2}H}{\sqrt{1-4H^2}}$$
that is, there are two numbers $\delta > 0$ and $R \gg 1$ such that
$$u(\rho) = \int_R^\rho \frac{g(r)\sqrt{1+4\tau^2\tanh^2(r/2)}}{\sqrt{f(r)}}dr \geq \int_R^\rho \delta dr = v(\rho)$$
since $\lim\limits_{\rho\to+\infty} v(\rho) = +\infty$ we have that, $\lim\limits_{\rho\to+\infty} u(\rho) = +\infty$.

□

Next Lemma, is analogous to Lemma 6.5.1 in the case $H = 1/2$. Observe that $f(\rho) = -2d\cosh^2\rho - (1+d^2)$, thus the set $\{\rho, f(\rho) > 0\}$ is nonempty if and only if $d < 0$

Lemma 6.5.2. *Assume $H = 1/2$ and $d < 0$. Then $f(\rho) \geq 0$ if and only if*
$$\cosh^2\rho \geq \frac{1+d^2}{-2d}$$
Let $\rho_1 \geq 0$ such that $\cosh\rho_1 = \dfrac{1+d^2}{-2d}$, then $f(\rho_1) = 0$ and $\rho_1 = 0$ if and only if $d = -1$.

1. *If $d \in (-1,0)$, then $\dfrac{-d}{2H} < \cosh\rho_1$. Consequently the function u is strictly increasing for $\rho \geq \rho_1 > 0$ and has a nonfinite derivative at ρ_1.*
2. *If $d = -1$, then $u'(\rho) = \dfrac{1}{\sqrt{2}}\sqrt{(\cosh\rho - 1)(1+4\tau^2\tanh^2(\rho/2))}$. Therefore the function u is defined for $\rho \geq 0$, it has a zero derivative at 0 and is strictly increasing for $\rho > 0$.*
3. *If $d < -1$ there exist $\rho_0 > \rho_1 > 0$ such that $\dfrac{-d}{2H} = \cosh\rho_0$. Consequently the function u is defined for $\rho \geq \rho_1 > 0$ with a nonfinite derivative at ρ_1, it is strictly increasing for $\rho_1 < \rho < \rho_0$, has a zero derivative at ρ_0 and it is strictly increasing for $\rho > \rho_0$.*
4. *For any d we have $\lim\limits_{\rho\to+\infty} u(\rho) = +\infty$.*

Proof. Observe that, $f(\rho \geq 0$ if and only if $\cosh\rho \geq \dfrac{1+d^2}{-2d}$, this least inequality is possible because $\dfrac{1+d^2}{-2d} \geq 1$. Let $\rho_1 \geq 0$ such that $\cosh\rho_1 = \dfrac{1+d^2}{-2d}$, then $f(\rho_1) = 0$ and $\rho_1 = 0$ if ad only if $d = -1$.

Now, we prove each statement.

1. If $d \in (-1,0)$, then $2H\cosh\rho_1 + d > 2H - 1 = 0$ this implies $\cosh\rho > \cosh\rho_1 > 0$, this is $u'(\rho) > 0$. Consequently the function u is strictly increasing for $\rho \geq \rho_1 > 0$ and has a nonfinite derivative at ρ_1.

2. if $d = -1$, then $u' = \frac{1}{\sqrt{2}}\sqrt{(\cosh\rho - 1)(1 + 4\tau^2\tanh^2(\rho/2))} = \frac{1}{\sqrt{2}}\sqrt{h(\rho)} \geq 0$,where $h(\rho) = \sqrt{(\cosh\rho - 1)(1 + 4\tau^2\tanh^2(\rho/2))}$, so $u'' = \frac{h'}{2\sqrt{2}\sqrt{h}}$, since
$$h' \geq 0$$
we obtain $u'' \geq 0$. Therefore the function u is defined for $\rho \geq 0$, it has a zero derivative at 0, is strictly increasing for $\rho > 0$, and is up concave.

3. If $d < -1$ there exist $\rho) > \rho_1$ such that $\cosh\rho_0 = \frac{-d}{2H}$ because $\frac{1+d^2}{-2d} < \frac{-d}{2H} \Leftrightarrow |d| > 1$. Consequently the function u is defined for $\rho \geq \rho_1 > 0$ with a nonfinite derivative at ρ_1, it is strictly decreasing for $\rho_1 < \rho < \rho_0$, has a zero derivative at ρ_0 and it is strictly increasing for $\rho > \rho_0$.

4. observe that,
$$\lim_{\rho\to+\infty} u' = +\infty$$
that is, there are two numbers $\delta > 0$ and $R \gg 1$ such that
$$u(\rho) = \int_R^\rho \frac{g(r)\sqrt{1 + 4\tau^2\tanh^2(r/2)}}{\sqrt{f(r)}}dr \geq \int_R^\rho \delta r dr = v(\rho)$$
since $\lim_{\rho\to+\infty} v(\rho) = +\infty$ we have that, $\lim_{\rho\to+\infty} u(\rho) = +\infty$.

□

Now we compute the curvature at the point $\rho = \rho_1$, remember that

$$u' = \frac{(d + 2H\cosh(\rho))\sqrt{1 + 4\tau^2\tanh^2(\rho/2)}}{\sqrt{\sinh^2(\rho) - (d + 2H\cosh(\rho))^2}} = \frac{g(\rho)\sqrt{h(\rho)}}{\sqrt{f(\rho)}}$$

Lemma 6.5.3. *Letting $\rho \longrightarrow \rho_1$, we infer by a computation that the curvature*

$$k(\rho) = \frac{u''}{(1 + (u')^2)^{3/2}}$$

goes to

– $k(\rho_1) = -\frac{f'(\rho_1)}{2h(\rho_1)g^2(\rho_1)}$ *when $\rho \to \rho_1$, if $d \neq -2H$,*

– $k(\rho_1) = H$ *when* $\rho \to \rho_1 = 0$, *if* $d \equiv -2H$,

Proof. Observe that

– If $d \neq -2H$,

$$\begin{aligned} g'(\rho) &= 2H\sinh(\rho) \\ h'(\rho) &= 4\tau^2\tanh(\rho/2)sech^2(\rho/2) \\ f'(\rho) &= \sinh(2\rho)(1-4H^2) - 4Hd\sinh(\rho) \end{aligned}$$

thus, $f'(\rho_1) \neq 0$, $g'(\rho_1) \neq 0$, $h'(\rho_1) \neq 0$. Since

$$u'' = \frac{2g'hf + gh'f - ghf'}{2f^{3/2}\sqrt{h}}$$

then

$$k(\rho) = \frac{u''}{(1+(u')^2)^{3/2}} = \frac{2g'hf + gh'f - ghf'}{2\sqrt{h}(f+g^2h)^{3/2}}$$

As $0 < \rho_1 < \infty$, we have $0 < h(\rho_1) < \infty$, $0 < g^2(\rho_1) < \infty$, $f(\rho_1) = 0$. This is clear that $k(\rho) \to k(\rho_1)$ when $\rho \to \rho_1$.

– If $d \equiv -2H$, then $f(\rho) = (\cosh(\rho)-1)((1-4H^2)(\cosh(\rho)-1)+2)$, so

$$u' = \frac{2H\sqrt{\cosh(\rho)-1}\sqrt{1+4\tau^2\tanh^2(\rho/2)}}{\sqrt{(1-4H^2)(\cosh(\rho)-1)+2}} = \frac{2H\sqrt{g}\sqrt{h}}{\sqrt{f}}$$

and

$$\begin{aligned} k(\rho) &= \frac{H}{\sqrt{g}\sqrt{h}(f+4H^2gh)^{3/2}}(g'hf + gh'f - ghf') \\ &= \frac{H}{\sqrt{h}(f+4H^2gh)^{3/2}}\left(\frac{g'}{\sqrt{g}}hf + \sqrt{g}(h'f - hf')\right) \end{aligned}$$

here, $f(0) = 2$, $h(0) = 1$, $g(0) = 0$, and $h'(0) = 0$; as

$$\frac{g'hf}{\sqrt{g}} = \sqrt{\cosh(\rho)-1}h(\rho)f(\rho) \to 2\sqrt{2}, \quad if \quad \rho \to 0$$

then $k(\rho) \to H$ when $\rho \to \rho_1 = 0$.

□

As a consequence of Lemma (6.5.1) and Lemma (6.5.2), we have the next results.

Theorem 6.5.1. *(Rotational H-surface with $0 < H \leq 1/2$). Assume $0 < H \leq 1/2$. there exist a one-parameter family $\mathfrak{H}_d$, $d \in \mathbb{R}$ for $H < 1/2$ and $d < 0$ for $H = 1/2$, of complete rotational H-surfaces.*

1. *For $d > -2H$, the surface $\mathfrak{H}_d$ is a properly embedded annulus, symmetric with respect to the slice $\{t = 0\}$, the distance between the "neck" and the rotational axis $R = \{(0,0) \times \mathbb{R}\}$ is arccosh($\frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}$) for $H < 1/2$ and arcosh($\frac{1 + d^2}{-2d}$) for $H = 1/2$. See Fig. 1.a*
2. *For $d = -2H$, the surface $\mathfrak{H}_{-2H}$ is an entire vertical graph, denoted by S^H. Moreover S^H is contained in the halfspace $\{t \geq 0\}$ and it is tangent to slice $\mathbb{D}^2 \times \{0\}$ at the point $(0,0,0)$. See Fig. 1.b*
3. *For $d < -2H$, the surface $\mathfrak{H}_d$ is a properly immersed (and nonembedded) annulus, it is symmetric with respect to slice $\{t = 0\}$, the distance between the "neck" and the rotational axis R is arcosh($\frac{2dH + \sqrt{1 - 4H^2 + d^2}}{1 - 4H^2}$) for $H < 1/2$ and arcosh($\frac{1+d^2}{-2d}$) for $H = 1/2$. See Fig. 1.c*
4. *In each of the previous case the surface is unbounded in the t-coordinate. When d tends to $-2H$ with either $d > -2H$ or $d < -2H$, then the surface $\mathfrak{H}_d$ tends toward the union of S^H and its symmetric with respect to the slice $\{t = 0\}$. Furthermore, any rotational H-surface with $0 < H \leq 1/2$ is up to an ambient isometry, a part of a surface of the family $\mathfrak{H}_d$.*

Proof. The result is a straightforward consequence of Lemma 6.5.1 and Lemma 6.5.2. For $d = -2H$, $\mathfrak{H}_{-2H}$ is the rotational surface generated by the graph of the function u.

For $d \neq -2H$, let γ be the union of the graph of u join with its symmetric with respect to the slice $\{t = 0\}$. Then $\mathfrak{H}_d$ is the rotational surface generated by the curve γ. □

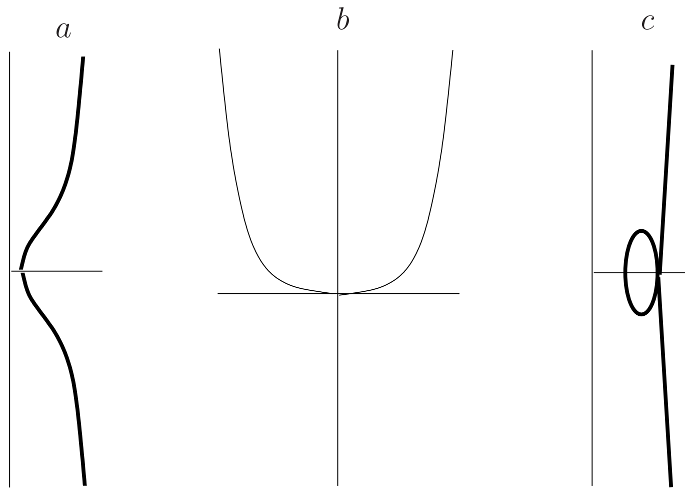


Figure 1.– Generating curve for rotational surfaces with $H \leq 1/2$.

Observe that, $f(\rho) = (1-4H^2)\cosh^2\rho - 4Hd\cosh\rho - (1+d^2)$, so for $H > 1/2$, the set $\{\rho, f(\rho) > 0\}$ is nonempty if and only if $d < 0$. Furthermore, $f\left(\dfrac{2dH \pm \sqrt{1-4H^2+d^2}}{1-4H^2}\right) = 0$, this least equality is possible since $1-4H^2+d^2 > 0$, this is $d < -\sqrt{4H^2-1}$.

Lemma 6.5.4. *Let H and d satisfying $H > 1/2$ and $d < -\sqrt{4H^2-1}$. Then, there exist two numbers $0 \leq \rho_1 < \rho_2$ such that*

$$\cosh\rho_1 = \frac{2dH + \sqrt{1-4H^2+d^2}}{1-4H^2}$$
$$\cosh\rho_2 = \frac{2dH - \sqrt{1-4H^2+d^2}}{1-4H^2}$$

Therefore, $f(\rho) > 0$ if and only if $\rho_1 < \rho < \rho_2$ and $f(\rho_1) = f(\rho_2) = 0$.

1. *If $d < -2H$, then $\rho_1 > 0$ and there exist a unique number $\rho_0 \in (\rho_1, \rho_2)$ satisfying $g(\rho_0) = 0$. Furthermore $g \leq 0$ on $[\rho_1, \rho_0)$ and $g \geq 0$ on $(\rho_0, \rho_2]$. Consequently, the function u is defined on $[\rho_1, \rho_2]$, has a nonfinite derivative at ρ_1 and ρ_2, has a zero derivative at ρ_0, is strictly decreasing on (ρ_1, ρ_0) and strictly increasing on (ρ_0, ρ_2).*

2. *If $d = -2H$, then $\rho_1 = 0$ and $u'(\rho) = \dfrac{2H\sqrt{\cosh\rho - 1}\sqrt{1+4\tau^2\tanh^2(\rho/2)}}{\sqrt{(1-4H^2)\cosh\rho + 4H^2 + 1}}$. Consequently, the function u is defined on $[0, \rho_2]$, is strictly increasing, has a zero derivative at 0 and a nonfinite derivative at ρ_2.*

3. *If* $-2H < d < -\sqrt{4H^2-1}$, *then* $\rho_1 > 0$ *and* $g \leq 0$ *on* $[\rho_1, \rho_2]$. *Therefore the function* u *is defined on* $[\rho_1, \rho_2]$, *is strictly increasing and has nonfinite derivative at* ρ_1 *and* ρ_2.

Proof. Following the ideas of the Lemma 6.5.1, we have

$$u(\rho) = \int_*^\rho \frac{(d+2H\cosh r)\sqrt{1+4\tau^2\tanh^2 r/2}}{\sqrt{\sinh^2 r - (d+2H\cosh r)^2}}dr$$

$$u'(\rho) = \frac{g(\rho)\sqrt{1+4\tau^2\tanh^2\rho/2}}{\sqrt{f(\rho)}}$$

as $f(\rho) = (1-4H^2)\cosh^2\rho - 4Hd\cosh\rho - (1+d^2)$ and $d < -\sqrt{4H^2-1}$, then there exist two numbers $\rho_1 < \rho_2$ such that $f(\rho_1) = f(\rho_2) = 0$, that is $\cosh\rho_1 = \dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}$ and $\cosh\rho_2 = \dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2}$, and for all $\rho_1 < \rho < \rho_2$, $f(\rho) > 0$.

Observe that

$$\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} \leq \frac{-d}{2H} < \frac{2dH-\sqrt{1-4H^2+d^2}}{1-4H^2} \qquad (6\text{-}9)$$

– The inequality $\dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} \leq \dfrac{-d}{2H}$ is true because it is equivalent to

$$\begin{aligned} 4H^2d^2 + 2H\sqrt{1-4H^2+d^2} &\geq -d + 4H^2d^2 \iff \\ 4H^2(1-4H^2+d^2) &\geq d^2 \iff \\ d^2 &\geq 4H^2 \iff \\ -2H &\geq d \end{aligned}$$

– The inequality $\dfrac{-d}{2H} < \dfrac{2dH-\sqrt{1-4H^2+d^2}}{1-4H^2}$ is true because it is equivalent to

$$\begin{aligned} -d + 4dH^2 &> 4dH^2 - 2H\sqrt{1-4H^2+d^2} \\ -d &> -2H\sqrt{1-4H^2+d^2} \end{aligned}$$

and it is true since $d < 0$.

Now, we prove each statement.

1. If $d < -2H$, then $\rho_1 > 0$ and the inequality (6-9) says that there exist a unique number $\rho_0 \in (\rho_1, \rho_1)$ satisfying $g(\rho_0) = 0$, that is $g(\rho_0) = \frac{-d}{2H}$. Furthermore $g \leq 0$ on $[\rho_1, \rho_0)$ and $g \geq 0$ on $(\rho_0, \rho_2]$. Consequently, the function u is defined on $[\rho_1, \rho_2]$, has a nonfinite derivative at ρ_1 and ρ_2, has a zero derivative at ρ_0, is strictly decreasing on (ρ_1, ρ_0) and strictly increasing on (ρ_0, ρ_2).

2. If $d = -2H$, so $\rho_1 = 0$ and $u' = 2H\sqrt{\frac{(\cosh\rho - 1)(1 + 4\tau^2\tanh^2(\rho/2))}{a\cosh\rho + (2-a)}}$, where $a = 1 - 4H^2 < 0$, then

$$\begin{aligned} u' &= 2H\sqrt{p(\rho)}\sqrt{1+\tanh^2(\rho/2)} > 0 \\ u'' &= \frac{Hp'}{\sqrt{p}} + H\sqrt{p(\rho)}\frac{\tanh(\rho/2)sech^2(\rho/2)}{\sqrt{1+\tanh^2(\rho/2)}} \end{aligned}$$

since

$$p' > 0$$

we have, $u'' > 0$, this is, the function u is defined for $\rho \geq 0$, it has a zero derivative at 0 and a nonfinite derivative at ρ_2, u is strictly increasing for $\rho > 0$, and it is up concave.

3. If $-2H < d < -\sqrt{4H^2 - 1}$, then $\rho_1 > 0$ and $g(\rho) = d + 2H\cosh_\rho \geq -2H + 2H = 0$ on $[\rho_1, \rho_2]$. Therefore the function u is defined on $[\rho_1, \rho_2]$, is strictly increasing and has nonfinite derivative at ρ_1 and ρ_2.

□

Lemma 6.5.5. *Letting $\rho \longrightarrow \rho_1, \rho_2$, we infer by a computation that the curvature*

$$k(\rho) = \frac{u''}{(1 + (u')^2)^{3/2}}$$

goes to

– $k(\rho_1) = -\frac{f'(\rho_1)}{2h(\rho_1)g^2(\rho_1)}$ *when $\rho \to \rho_1$, if $d \neq -2H$,*

– $k(\rho_2) = -\frac{f'(\rho_2)}{2h(\rho_2)g^2(\rho_2)}$ *when $\rho \to \rho_2$, if $d \neq -2H$,*

– $k(\rho_1) = H$ *when $\rho \to \rho_1 = 0$, if $d \equiv -2H$,*

Proof. The proof is analogous to this one on Lemma (6.5.3) □

As an immediate consequence of Lemma 6.5.4 we obtain the next Theorem.

Theorem 6.5.2. *(Rotational surfaces with $H > 1/2$) Assume $H > 1/2$. There exist a one-parameter family $\mathfrak{D}_d$ of complete rotational H-surfaces, $d \leq -\sqrt{4H^2-1}$.*

1. *For $d < -2H$, the surface $\mathfrak{D}_d$ is an immersed (and nonembedded) annulus, invariant by a vertical translation and is contained in the closed region bounded by the two vertical cylinders $\rho = \rho_1$ and $\rho = \rho_2$. Furthermore $\rho_1 \to +\infty$ and $\rho_2 \to +\infty$ when $d \to -\infty$ and $\rho_1 \to 0$ and $\rho_2 \to arcosh\left(\dfrac{4H^2+1}{4H^2-1}\right)$ when $d \to -2H$. Such surfaces are analogous to the nodoids of Delaunay in $\mathbb{R}^3$. See Fig. 2.a*

2. *For $d = -2H$, the surface $\mathfrak{D}_{-2H}$ is an embedded sphere and the maximal distance from the rotational axis is $\rho_2 = arcosh\left(\dfrac{4H^2+1}{4H^2-1}\right)$. See Fig. 2.b*

3. *For $-2H < d < -\sqrt{4H^2-1}$; the surface $\mathfrak{D}_d$ is an embedded annulus, invariant by a vertical translation and is contained in the closed region bounded by the two vertical cylinders $\rho = \rho_1$ and $\rho = \rho_2$. Furthermore $\rho_1 \to 0$ and $\rho_2 \to arcosh\left(\dfrac{4H^2+1}{4H^2-1}\right)$ when $d \to -2H$ and both $\rho_1, \rho_2 \to arcosh\left(\dfrac{2H}{\sqrt{4H^2-1}}\right)$ when $d \to -\sqrt{4H^2-1}$. Moreover $\rho_2 \to arcosh\left(\dfrac{2H}{\sqrt{4H^2-1}}\right) < \rho_2$. Such surfaces are analogous to the undoloids of Delaunay in $\mathbb{R}^3$. See Fig. 2.c*

4. *For $d = -\sqrt{4H^2-1}$, the surface $\mathfrak{D}_{-\sqrt{4H^2-1}}$ is the vertical cylinder over the circle with hyperbolic radius $arcosh\left(\dfrac{2H}{\sqrt{4H^2-1}}\right)$.*

Proof. Here we assume the notation of Lemma 6.5.4.

1. For $d < -2H$, we have $\cosh\rho_2 = \dfrac{-2dH+\sqrt{1-4H^2+d^2}}{4H^2-1} \longrightarrow +\infty$ when $d \to -\infty$, and $\cosh\rho_1 = \dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} = \dfrac{d^2+1}{-2dH+\sqrt{1-4H^2+d^2}} \to \dfrac{+\infty}{+\infty}$ when $d \to -\infty$, by using L'Hospital we obtain that $\cosh\rho_1 \to +\infty$. this is clear that $\rho_1 \to 0$ and $\rho_2 \to arcosh\left(\dfrac{4H^2+1}{4H^2-1}\right)$ when $d \to -2H$.

2. For $d = -2H$, then from the Lemma 6.5.4 and 1. the surface $\mathfrak{D}_{-2H}$ is an embedded sphere and the maximal distance from the rotational axis is $\rho_2 = arcosh\left(\dfrac{4H^2+1}{4H^2-1}\right)$.

3. Observe that $\cosh\rho_1, \cosh\rho_2 \to \dfrac{2H}{\sqrt{4H^2-1}}$ when $d \to -\sqrt{4H^2-1}$. Furthermore, the inequality

$$\frac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} < \frac{2H}{\sqrt{4H^2-1}} < \frac{2dH-\sqrt{1-4H^2+d^2}}{1-4H^2}$$

hold, because

– $\dfrac{2dH+\sqrt{1-4H^2+d^2}}{1-4H^2} < \dfrac{2H}{\sqrt{4H^2-1}}$ is equivalent to

$$\frac{-2dH-\sqrt{1-4H^2+d^2}}{\sqrt{4H^2-1}} < 2H \iff$$

$$-2dH < 2H\sqrt{4H^2-1}+\sqrt{1-4H^2+d^2} \iff$$

$$4H^2d^2 < 4H^2(4H^2-1)+1-4H^2+2H\sqrt{4H^2-1}\sqrt{1-4H^2+d^2} \iff$$

$$0 < 4H^2(4H^2-d^2)+1+d^2+2H\sqrt{4H^2-1}\sqrt{1-4H^2+d^2} \iff$$

and $4H^2-d^2>0$ hold, since

$$-2H<d.$$

– Furthermore, $\dfrac{2H}{\sqrt{4H^2-1}} < \dfrac{2dH-\sqrt{1-4H^2+d^2}}{1-4H^2}$ is equivalent to

$$d+\sqrt{4H^2-1}<0.$$

4. For $d = -\sqrt{4H^2-1}$, then $\cosh\rho_1 = \cosh\rho_2 = \dfrac{2H}{\sqrt{4H^2-1}}$, then $\mathfrak{D}_{-\sqrt{4H^2-1}}$ is the vertical cylinder over the circle hyperbolic with hyperbolic radius $\rho = arcosh\left(\frac{2H}{\sqrt{4H^2-1}}\right)$, so its geodesic curvature k_g is given by $k_g = \dfrac{1}{\tanh\rho} = 2H$.

□

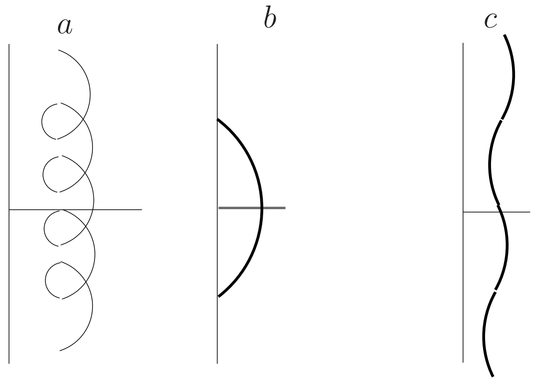


Figure 2.– Generating curve for rotational surfaces with $H > 1/2$

6.6 Applications

In this section we use the study of rotational surfaces as well as the examples constructed to give some intrusting applications, here we follow ideas from (14).

6.6.1 Asymptotic behavior of rotational surfaces

In the paper: "A half-space theorem for mean curvature $H = \frac{1}{2}$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$", the authors studied the asymptotic behavior for rotational $H = 1/2$ surfaces immersed into $\mathbb{H}^2 \times \mathbb{R}$. There, they have proved a half-space theorem for rotational $H = 1/2$ surfaces in $\mathbb{H}^2 \times \mathbb{R}$ (see (14, Theorem 1)).

Since the behavior of rotational $H = \frac{1}{2}$ surfaces immersed into $\widetilde{PSL}_2(\mathbb{R},\tau)$ is similar to the rotational surfaces in $\mathbb{H}^2 \times \mathbb{R}$ [see Theorem 6.5.1], it is natural asks for a half-space theorem (in the same sense) for mean curvature $H = 1/2$ surfaces in $\widetilde{PSL}_2(\mathbb{R},\tau)$.

Following the notations from Lemma 6.1.2, the function u which describes a rotational surface having constant mean curvature H satisfies the next expression:

$$u(\rho) = \int \frac{(2H\cosh(\rho) + d)\sqrt{1 + 4\tau^2 \tanh^2\left(\frac{\rho}{2}\right)}}{\sqrt{\sinh^2(\rho) - (2H\cosh(\rho) + d)^2}} d\rho$$

where d is a real number.

Denoting by $\alpha = -d$, then for any $\alpha \in \mathbb{R}_+$, there exist a rotational surface $\mathfrak{H}_\alpha$ of constant mean curvature $H = \dfrac{1}{2}$.

$$u_\alpha(\rho) = \int \frac{(\cosh(\rho)-\alpha)\sqrt{1+4\tau^2\tanh^2\left(\frac{\rho}{2}\right)}}{\sqrt{\sinh^2(\rho)-(\cosh(\rho)-\alpha)^2}}d\rho.$$

Recall that:

(i) For $\alpha \neq 1$, the surface $\mathfrak{H}_\alpha$ has two vertical ends (where a vertical end is a topological annulus, with no asymptotic point at finite height) that are vertical graph over the exterior of a disk D_α.

(ii) For $\alpha \equiv 1$, the surface $\mathfrak{H}_1$, which we denote simply by S, has only one end and it is a graph over $\mathbb{D}^2$ the hyperbolic disk.

Up to vertical translation, one can assume that $\mathfrak{H}_\alpha$ is symmetric with respect to the horizontal plane $t = 0$. For $\alpha = 1$, the surface $\mathfrak{H}_1$ has only one end, it is a graph over $\mathbb{D}^2$ and it is denoted by S.

For any $\alpha > 1$ the surface $\mathfrak{H}_\alpha$ is not embedded. The self-intersection set is a horizontal circle on the plane $t = 0$. Denote by ρ_α the radius of the intersection circle. For $\alpha < 1$ the surface $\mathfrak{H}_\alpha$ is embedded.

For any $\alpha \in \mathbb{R}_+$, let $u_\alpha : \mathbb{D}^2 \times \{0\} \backslash D_\alpha \longrightarrow \mathbb{R}$ be the function such that the end of the surface $\mathfrak{H}_\alpha$ is the vertical graph of u_α. For any $\alpha \neq 1$, the graph of u_α is defined outside the disk D_α of radius ρ_α and it is vertical along the boundary ∂D_α. We choose the integration constant such that the graph of u_α is contained in the halfspace $t \geq 0$. The radius of the disk D_α is $\rho_\alpha = -\ln(\alpha)$. Furthermore ρ_α is always greater o equal to zero, it is zero if and only if $\alpha = 1$ and tends to infinity as $\alpha \to 0$

Lemma 6.6.1. *The asymptotic behavior of the function u_α has the following form:*

$$u_\alpha(\rho) \simeq \frac{\sqrt{1+4\tau^2}}{\sqrt{\alpha}}\exp^{\left(\frac{\rho}{2}\right)}, \qquad \rho \to \infty$$

where ρ is the hyperbolic distance from the origin.

Proof. Observe that:

Setting:

$$j(\rho) = \frac{(\cosh(\rho)-\alpha)}{\sqrt{\sinh^2(\rho)-(\cosh(\rho)-\alpha)^2}}$$

$$l(\rho) = \sqrt{1+4\tau^2\tanh^2\left(\frac{\rho}{2}\right)}$$

and by using the identity $\dfrac{1}{1-r} = 1 - r + r^2 - r^3 + ...$, we obtain:

$$\begin{aligned} j(\rho) &= \frac{\cosh(\rho)-\alpha}{\sqrt{2\alpha\cosh(\rho)-c}} \\ &= \frac{\sqrt{2\alpha\cosh(\rho)-c}}{2\alpha} + \frac{c-2\alpha^2}{2\alpha}\sqrt{\frac{1}{2\alpha\cosh(\rho)-c}} \end{aligned}$$

where $c = 1+\alpha^2$. Furthermore:

$$\begin{aligned} A(\rho) &:= \sqrt{2\alpha\cosh(\rho)-c} \\ &\approx \sqrt{\alpha}\exp\left(\frac{\rho}{2}\right) + O\left(-\frac{\rho}{2}\right) \end{aligned}$$

and,

$$\begin{aligned} B^2(\rho) &:= \frac{1}{2\alpha\cosh(\rho)-c} \\ &= \frac{\exp(-\rho)}{\alpha}\frac{1}{1+\exp(-2\rho)-(\frac{c}{\alpha})\exp(-\rho)} \\ &= \frac{\exp(-\rho)}{\alpha}\left[1-\left(\exp(-2\rho)-\frac{c}{\alpha}\exp(-\rho)\right)+...\right] \\ &\approx \frac{1}{\alpha}\left[\exp(-\rho)+\frac{d}{\alpha}\exp(-2\rho)+O\left(\exp(-3\rho)\right)\right] \end{aligned}$$

this implies:

$$B(\rho) \approx \frac{1}{\sqrt{\alpha}}\exp\left(-\frac{\rho}{2}\right) + O(\exp(-\rho)).$$

Analogously:

$$\sqrt{1+4\tau^2\tanh^2(\rho/2)} \approx \sqrt{1+4\tau^2} + O(\exp(-\rho/2))$$

Since the integrand function of $u_\alpha(\rho)$ is $j(\rho)l(\rho)$. The asymptotic behavior of the function $u_\alpha(\rho)$ is:

$$u_\alpha(\rho) \approx \frac{\sqrt{1+4\tau^2}}{\sqrt{\alpha}}\exp(\rho/2) + k + O(\exp(\rho/2)).$$

where k is the integration constant, $\rho \to +\infty$. □

This motives the following definition:

Definition 6.6.1. *We define* $\dfrac{\sqrt{1+4\tau^2}}{\sqrt{\alpha}} \in \mathbb{R}_+$ *the (exponential) growth of the end.*

Recall that for $\alpha > 1$ the surface $\mathfrak{H}_\alpha$ are immersed, while they are embedded for $0 < \alpha \leq 1$.

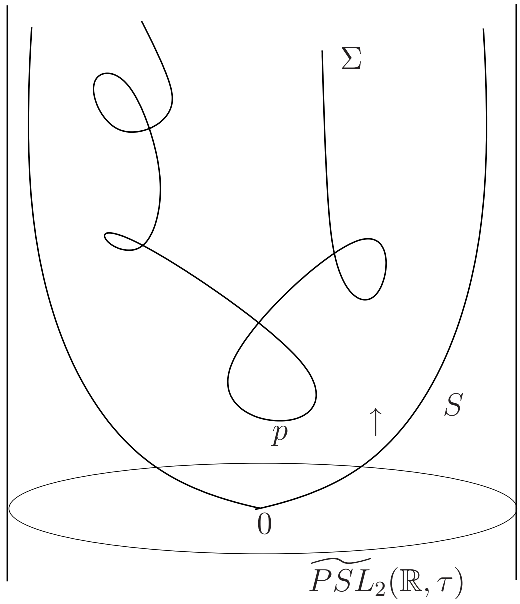
Remark 6.6.1. *The growth of any immersed rotational surface is smaller than the growth of the simply connected surface* S*, and the growth of any embedded rotational surface is greater than the growth of* S*.*

Theorem 6.6.1. *Let* S *be a simply connected rotational surface in* $\widetilde{PSL}_2(\mathbb{R},\tau)$*, having constant mean curvature* $H = \frac{1}{2}$*. Let* Σ *be a complete surface having constant mean curvature* $H = \frac{1}{2}$*, different from a rotational simply connected one. Then,* Σ *cannot be properly immersed in the mean convex side of* S

Proof. One can assume that the surface S is tangent at the origin of $\widetilde{PSL}_2(\mathbb{R},\tau)$ to the slice $t = 0$, and it is contained in $\{t \geq 0\}$.

Suppose by contradiction, that Σ is contained in the mean convex side of S. We will divide the proof in two step. First, we will suppose that Σ is not asymptotic to S and we will get to a contradiction. Second, we will assume that Σ is asymptotic to S, again we will get to a contradiction. This two steps, let us conclude the theorem.

First step. If Σ is not asymptotic to S, lifts vertically S. Then, there exist a first contact point p between Σ and the translation of S, this means that, in a neighborhood of $p\Sigma$, Σ is in the mean convex side of S and both mean curvature vectors coincides at p. In this case, one has a contradiction by the maximum principle.



Second step. Now, we can assume that Σ is asymptotic to S when t goes to $+\infty$.

Since that, Σ is properly immersed, there is a point $q \in \Sigma$, which has lowest height. To see this, consider a small geodesic all $B(0) \subset \widetilde{PSL}_2(\mathbb{R},\tau)$ in $\widetilde{PSL}_2(\mathbb{R},\tau)$ centered at the origin. Then, the intersection $B(0)\cap\Sigma$ is a compact set lying in $\{t > 0\}$. Hence, there is such point $q \in \Sigma$.

Let h be the height of q. Denote by $S(h)$ the vertical lifting of S of hight h. Now, consider $0 < \epsilon < h$ fixed, and denote by $S(h-\epsilon)$, the translation of S to height $t = h - \epsilon$.

Denote by W the noncompact slab bounded by S and $S(h-\epsilon)$. From the maximum principle, there are no compact component of Σ contained in W.

Denote by Σ_1 the noncompact connected component of Σ contained in W. By definition of Σ_1, the boundary of Σ_1 is contained in $\partial W - S = S(h-\epsilon)$.

Now, consider the family of rotational non-embedded surfaces $\mathfrak{H}_\alpha$, $\alpha > 1$. Translates each $\mathfrak{H}_\alpha$ vertically in order to have the waist on the plane $t = h - \epsilon$.

By abuse of notation, we continue to call the translation $\mathfrak{H}_\alpha$. Denote by $\mathfrak{H}_\alpha^+$ the part of the surface outside the cylinder of radius ρ_α. Note that $\mathfrak{H}_\alpha^+$ is embedded and is a vertical graph. When $\alpha \to +\infty$, then $\rho_\alpha \to +\infty$ as well. Furthermore, the growth of the end of $\mathfrak{H}_\alpha^+$ is smaller than the growth of S.

Hence, when α is great enough, say α_0, $\mathfrak{H}_\alpha^+$ is outside of the mean convex side of S. Thus, $\mathfrak{H}_{\alpha_0}^+$ does not intersect Σ. Furthermore, when $\alpha \to 1$, $\mathfrak{H}_\alpha^+$ converge to $S(h-\epsilon)$.

Now, start to decrease α from α_0 to one. Before reaching $\alpha = 1$, the surface $\mathfrak{H}_\alpha^+$ first meet S and then touches Σ tangentially at an interior point above $\mathfrak{H}_\alpha^+$, it because $\partial\Sigma_1$ lies on $S(h-\epsilon)$ and the growth of any $\mathfrak{H}_\alpha^+$ is smaller than the growth of S.

Again, the existence of such an interior tangency point is a contradiction with the maximum principle.

□