



Carlos Diosdado Espinoza Peñafiel

**Surfaces of Constant Mean Curvature in
Homogeneous Three Manifolds with Emphasis
in $\widetilde{\text{PSL}}_2(\mathbb{R}, \tau)$**

Tese de Doutorado

Thesis presented to the Postgraduate Program in Mathematics of
the Departamento de Matemática, PUC–Rio as partial fulfillment
of the requirements for the degree of Doutor em Matemática

Advisor : Prof. Ricardo Sá Earp
Co–Advisor: Prof. Harold Rosenberg

Rio de Janeiro
July 2010



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Abstract

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In this thesis we study H -surfaces, that is, surfaces having constant mean curvature, immersed in homogeneous simply connected 3-manifold. We focus our attention in the study of existence of H multigraphs. We also study the H -surfaces invariant by one-parameter group of isometries which are immersed in the space $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Keywords

Constant Mean Curvature. Homogeneous Manifolds. Invariant Surfaces. One-parameter Group of Isometries. Multigraphs.

Resumo

Espinoza Peñafiel, Carlos Diosdado; Sá Earp, Ricardo; Rosenberg, Harold. **Superfícies de Curvatura Média Constante em Variedades Homogêneas de Dimensão 3 com Ênfase em $\widetilde{PSL}_2(\mathbb{R}, \tau)$** . Rio de Janeiro, 2010. 138p. Tese de Doutorado — Departamento de Matemática, Pontifícia Universidade Católica do Rio de Janeiro.

Nesta tese, nós estudamos H -superfícies, isto é, superfícies tendo curvatura média constante, imersas em variedades homogêneas simplesmente conexas de dimensão 3. Nós focamos nossa atenção no estudo de existência de H multigráficos. Também estudamos a H -superfícies invariantes por um grupo a um parâmetro de isometrias que estão imersas no espaço $\widetilde{PSL}_2(\mathbb{R}, \tau)$.

Palavras-chave

Curvatura Média Constante. Variedades Homogêneas. Superfícies Invariantes. Grupo a Um-parâmetro de Isometrias. Multigráficos.

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*Inclina hoy la cabeza ante los libros para que
mañana no la inclines ante los hombres.*

Mi madre Rosa y mi abuela Delia, Enseñanzas que valen oro.