7 Surfaces invariant by one-parameter group of parabolic isometries having constant mean curvature in $\widetilde{\mathbf{PSL}}_{\mathbf{2}}(\mathbb{R},\tau)$

On (3) Ricardo Sa Earp gave explicit formulas for parabolic and hyperbolic screw motions surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$. There, they gave several examples.

In this chapter we only consider surfaces invariant by one-parameter group of parabolic isometries having constant mean curvature immersed in $\widetilde{PSL}_2(\mathbb{R},\tau)$. Since for $\tau \equiv 0$ we are in $\mathbb{H}^2 \times \mathbb{R}$ then we have generalized the result obtained by Ricardo Sa Earp when the surface is invariant by one-parameter group of parabolic isometries having constant mean curvature. In this chapter we also give explicit formulas and we give the geometric behavior for surfaces invariant by one-parameter group of parabolic isometries having constant mean curvature immersed in $\widetilde{PSL}_2(\mathbb{R},\tau)$.

In this chapter we focus our attention on surfaces invariant by one-parameter group of parabolic isometries. To study this kind of surface we take the half space model for the hyperbolic space, that is, $M^2 = \mathbb{H}^2$.

By Proposition 5.1.1, we know that, to obtain a parabolic motions on $\widetilde{PSL}_2(\mathbb{R},\tau)$, it is necessary consider a parabolic isometry on $\mathbb{H}^2$.

7.1 Surfaces invariant by one-parameter group of parabolic isometries main lemma

The idea to obtain surfaces invariant by one-parameter group of parabolic isometries is simple. We will take a curve in the yt plane and we will apply one-parameter group Γ of parabolic isometries on $\widetilde{PSL}_2(\mathbb{R},\tau)$. We denote by $\alpha(y) = (0, y, u(y))$ the curve in the yt plane and by $S = \Gamma(\alpha)$, the surfaces invariant by one-parameter group of parabolic isometries generated by α.

Since the most simple parabolic isometry on $\mathbb{H}^2$ is the horizontal translation, then our surface S is parameterized by,

$$\varphi(x, y) = (x, y, u(y))$$

Lemma 7.1.1. *With the notations above, and denoting by H the mean curvature of S, then the function u satisfies*

$$u(y) = \int \frac{(dy - 2H)\sqrt{1+4\tau^2}}{y\sqrt{1-(dy-2H)^2}}dy$$

where d is a real number.

Proof. The proof is analogous to the case of rotational surfaces. By completeness we give it. Since S has mean curvature H, then by lemma 5.2.1 the function u satisfies the equation

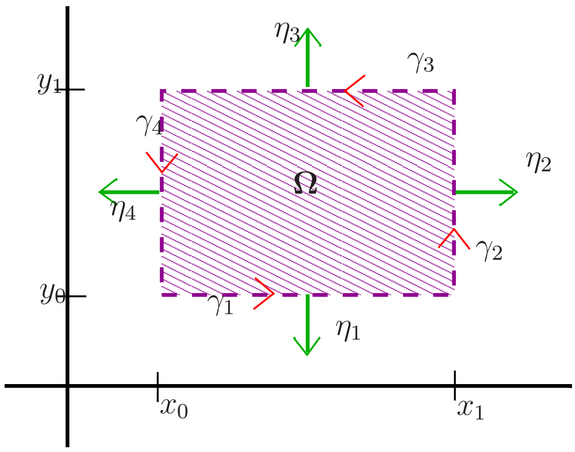
$$2H = div_{\mathbb{H}^2}\left(\frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2\right), \tag{7-1}$$

where $W = \sqrt{1+\alpha^2+\beta^2}$, $\alpha = -2\tau$, and $\beta = \frac{u_y}{\lambda}$.

Making:

$$X_u = \frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2$$

And let $x_0, x_1 \in \mathbb{R}$ with $x_0 < x_1$ and $0 < y_0, y_1 \in \mathbb{R}$ with $y_0 < y_1$ and consider the domain $\Omega = [x_0, x_1] \times [y_0, y_1]$ in the plane xy.



By integrating the equation (7-1), we obtain

$$\int_{\partial(\Omega)} \langle X_u, \eta\rangle = 2H Area([x_0, x_1] \times [y_0, y_1])$$

where η is the outer co-normal.

Since

$$\int_{\partial\Omega} \langle X_u, \eta\rangle = \int_{\gamma_1} \langle X_u, \eta_1\rangle + \int_{\gamma_2} \langle X_u, \eta_2\rangle + \int_{\gamma_3} \langle X_u, \eta_3\rangle + \int_{\gamma_4} \langle X_u, \eta_4\rangle$$

we compute each integral.

For the fist integral: Observe that,

$$\gamma_1(s) = (s, y_0), \quad x_0 \leq s \leq x_1.$$

This implies,

$$\gamma_1' = \partial_x,$$

thus

$$|\gamma_1'| = \lambda,$$

and

$$\eta_1 = -e_2.$$

Furthermore

$$\langle X_u, \eta_1 \rangle = \frac{-\beta(y_0)}{W},$$

hence

$$\int_{\gamma_1} \langle X_u, \eta_1 \rangle = \int_{x_0}^{x_1} -\frac{\beta\lambda}{W}(x_0)dx$$

For the third integral: Observe that,

$$\gamma_3(s) = (x_1 - s, y_1), \quad 0 \leq s \leq x_1 - x_0.$$

This implies,

$$\gamma_3' = -\partial_x,$$

thus

$$|\gamma_3'| = \lambda,$$

and

$$\eta_3 = e_2.$$

Furthermore

$$\langle X_u, \eta_3 \rangle = \frac{\beta(y_1)}{W},$$

hence

$$\int_{\gamma_3} \langle X_u, \eta_3 \rangle = \int_0^{x_1 - x_0} \frac{\beta\lambda}{W}(y_1)ds = \int_{x_0}^{x_1} \frac{\beta\lambda}{W}(y_1)dx$$

For the second integral: Observe that,

$$\gamma_2(s) = (x_1, s), \quad y_0 \leq s \leq y_1.$$

This implies,

$$\gamma_2' = \partial_y,$$

thus

$$|\gamma_2'| = \lambda,$$

and

$$\eta_2 = e_1.$$

Furthermore

$$\langle X_u, \eta_2 \rangle = \frac{\alpha}{W},$$

hence

$$\int_{\gamma_2} \langle X_u, \eta_2 \rangle = \int_{y_0}^{y_1} \frac{\alpha\lambda}{W} dy$$

For the four integral: Observe that,

$$\gamma_4(s) = (x_0, x_1 - s), \quad 0 \leq s \leq x_1 - x_0.$$

This implies,

$$\gamma_4' = \partial_y,$$

thus

$$|\gamma_4'| = \lambda,$$

and

$$\eta_4 = -e_1.$$

Furthermore

$$\langle X_u, \eta_4 \rangle = -\frac{\alpha}{W},$$

hence

$$\int_{\gamma_4} \langle X_u, \eta_4 \rangle = \int_0^{y_1 - y_0} -\frac{\alpha\lambda(y_1 - s)}{W} ds = \int_{x_0}^{y_1} -\frac{\alpha\lambda}{W} dy$$

Taking into account this four integrals, we obtain:

$$\int_{\partial\Omega} \langle X_u, \eta \rangle = \int_{x_0}^{x_1} \frac{\beta\lambda}{W}(y_1) dx - \int_{x_0}^{x_1} \frac{\beta\lambda}{W}(y_0) dx$$

Observe that,

$$\begin{aligned} Area(\Omega) &= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\sqrt{\det(g_{ij})}dydx \\ &= \int_{x_0}^{x_1}\int_{y_0}^{y_1}\frac{1}{y^2}dydx \end{aligned}$$

Thus, we conclude that,

$$\int_{x_0}^{x_1}\frac{\beta\lambda}{W}(y_1)dx - \int_{x_0}^{x_1}\frac{\beta\lambda}{W}(y_0)dx = 2H\int_{x_0}^{x_1}\int_{y_0}^{y_1}\lambda^2 dydx$$

which we can write in the form,

$$\int_{y_0}^{y_1}\partial_y\left(\frac{\beta\lambda}{W}(y)\right)dy = 2H\int_{y_0}^{y_1}\lambda^2 dy$$

As Ω is any domain in the plane xy, and taking the derivative with respect to y we obtain:

$$\partial_y\left(\frac{\lambda\beta}{W}\right) = 2H\lambda^2$$

by integrating this expression,

$$\frac{\lambda^2\beta}{\sqrt{\lambda^2+4\tau^2\lambda^2+u_y^2}} = -2H\lambda + d$$

where $d\in\mathbb{R}$. This implies,

$$u_y^2[1-(dy-2H)^2] = (dy-2H)^2[\lambda^2+4\tau^2\lambda^2]$$

thus the function u satisfies,

$$u(\rho) = \int\lambda\frac{(dy-2H)\sqrt{1+4\tau^2}}{\sqrt{1-(dy-2H)^2}}dy$$

□

7.2
Examples of surfaces invariant by one-parameter group of parabolic isometries in $\widetilde{\mathbf{PSL}}_{\mathbf{2}}(\mathbb{R},\tau)$

After a straightforward computation we obtain the next lemma.

Lemma 7.2.1. *The solution of the integral is given by*

– *If* $H\equiv 0$, *then*

$$u(y) = \sqrt{1+4\tau^2}\arcsin(dy)$$

– *If* $H = \frac{1}{2}$, *then*

$$u(y) = \sqrt{1+4\tau^2}\arcsin(dy-1) + \frac{2\sqrt{1+4\tau^2}}{\tan(\frac{\arcsin(cy-1)}{2})+1}$$

– *If* $H > \frac{1}{2}$, *then*

$$\begin{aligned} u(y) &= \sqrt{1+4\tau^2}\arcsin(dy-2H) + \\ & -\frac{4\sqrt{1+4\tau^2}H}{\sqrt{4H^2-1}}\arctan\left(\frac{2H\tan(\frac{\arcsin(dy-2H)}{2})+1}{\sqrt{4H^2-1}}\right) \end{aligned}$$

where $d \in \mathbb{R}$.

Proof. We integrate each expression.

1. First, we consider the case $H \equiv 0$, then the integral

$$h(y) = \sqrt{1+4\tau^2}\int \frac{dy-2H}{y}\frac{1}{\sqrt{1-(dy-2H)^2}}dy$$

becomes

$$h(y) = \sqrt{1+4\tau^2}\int d\frac{1}{\sqrt{1-d^2y^2}}dy$$

that is

$$h(y) = \sqrt{1+4\tau^2}\arcsin(dy)$$

2. Second, we consider the case $H = 1/2$, but we are doing the same computation to the case $H \neq 0$, then the integral

$$h(y) = \sqrt{1+4\tau^2}\int \frac{dy-2H}{y}\frac{1}{\sqrt{1-(dy-2H)^2}}dy$$

we can write as

$$h(y) = \sqrt{1+4\tau^2}\int \frac{1}{y}\frac{dy-2H}{\sqrt{1-(dy-2H)^2}}dy$$

now, we integrate by part, set

– $u = \frac{1}{y}$, then $du = -\frac{1}{y^2}$

– $dv = \frac{dy-2H}{\sqrt{1-(dy-2H)^2}}$, then $v = -\frac{\sqrt{1-(dy-2H)^2}}{d}$

so, the integral becomes,

$$h(y) = \sqrt{1+4\tau^2}\left[-\frac{\sqrt{1-(dy-2H)^2}}{dy} - \frac{1}{d}\int \frac{\sqrt{1-(dy-2H)^2}}{y^2}dy\right]$$

now, we are going to integrate the integral

$$\int \frac{\sqrt{1-(dy-2H)^2}}{y^2}dy$$

by substitution, making $s = dy - 2H$ then $ds = d.dy$, and $\frac{1}{y^2} = \left(\frac{d}{s+2H}\right)^2$, thus

$$\int \frac{\sqrt{1-(dy-2H)^2}}{y^2}dy = d\int \frac{\sqrt{1-s^2}}{(s+2H)^2}ds$$

this least integral, we integrate by parts

- $m = \sqrt{1-s^2}$, then $dm = \frac{-s}{\sqrt{1-s^2}}ds$
- $dn = \frac{ds}{(s+2H)^2}$, then $n = \frac{-1}{s+2H}$

so,

$$\int \frac{\sqrt{1-s^2}}{(s+2H)^2}ds = -\frac{\sqrt{1-s^2}}{s+2H} - \int \frac{s}{(s+2H)}\frac{ds}{\sqrt{1-s^2}}$$

thus, we obtain,

$$h = \sqrt{1+4\tau^2}\int \frac{s}{(s+2H)}\frac{ds}{\sqrt{1-s^2}}$$

we make

$$\begin{aligned} h &= \sqrt{1+4\tau^2}\int \frac{s}{(s+2H)}\frac{ds}{\sqrt{1-s^2}} \\ &= \sqrt{1+4\tau^2}\int \frac{s+2H-2H}{(s+2H)}\frac{ds}{\sqrt{1-s^2}} \\ &= \sqrt{1+4\tau^2}\int \frac{ds}{\sqrt{1-s^2}} - \sqrt{1+4\tau^2}\int \frac{2H}{(s+2H)}\frac{ds}{\sqrt{1-s^2}} \\ &= \sqrt{1+4\tau^2}\arcsin(s) - 2\sqrt{1+4\tau^2}H\int \frac{1}{s+2H}\frac{ds}{\sqrt{1-s^2}} \end{aligned}$$

that is

$$h = \sqrt{1+4\tau^2}\arcsin(dy-2H) - 2\sqrt{1+4\tau^2}H\int \frac{1}{s+2H}\frac{ds}{\sqrt{1-s^2}}$$

to integrate the least integral, we set $s=\sin(p)$, then $ds=\cos(p)dp$, so

$$h=\sqrt{1+4\tau^2}\arcsin(dy-2H)-2\sqrt{1+4\tau^2}H\int\frac{1}{\sin(p)+2H}dp \quad (7\text{-}2)$$

here, we consider that $H=1/2$, then

$$h(y)=\sqrt{1+4\tau^2}\arcsin(dy-1)+\frac{2\sqrt{1+4\tau^2}}{\tan(\frac{\arcsin(dy-1)}{2})+1}$$

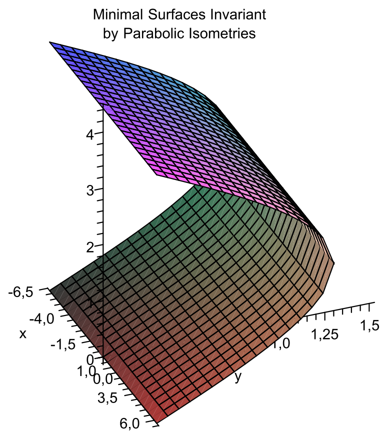
3. Third, we consider $H\neq 0$, we follow the proof in the second case, thus by using the equation (7-2), we obtain

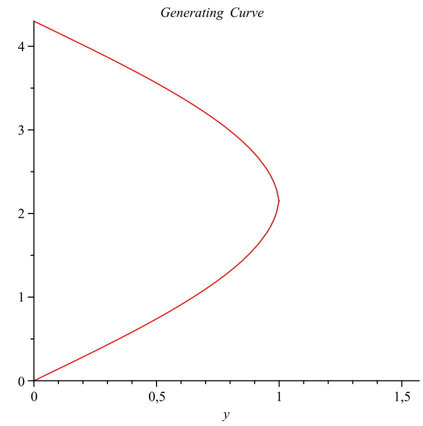
$$\begin{aligned} h(y) &= \sqrt{1+4\tau^2}\arcsin(dy-2H)+ \\ &\quad -\frac{4\sqrt{1+4\tau^2}H}{\sqrt{4H^2-1}}\arctan\left(\frac{2H\tan(\frac{\arcsin(dy-2H)}{2})+1}{\sqrt{4H^2-1}}\right)\end{aligned}$$

□

This Lemma gives an immediate examples:

Example 7.2.1. *Considering* $H\equiv 0$, $\tau=-1/2$ *and* $d=1$, *we obtain a minimal surface invariant by one-parameter group of parabolic isometries which is a vertical graph, by considering the rotation by* π *around the* y *axis we obtain a complete embedded minimal surfaces invariant by parabolic isometries in* $\widetilde{PSL}_2(\mathbb{R},\tau)$.

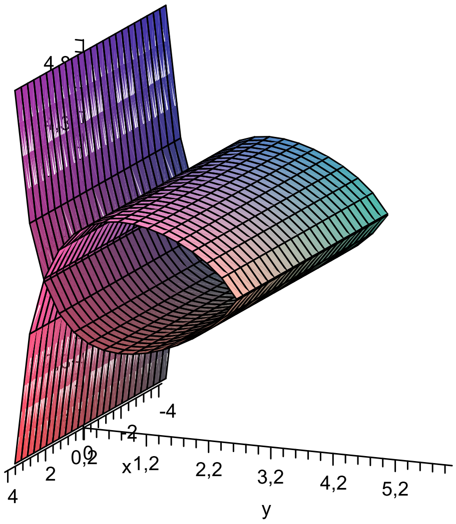


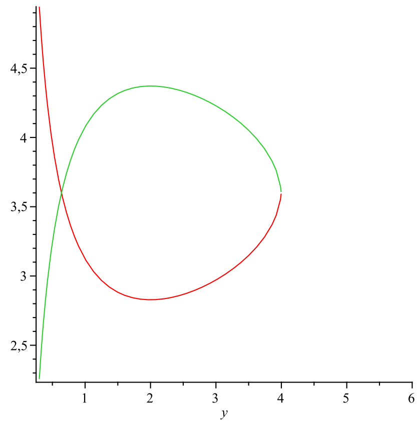


Example 7.2.2. *Considering* $H=1/2$, $\tau=-1/2$ *and* $d=1/2$, *we obtain*

$$u(y)=\sqrt{2}\arcsin(dy-1)+\frac{2\sqrt{2}}{\tan(\frac{\arcsin(dy-1)}{2})+1}$$

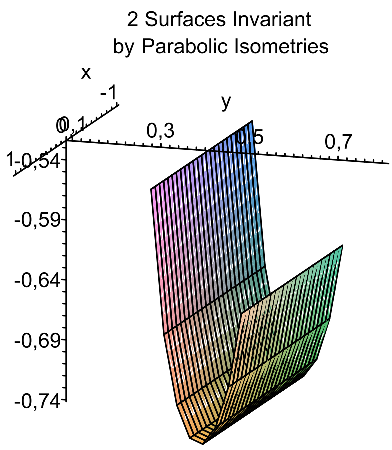
with Maple's help:

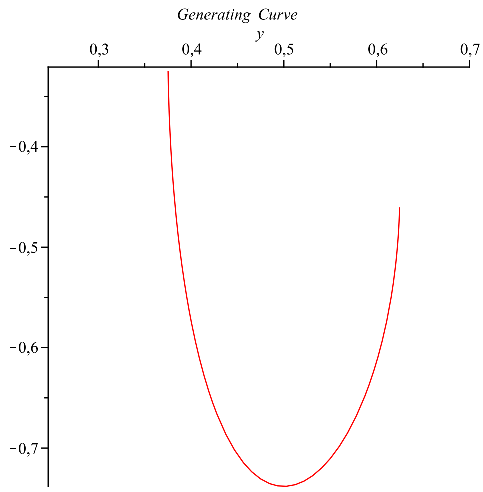




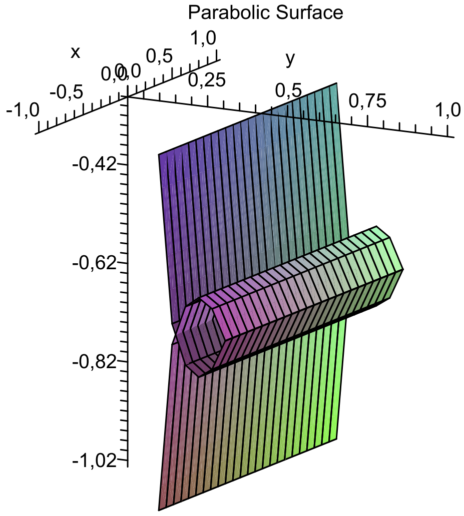
Example 7.2.3. *Finally, we plot a* $H = 2$ *surfaces invariant by parabolic isometries. Putting* $d = 8$, $\tau = -1/2$ *and* $H = 2$, *we obtain:*

$$u(y) = \sqrt{2}\arcsin(8y - 2H) - \frac{4\sqrt{2}H}{\sqrt{4H^2-1}}\arctan\left(\frac{2H\tan(\frac{\arcsin(8y-2H)}{2}) + 1}{\sqrt{4H^2-1}}\right)$$





By considering the rotation around the y *axes we have a complete surface, here we give part of this surface.*



7.3 Surfaces invariant by parabolic isometries in $\widetilde{\mathbf{PSL_2}}(\mathbb{R},\tau)$ having constant mean curvature $\mathbf{H} \neq 0$

In this section we describe the behavior of surfaces invariant by parabolic isometries, which have constant mean curvature $H \neq 0$. For later use we define the function $g(y) = dy - 2H$.

Taking into account Formula (7.1.1), we obtain the next Lemma.

Lemma 7.3.1. *Let H be the mean curvature of a surface invariant by parabolic isometries. Then from the Formula (7.1.1) we obtain,*

1. *If $d > 0$, we have*
   - *If $1/2 < H$, then $y_1 < y < y_2$ where $y_1 = \dfrac{2H-1}{d}$ and $y_2 = \dfrac{2H+1}{d}$ and there exist a unique number $y_0 = \dfrac{2H}{d} \in (y_1, y_2)$ satisfying $g(y_0)$=0. Furthermore $g \leq 0$ on $[y_1, y_0)$ and $g \geq 0$ on $(y_0, y_1]$. Consequently, the function $h(y)$ is defined on $[y_1, y_2]$, has a nonfinite derivative at y_1 and y_2, is strictly decreasing on (y_1, y_0) and strictly creasing on (y_0, y_2).*
   - *If $0 < H < 1/2$, then $0 < y < y_2$ and there exist a unique number $y_0 = \dfrac{2H}{d} \in (0, y_2)$ satisfying $g(y_0)$=0. Furthermore $g \leq 0$ on $(0, y_0)$ and $g \geq 0$ on $(y_0, y_1]$. Consequently, the function $u(y)$ is defined on $(0, y_2]$. The function u has a nonfinite derivative at y_2, is strictly decreasing on $(0, y_0)$ and strictly creasing on (y_0, y_2).*

2. *If $d < 0$, we have*

– *Here, necessarily* $0 < H < 1/2$. *Setting* $d = -c$, *we have that,* $0 < y < y_2$, *where* $y_2 = \dfrac{1-2H}{c}$. *Consequently, the function* $u(y)$ *is defined on* $(0, y_2]$, *has a nonfinite derivative* y_2, *is strictly decreasing on* $(0, y_2)$.

Proof. Setting $f(y) = 1-(dy-2H)^2$, then $f(y) = 0 \Longleftrightarrow y = \dfrac{2Hd \pm |d|}{d^2}$. So

1. If $d > 0$, then $y = \dfrac{2H \pm 1}{d}$, thus we consider two cases,

   – If $2H-1 > 0$, then $f(y) > 0 \Longleftrightarrow y_1 = \dfrac{2H-1}{d} < y < y_2 = \dfrac{2H+1}{d}$, and this is clear that $y_1 < y_0 = \dfrac{2H}{d} < y_2$. So the affirmation holds.

   – If $2H-1 < 0$, then $f(y) > 0 \Longleftrightarrow 0 < y < y_2 = \dfrac{2H+1}{d}$, and this is clear that $0 < y_0 = \dfrac{2H}{d} < y_2$. So the affirmation holds.

2. If $d < 0$. Setting $d = -c$, with $c > 0$, then $y = \dfrac{-2Hc \mp c}{c^2} = \dfrac{-2H \mp 1}{c}$, since that $y > 0$, this implies that $1-2H > 0$. So $u'(y) = -\sqrt{2}\dfrac{cy+2H}{y\sqrt{1-(cy+2H)^2}}$, and $0 < y < y_2 = \dfrac{1-2H}{c}$, thus the function $u(y)$ is strictly decreasing and has a nonfinite derivative at y_2. So the affirmation holds.

□

Lemma 7.3.2. *Letting* $y \longrightarrow y_1, y_2$, *we infer by a computation that the curvature*

$$k(\rho) = \frac{u''}{(1+(u')^2)^{3/2}}$$

goes to

– *If* $d > 0$ *and* $H > 1/2$,

$$\begin{aligned} k(y_1) &= -\frac{y_1^2 f'(y_1)}{(1+4\tau^2)g(y_1)} \\ k(y_2) &= -\frac{y_2^2 f'(y_2)}{(1+4\tau^2)g(y_2)} \end{aligned}$$

– *If* $d > 0$ *and* $0 < H < 1/2$

$$k(y_2) = -\frac{2dy_2^2}{1+4\tau^2}$$

– *If* $d<0$, *this is* $d=-c$, $c>0$,

$$k(y_2) \;=\; \frac{2cy_2^2}{1+4\tau^2}$$

Proof. The proof is analogous to this one on Lemma (6.5.3). By considering

$$k(y)=\frac{y\sqrt{1+4\tau^2}}{[y^2f+(1+4\tau^2)g^2]^{3/2}}[2yfg'-2gf-ygf']$$

□

As a consequence of Lemma (7.3.1), we have the next results.

Theorem 7.3.1. *Let* S *be the* H *surface invariant by parabolic isometries immersed into* $\widetilde{PSL}_2(\mathbb{R},\tau)$. *Then, there exist a one-parameter family* $\mathcal{P}_d$, $d\in\mathbb{R}$ *of complete* H*-surfaces invariant by one-parameter group of parabolic isometries such that,*

1. *For* $d>0$, *and* $H>1/2$ *the surface* $\mathcal{P}_d$ *is immersed (and nonembedded) annulus, invariant by vertical translation, and is contained in the closed region bounded by the vertical cylinders* $y=y_1$ *and* $y=y_2$. *See Fig. 3.a*
2. *For* $d>0$, *and* $0<H<1/2$ *the surface* $\mathcal{P}_d$ *is a properly immersed (and nonembedded) annulus, it is symmetric with respect to slice* $t=0$, *the maximum value of* y *is* $y=y_2$. *See Fig. 3.b*
3. *For* $d=-c<0$ *and* $0<H<1/2$ *the surface* $\mathcal{P}_d$ *is a properly embedded annulus symmetric with respect to the slice* $t=0$, *and the maximum value of* y *is* $y=y_2$. *See Fig. 3.c*
4. *When* d *tends to* 0, *then the surface* $\mathcal{P}_d$ *tends toward the surface*

$$F(y)=\frac{-2\sqrt{1+4\tau^2}H\ln(y)}{\sqrt{1-4H^2}}$$

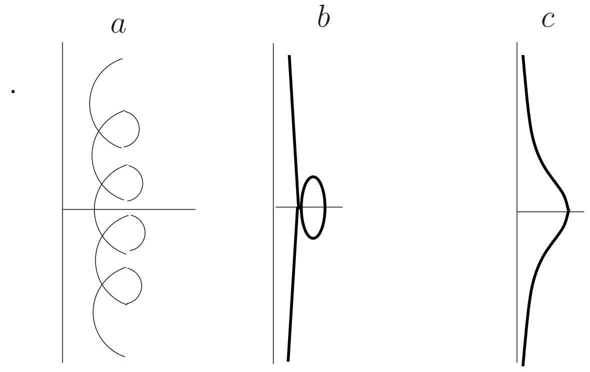


Figure 3.– Generating curve for surfaces invariant by one-parameter group of parabolic isometries with $H \neq 1/2$

Now, we consider the case $H \equiv 1/2$. In this case the function u from the Lemma 7.1.1 become

$$u(y) = \sqrt{1+4\tau^2}\int \frac{dy-1}{y}\frac{1}{\sqrt{1-(dy-1)^2}}dy \tag{7-3}$$

We denote by $f(y) = 1-(dy-1)^2$ and $g(y) = dy-1$, so we obtain the next lemma.

Lemma 7.3.3. *By considering the surface S invariant by one-parameter group of parabolic isometries with constant mean curvature $H = 1/2$, we obtain that $d > 0$ and the function $h(y)$ is defined for $0 < y < y_2 = \frac{2}{d}$. Furthermore, there exist a number $y_0 = \frac{1}{d}$ with $0 < y_0 < y_2$ such that $g(y)$ is positive for $0 < y < y_0$, $g(y_0) = 0$ and $g(y)$ is negative for $y_0 < y < y_2$. Consequently the function $h(y)$ is strictly decreasing for $0 < y < y_0$, has a horizontal tangent at $y = y_0$ and is strictly increasing for $y_0 < y < y_2$.*

Proof. The function $f(y) > 0 \Leftrightarrow (dy-1)^2 < 1 \Leftrightarrow -1 < dy-1 < 1 \Leftrightarrow 0 < dy < 2$, since $y > 0$ this implies that $d > 0$, so $0 < y < y_2 = \frac{2}{d}$, observe that $f(y) = 0 \Leftrightarrow$ either $y = 0$ or $y = \frac{2}{d}$. Observe that $0 < y_0 = \frac{1}{d} < \frac{2}{d}$, then $g(y)$ is positive for $0 < y < y_0$, $g(y_0) = 0$ and $g(y)$ is negative for $y_0 < y < y_2$. Consequently the function $h(y)$ is strictly decreasing for $0 < y < y_0$, has a horizontal tangent at $y = y_0$ and is strictly increasing for $y_0 < y < y_2$.

□

Considering $H = 1/2$, we have the next Lemma.

Lemma 7.3.4. *Letting $y \longrightarrow y_2$, we infer by a computation that the curvature*

$$k(\rho) = \frac{u''}{(1+(u')^2)^{3/2}}$$

goes to

$$k(y_2) = -\frac{2dy_2^2}{1+4\tau^2}$$

Proof. The proof is analogous to this one on Lemma (6.5.3). □

As a consequence of Lemma 7.3.3 we have the next result.

Theorem 7.3.2. *Let S be the $H = 1/2$ surface invariant by parabolic isometries immersed into $\widetilde{PSL}_2(\mathbb{R},\tau)$. Then, there exist a one-parameter family $\mathcal{J}_d$, $d \in \mathbb{R}_+$ of complete H-surfaces invariant by one-parameter group of parabolic isometries such that the surface $\mathcal{J}_d$ is a properly immersed (and non-embedded) annulus, it is symmetric with respect to slice $t = 0$, the maximum value of y is $y = y_2$. See Fig. 4*

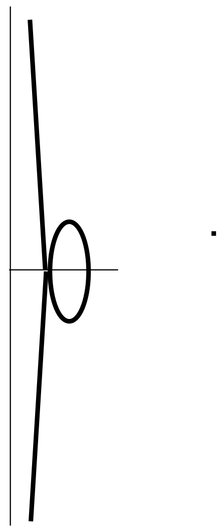


Figure 4..- Generating curve for surfaces invariant by one-parameter group of parabolic isometries with $H \equiv 1/2$