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Surfaces invariant by one-parameter group of hyperbolic isometries having constant mean curvature in $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$

On (3) Ricardo Sa Earp gave explicit formulas for parabolic and hyperbolic screw motions surfaces immersed in $\mathbb{H}^2 \times \mathbb{R}$. There, they gave several examples.

In this chapter we only consider surfaces invariant by one-parameter group of hyperbolic isometries having constant mean curvature immersed in $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$. Since for $\tau \equiv 0$ we are in $\mathbb{H}^2 \times \mathbb{R}$ then we have generalized the result obtained by Ricardo Sa Earp when the surface is invariant by one-parameter group of hyperbolic isometries having constant mean curvature.

In this chapter we focus our attention on surfaces invariant by one-parameter group of hyperbolic isometries. To study this kind of surfaces, we take $M^2 = \mathbb{H}^2$ that is the half plane model for the hyperbolic space. Thus, the metric of M^2 is given by:

$$d\sigma^2 = \lambda^2(dx^2 + dy^2), \quad \lambda = \frac{1}{y}.$$

From Proposition 5.1.1, we know that to obtain a hyperbolic motion on $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$, it is necessary consider a hyperbolic motion (hyperbolic isometry) on \mathbb{H}^2 .

8.1

Surfaces invariant by one-parameter group of hyperbolic isometries main lemma

The idea to obtain a surface invariant by one-parameter group of hyperbolic isometries is simple, we will take a curve in the xt plane and we will apply one-parameter group Γ of hyperbolic isometries on $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$. We denote by $\alpha(x) = (x, 1, u(x))$ the curve in the xt plane and by $S = \Gamma(\alpha)$, the surface invariant by one-parameter group of hyperbolic isometries generate by α .

Since the most simple hyperbolic isometry on \mathbb{H}^2 is the homotetia from the origin, we re-parameterized the hyperbolic plane whit coordinates R^+ and

$\theta \in (0, \pi)$, so that

$$\begin{aligned}x &= R \cos(\theta) \\y &= R \sin(\theta)\end{aligned}$$

where $R \in \mathbb{R}$ and $\theta \in (0, \pi)$.

Thus, the surface S is parameterized by,

$$\varphi(\rho, \theta) = (R \cos(\theta), R \sin(\theta), u(\theta))$$

Now, we rewrite all the expressions in this news coordinates, thus we obtain the next lemma.

Lemma 8.1.1. *By considering the above re-parametrization to the hyperbolic plane, we may rewrite all terms in the form,*

$$\begin{aligned}- \partial_x &= \cos(\theta) \partial_R - \frac{\sin(\theta)}{R} \partial_\theta \\- \partial_y &= \sin(\theta) \partial_R + \frac{\cos(\theta)}{R} \partial_\theta \\- R_x &= \cos(\theta) \\- R_y &= \sin(\theta) \\- \theta_x &= -\frac{\sin(\theta)}{R} \\- \theta_y &= \frac{\cos(\theta)}{R} \\- \lambda &= \frac{1}{R \sin(\theta)} \\- d\sigma^2 &= \frac{1}{R^2 \sin^2(\theta)} (dR^2 + R^2 d\theta^2)\end{aligned}$$

Proof. From,

$$\begin{cases} x = R \cos(\theta), \\ y = R \sin(\theta), \end{cases} \quad (8-1)$$

we obtain,

$$\begin{cases} dx = \cos(\theta) dR - R \sin(\theta) d\theta, \\ dy = \sin(\theta) dR + R \cos(\theta) d\theta, \end{cases} \quad (8-2)$$

since,

$$\begin{aligned}dx^2 + dy^2 &= dR^2 + R^2 d\theta^2, \\ \lambda &= \frac{1}{R \sin(\theta)}\end{aligned}$$

then $\lambda^2 = \frac{1}{R^2 \sin^2(\theta)}$, thus,

$$\begin{aligned} d\sigma^2 &= \lambda^2(dx^2 + dy^2) \\ &= \frac{1}{R^2 \sin^2(\theta)} (dR^2 + R^2 d\theta^2) \end{aligned}$$

Furthermore, setting

$$\partial_x = a\partial_R + b\partial_\theta$$

and evaluating in equation (8-2). We obtain,

$$\begin{cases} 1 = \cos(\theta)a - R \sin(\theta)b, \\ 0 = \sin(\theta)a + R \cos(\theta)b, \end{cases}$$

hence,

$$a = \cos(\theta), \quad b = -\frac{\sin(\theta)}{R}$$

that is,

$$\partial_x = \cos(\theta)\partial_R - \frac{\sin(\theta)}{R}\partial_\theta.$$

By considering,

$$\partial_y = a\partial_R + b\partial_\theta$$

and evaluating in equation (6-2). We obtain,

$$\begin{cases} 0 = \cos(\theta)a - R \sin(\theta)b, \\ 1 = \sin(\theta)a + R \cos(\theta)b, \end{cases}$$

so

$$a = \sin(\theta), \quad b = \frac{\cos(\theta)}{R}$$

that is,

$$\partial_y = \sin(\theta)\partial_R + \frac{\cos(\theta)}{R}\partial_\theta.$$

From $x^2 + y^2 = R^2$, we obtain,

$$2x = 2RR_x$$

$$2y = 2RR_y$$

Considering the equation (8-1), we obtain,

$$R_x = \cos(\theta)$$

$$R_y = \sin(\theta)$$

Derivation of equation (8-1) with respect to x and y respectively gives,

$$1 = R_x \cos(\theta) - R \sin(\theta)\theta_x$$

$$1 = R_y \sin(\theta) + R \cos(\theta)\theta_y$$

Now, by using R_x and R_y , we obtain,

$$\begin{aligned} \theta_x &= -\frac{\sin(\theta)}{R} \\ \theta_y &= \frac{\cos(\theta)}{R} \end{aligned}$$

This completes the proof. □

The next Lemma is crucial for our study. We follow ideas of Appendix A of (11). Consider the graph $t = u(\theta)$ in the plane $x t$, and denote by $S = \text{grap}(\Gamma u)$. Supposing that S has constant mean curvature H , we have the lemma:

Lemma 8.1.2. *The function u satisfies*

$$u(\theta) = \int \frac{(d - 2H \cot(\theta))\sqrt{1 + 4\tau^2 \cos^2(\theta)}}{\sqrt{1 - \sin^2(\theta)(d - 2H \cot(\theta))^2}} d\theta - 2\tau\theta \quad (8-3)$$

where d is a real number.

Proof. Since S has mean curvature H , then by lemma 5.2.1 the function u satisfies the equation

$$2H = \text{div}_{\mathbb{H}^2} \left(\frac{\alpha}{W} e_1 + \frac{\beta}{W} e_2 \right), \quad (8-4)$$

where $W = \sqrt{1 + \alpha^2 + \beta^2}$, $\alpha = \frac{u_x}{\lambda} - 2\tau$, and $\beta = \frac{u_y}{\lambda}$.

By abuse of notation we consider $u(x, y) = u(R(x, y), \theta(x, y))$, since S is a surface invariant by one-parameter group of hyperbolic isometries, then the function u does not depend of R , so,

$$\begin{cases} u_x = u_R R_x + u_\theta \theta_x = u_\theta \theta_x, \\ u_y = u_R R_y + u_\theta \theta_y = u_\theta \theta_y, \end{cases}$$

By considering the lemma 8.1.1, this implies,

$$\begin{aligned}\frac{u_x}{\lambda} &= -u_\theta \sin^2(\theta) \\ \frac{u_y}{\lambda} &= u_\theta \sin(\theta) \cos(\theta)\end{aligned}$$

Thus

$$\begin{aligned}\alpha &= -u_\theta \sin^2(\theta) - 2\tau \\ \beta &= u_\theta \sin(\theta) \cos(\theta)\end{aligned}$$

this implies,

$$\begin{aligned}\alpha^2 + \beta^2 &= \sin^2(\theta)(u_\theta + 2\tau)^2 + 4\tau^2 \cos^2(\theta) \\ W^2 &= 1 + \sin^2(\theta)(u_\theta + 2\tau)^2 + 4\tau^2 \cos^2(\theta)\end{aligned}$$

$$\text{Setting } X_u = \frac{\alpha}{W}e_1 + \frac{\beta}{W}e_2 = \frac{1}{W} \left(\alpha \frac{\partial_x}{\lambda} + \beta \frac{\partial_y}{\lambda} \right).$$

We need express X_u in coordinates R and θ . Observe that,

$$\frac{\partial_x}{\lambda} = R \sin \theta \cos \theta \partial_R - \sin^2 \theta \partial_\theta$$

and

$$\frac{\partial_y}{\lambda} = R \sin^2 \theta \partial_R + \sin \theta \cos \theta \partial_\theta$$

Furthermore,

$$\begin{aligned}\alpha \frac{\partial_x}{\lambda} + \beta \frac{\partial_y}{\lambda} &= -(\sin^2 \theta u_\theta + 2\tau)(R \sin \theta \cos \theta \partial_R - \sin^2 \theta \partial_\theta) + \\ &\quad + \sin \theta \cos \theta u_\theta (R \sin^2 \theta \partial_R + \sin \theta \cos \theta \partial_\theta) \\ &= -[R \sin^3 \theta \cos \theta u_\theta + 2\tau R \sin \theta \cos \theta] \partial_R + [\sin^4 \theta u_\theta + 2\tau \sin^2 \theta] \partial_\theta \\ &\quad + R \sin^3 \theta \cos \theta u_\theta \partial_R + \sin^2 \theta \cos^2 \theta u_\theta \partial_\theta \\ &= -2\tau R \sin \theta \cos \theta \partial_R + \sin^2 \theta (u_\theta + 2\tau) \partial_\theta\end{aligned}$$

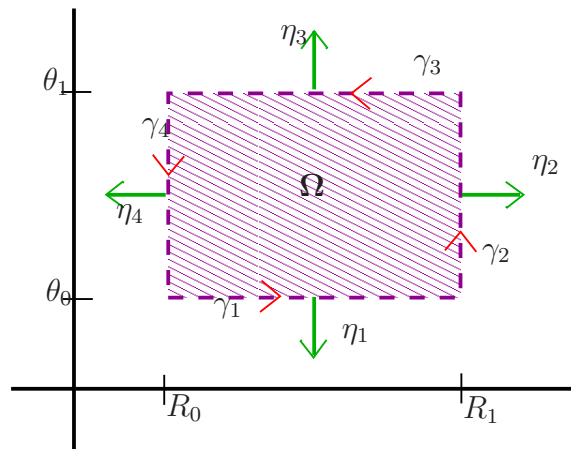
So, we conclude that,

$$X_u = \frac{1}{W} [\sin^2 \theta (u_\theta + 2\tau) \partial_\theta - 2\tau R \sin \theta \cos \theta \partial_R]$$

and

$$W = \sqrt{1 + 4\tau^2 \cos^2 \theta + \sin^2 \theta (u_\theta + 2\tau)^2}$$

Let $\theta_0, \theta_1 \in (0, \pi)$ with $\theta_0 < \theta_1$ and $R_0, R_1 \in \mathbb{R}^+$ with $R_0 < R_1$ and consider the domain $\Omega = [\theta_0, \theta_1] \times [R_0, R_1]$ in the plane $R^+\theta$.



By integrating the equation (8-4), we obtain

$$\int_{\partial(\Omega)} \langle X_u, \eta \rangle = 2H \text{Area}([\theta_0, \theta_1] \times [R_0, R_1])$$

where η is the outer co-normal.

Since

$$\int_{\partial\Omega} \langle X_u, \eta \rangle = \int_{\gamma_1} \langle X_u, \eta_1 \rangle + \int_{\gamma_2} \langle X_u, \eta_2 \rangle + \int_{\gamma_3} \langle X_u, \eta_3 \rangle + \int_{\gamma_4} \langle X_u, \eta_4 \rangle$$

we compute each integral:

For the first integral: Observe that,

$$\gamma_1(s) = (s, \theta_0), \quad R_0 \leq s \leq R_1.$$

This implies,

$$\gamma_1' = \partial_R,$$

thus

$$|\gamma_1'| = \frac{1}{R \sin \theta},$$

and

$$\eta_1 = -\sin \theta \partial_\theta.$$

Furthermore

$$\langle X_u, \eta_1 \rangle = -\frac{\sin^3 \theta (u_\theta + 2\tau)(\theta_0)}{W} \frac{1}{\sin^2 \theta} = -\frac{\sin \theta (u_\theta + 2\tau)}{W},$$

hence

$$\int_{\gamma_1} \langle X_u, \eta_1 \rangle = \int_{R_0}^{R_1} -\frac{u_\theta + 2\tau}{RW}(\theta_0) dR$$

For the third integral: Observe that,

$$\gamma_3(s) = (R_1 - s, \theta_1), \quad 0 \leq s \leq R_1 - R_0.$$

This implies,

$$\gamma_3' = -\partial_R,$$

thus

$$|\gamma_3'| = \frac{1}{R \sin \theta},$$

and

$$\eta_3 = \sin \theta \partial_\theta.$$

Furthermore

$$\langle X_u, \eta_3 \rangle = \frac{\sin^3 \theta (u_\theta + 2\tau)(\theta_1)}{W} \frac{1}{\sin^2 \theta} = \frac{\sin \theta (u_\theta + 2\tau)}{W},$$

hence,

$$\int_{\gamma_3} \langle X_u, \eta_3 \rangle = \int_0^{R_1 - R_0} \frac{\sin \theta (u_\theta + 2\tau)}{W} \frac{1}{(R_1 - s) \sin \theta} ds = \int_{R_0}^{R_1} \frac{(u_\theta + 2\tau)}{RW}(\theta_1) dR$$

For the second integral: Observe that,

$$\gamma_2(s) = (R_1, s), \quad \theta_0 \leq s \leq \theta_1.$$

This implies,

$$\gamma_2' = \partial_\theta,$$

thus

$$|\gamma_2'| = \frac{1}{\sin \theta},$$

and

$$\eta_2 = R \sin \theta \partial_R.$$

Furthermore

$$\langle X_u, \eta_2 \rangle = -\frac{2\tau R^2 \sin^2 \theta \cos \theta}{W} \frac{1}{R^2 \sin^2 \theta} = -\frac{2\tau \cos \theta}{W},$$

hence,

$$\int_{\gamma_2} \langle X_u, \eta_2 \rangle = \int_{\theta_0}^{\theta_1} -\frac{2\tau \cos \theta}{W \sin \theta} d\theta$$

For the four integral: Observe that,

$$\gamma_4(s) = (R_0, \theta_1 - s), \quad 0 \leq s \leq \theta_1 - \theta_0.$$

This implies,

$$\gamma_4' = -\partial_\theta,$$

thus

$$|\gamma_4'| = \frac{1}{\sin \theta},$$

and

$$\eta_4 = -R \sin \theta \partial_R.$$

Furthermore

$$\langle X_u, \eta_4 \rangle = \frac{2\tau R^2 \sin^2 \theta \cos \theta}{W} \frac{1}{R^2 \sin^2 \theta} = \frac{2\tau \cos \theta}{W},$$

hence,

$$\int_{\gamma_4} \langle X_u, \eta_4 \rangle = \int_0^{\theta_1 - \theta_0} \frac{2\tau \cos(\theta_1 - s)}{W} \frac{1}{\sin(\theta_1 - s)} ds = \int_{\theta_0}^{\theta_1} \frac{2\tau \cos \theta}{W \sin \theta} d\theta$$

Taking into account this four integrals, we obtain

$$\int_{\partial\Omega} \langle X_u, \eta \rangle = \int_{R_0}^{R_1} \frac{(u_\theta + 2\tau)(\theta_1)}{W} \frac{dR}{R} - \int_{R_0}^{R_1} \frac{(u_\theta + 2\tau)(\theta_0)}{W} \frac{dR}{R}$$

Observe that,

$$\begin{aligned} \text{Area}(\Omega) &= \int_{\theta_0}^{\theta_1} \int_{R_0}^{R_1} \sqrt{\det(g_{ij})} dR d\theta \\ &= \int_{\theta_0}^{\theta_1} \int_{R_0}^{R_1} \frac{1}{R \sin^2 \theta} dR d\theta \end{aligned}$$

So, we conclude that

$$\int_{R_0}^{R_1} \frac{(u_\theta + 2\tau)(\theta_1)}{W} \frac{dR}{R} - \int_{R_0}^{R_1} \frac{(u_\theta + 2\tau)(\theta_0)}{W} \frac{dR}{R} = \int_{\theta_0}^{\theta_1} \int_{R_0}^{R_1} \frac{1}{R \sin^2 \theta} dR d\theta$$

which we can write in the form:

$$\int_{\theta_0}^{\theta_1} \partial_{\theta} \left(\frac{(u_{\theta} + 2\tau)}{W} \right) d\theta = 2H \int_{\theta_0}^{\theta_1} \frac{1}{\sin^2 \theta} d\theta$$

By considering that Ω is any domain in the plane $R\theta$, and taking the derivative with respect to ρ we obtain:

$$\partial_{\theta} \left(\frac{u_{\theta} + 2\tau}{W} \right) = \frac{2H}{\sin^2 \theta}$$

by integrating this expression:

$$\frac{u_{\theta} + 2\tau}{\sqrt{1 + 4\tau^2 \cos^2 \theta + \sin^2 \theta (u_{\theta} + 2\tau)^2}} = -2H \cot \theta + d$$

where $d \in \mathbb{R}$. This implies:

$$(u_{\theta} + 2\tau)^2 [1 - \sin^2 \theta (d - 2H \cot \theta)^2] = (d - 2H \cot \theta)^2 [1 + 4\tau^2 \cos^2 \theta]$$

so the function u satisfies:

$$u(\theta) = \int \frac{(d - 2H \cot \theta) \sqrt{1 + 4\tau^2 \cos^2 \theta}}{\sqrt{1 - \sin^2 \theta (d - 2H \cot \theta)^2}} d\theta - 2\tau\theta$$

□

8.2

Examples of surfaces invariant by one-parameter group of hyperbolic isometries in $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$

Now, we explore the Lemma 8.1.2. By considering $\tau = -1/2$ we obtain the next consequences.

Lemma 8.2.1. *Making $d \equiv 0$ and $\tau = -1/2$, then the integral*

$$u = - \int \frac{2H \cot(\theta) \sqrt{1 + \cos^2(\theta)}}{\sqrt{1 - \sin^2(\theta) (2H \cot(\theta))^2}} d\theta + \theta$$

has the next solution,

$$\begin{aligned}
 u(\theta) = & \frac{1}{2} \arctan \left(\frac{\sin^2(\theta) - 1 + (1 - 4H^2)/8H^2}{\sqrt{(\sin^2(\theta) + (1 - 4H^2)/4H^2)(2 - \sin^2(\theta))}} \right) + \\
 & + \frac{\sqrt{2}H}{\sqrt{1 - 4H^2}} \ln \left\{ \frac{1 - 4H^2}{H^2} + \frac{(4H^2 - 1) \sin^2 \theta}{4H^2} + \right. \\
 & + \frac{\sqrt{2(1 - 4H^2)}}{H} \sqrt{((1 - 4H^2)/4H^2 + \sin^2 \theta) (2 - \sin^2(\theta))} \left. \right\} + \\
 & - \frac{\sqrt{2}H}{\sqrt{1 - 4H^2}} \ln \{ \sin^2 \theta \} + \theta
 \end{aligned}$$

where $0 < H < 1/2$.

Proof. Taking $\tau = -1/2$, $d \equiv 0$, the expression

$$f(\theta) = -2H \int \frac{\cot(\theta) \sqrt{1 + \cos^2(\theta)}}{\sqrt{1 - \sin^2(\theta)(2H \cot(\theta))^2}} d\theta$$

can be write as:

$$f(\theta) = - \int \frac{\sqrt{1 + \cos^2(\theta)}}{\sin \theta \sqrt{(1 - 4H^2)/4H^2 + \sin^2 \theta}} \cos \theta d\theta$$

making $b = (1 - 4H^2)4H^2$, $v = \sin \theta$, then $dv = \cos \theta d\theta$, so the integral becomes:

$$f \equiv - \int \frac{\sqrt{2 - v^2}}{v \sqrt{b + v^2}} dv$$

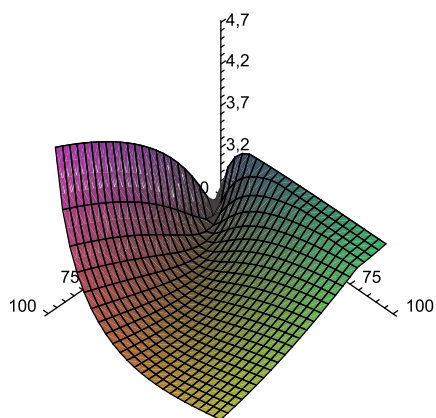
whose solution is given by:

$$\begin{aligned}
 f = & \frac{1}{2} \arctan \left(\frac{(v^2 - 1) + b/2}{\sqrt{2 - v^2} \sqrt{b + v^2}} \right) + \\
 & + \frac{\sqrt{2}}{2\sqrt{b}} \ln \left(\frac{4b + v^2(2 - b) + 2\sqrt{2b} \sqrt{(2 - v^2)(b + v^2)}}{v^2} \right)
 \end{aligned}$$

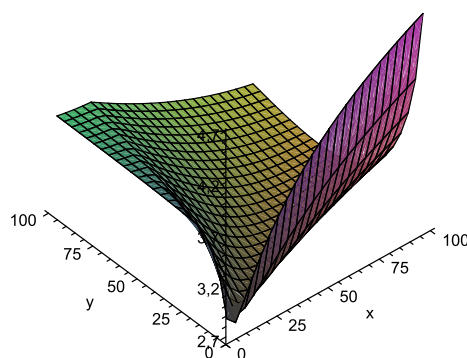
by substitution the values of v and b , we obtain the result. □

Example 8.2.1. Making $H = 0.3$ in the lemma 8.2.1, since $\theta = \arctan \left(\frac{y}{x} \right)$, so in coordinates XYT , the graph has two sheet. The first sheet is given by $x > 0$ (here we have two views to the first sheet),

H Surfaces, Invariant by Hyperbolic Isometries, with $x > 0$

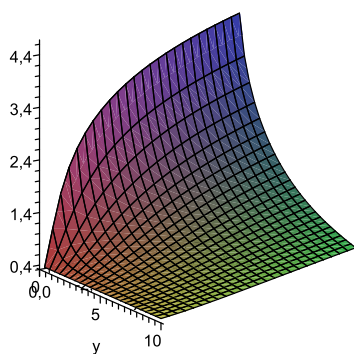


H Surfaces, Invariant by Hyperbolic Isometries, with $x > 0$

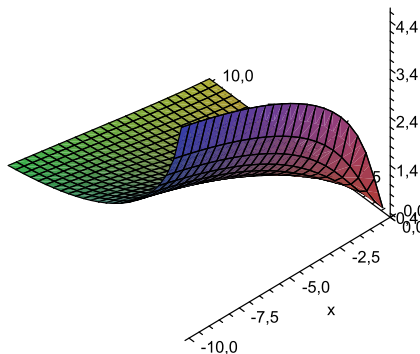


Now, we consider the sheet given by $x < 0$,

H Surfaces, Invariant by Hyperbolic Isometries, with $x < 0$

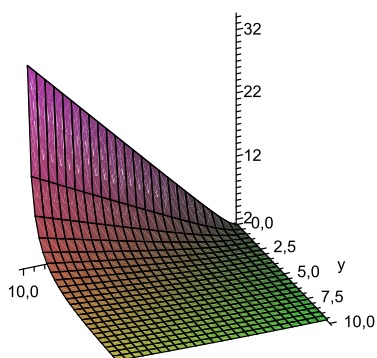


H Surfaces, Invariant by Hyperbolic Isometries, with $x < 0$

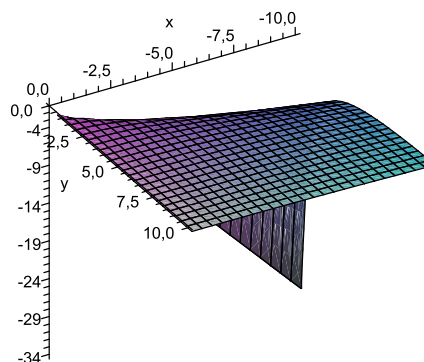


Example 8.2.2. Making $H = 1/2$, $\tau = -1/2$, $d \equiv 0$, and by following the ideas of the proof of the Lemma 8.2.1, we obtain a surface invariant by one-parameter group of hyperbolic isometries having constant mean curvature $H = 1/2$. We plot the surface with Maple's help.

1/2 Surface, Invariant by Hyperbolic Isometries, with $x > 0$



1/2 Surface, Invariant by Hyperbolic Isometries, with $x < 0$



Example 8.2.3. Making $H \equiv 0$, $d \equiv 1$, we obtain a minimal surface invariant by one-parameter group of hyperbolic isometries. To see this, observe that

$$u = \int \frac{(d - 2H \cot(\theta))\sqrt{1 + 4\tau^2 \cos^2(\theta)}}{\sqrt{1 - \sin^2(\theta)(d - 2H \cot(\theta))^2}} d\theta - 2\tau\theta$$

becomes:

$$u = \int \frac{\sqrt{1 + 4\tau^2 \cos^2(\theta)}}{\cos \theta} d\theta - 2\tau\theta$$

Making:

$$f(\theta) = \int \frac{\sqrt{1 + 4\tau^2 \cos^2(\theta)}}{\cos \theta} d\theta$$

then,

$$f(\theta) = \int \frac{\sqrt{1 + 4\tau^2 \cos^2(\theta)}}{\cos^2 \theta} \cos \theta d\theta$$

let $w = \sin \theta$ so $dw = \cos \theta d\theta$. Setting $a = \frac{1 + 4\tau^2}{4\tau^2}$, the last integral becomes:

$$f(\theta) = |2\tau| \int \frac{\sqrt{a - w^2}}{1 - w^2} dw$$

which has the next solution:

$$f = \arctan\left(\frac{w}{\sqrt{a - w^2}}\right) + \frac{1}{2}\sqrt{-1 + a} \ln\left(\frac{a - w + \sqrt{-1 + a}\sqrt{a - w^2}}{-1 + w}\right) + \frac{\sqrt{-1 + a}}{2} \ln\left(\frac{a + w + \sqrt{-1 + a}\sqrt{a - w^2}}{w + 1}\right)$$

which can be write in the form:

$$f = \arctan\left(\frac{w}{\sqrt{a - w^2}}\right) + \frac{\sqrt{a - 1}}{2} \ln\left(\frac{(a - w + \sqrt{-1 + a}\sqrt{a - w^2})(w + 1)}{(a + w + \sqrt{-1 + a}\sqrt{a - w^2})(w - 1)}\right)$$

so,

$$f = \arctan\left(\frac{|2\tau| \sin \theta}{\sqrt{1 + 4\tau^2 \cos^2 \theta}}\right) + \frac{1}{2|2\tau|} \ln\left(\frac{(1 + 4\tau^2 - 4\tau^2 \sin \theta + \sqrt{1 + 4\tau^2 \cos^2 \theta})(\sin \theta + 1)}{(1 + 4\tau^2 + 4\tau^2 \sin \theta + \sqrt{1 + 4\tau^2 \cos^2 \theta})(\sin \theta - 1)}\right)$$

So, our integral has the solution:

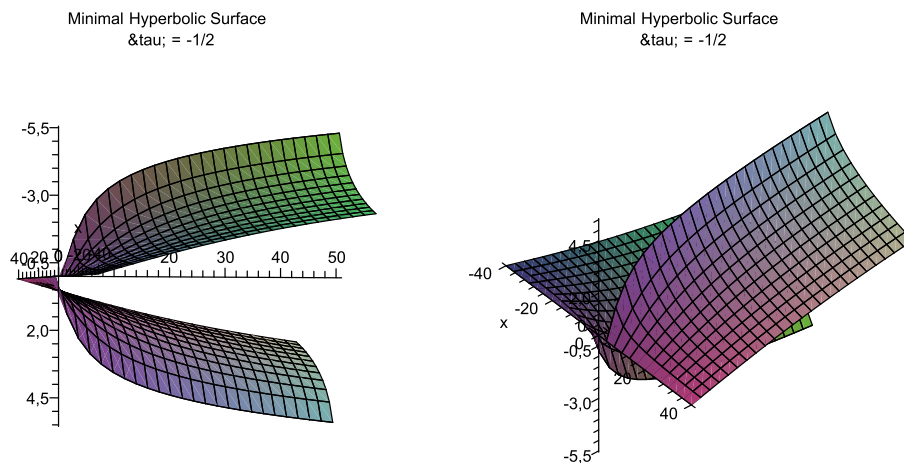
$$u(\theta) = \arctan\left(\frac{|2\tau| \sin \theta}{\sqrt{1 + 4\tau^2 \cos^2 \theta}}\right) - 2\tau\theta + \frac{1}{2|2\tau|} \ln\left(\frac{(1 + 4\tau^2 - 4\tau^2 \sin \theta + \sqrt{1 + 4\tau^2 \cos^2 \theta})(\sin \theta + 1)}{(1 + 4\tau^2 + 4\tau^2 \sin \theta + \sqrt{1 + 4\tau^2 \cos^2 \theta})(\sin \theta - 1)}\right)$$

Observe o domain.

Observe that, by considering $\tau = -1/2$ we obtain (following the same idea):

$$u = \arcsin\left(\frac{\sin \theta}{\sqrt{2}}\right) - \frac{1}{2} \operatorname{arctanh}\left(\frac{2 + \sin \theta}{\sqrt{2 - \sin^2 \theta}}\right) - \frac{1}{2} \operatorname{arctanh}\left(\frac{\sin \theta - 2}{\sqrt{2 + \sin^2 \theta}}\right) + \theta$$

This surface goes to $x = 0$ when y goes to 0, and when $x \rightarrow 0^+$ the surface goes to $-\infty$, if $x \rightarrow 0^-$ the surface goes to $-\infty$. See figure:

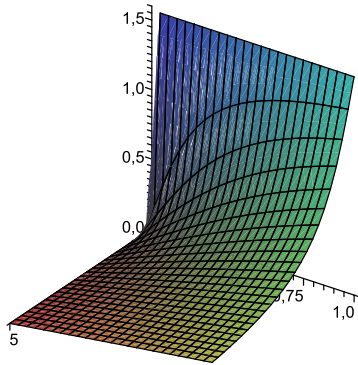


Example 8.2.4. Making $H \equiv 0$, $\tau = -1/2$, $d \equiv 0$, we obtain a minimal surfaces which consist of two sheet (see figure). When y goes to 0 the surface goes to $x = 0$. When $x \rightarrow 0^-$ the function u goes to a negative finite value, and when $x \rightarrow 0^+$ the function goes to a positive finite value. Observe that:

$$u(x, y) = \arctan\left(\frac{y}{x}\right)$$

We plot the surface with Maple's help.

Minimal Hyperbolic Surface
with Heigh Bounded



Minimal Hyperbolic Surface
with Heigh Bounded

