1 Introduction

Currently hyperbolic dynamical systems are quite well understood from the topological and statistical perspectives. A natural objective is to characterize the obstructions to hyperbolicity. In [21], Palis conjectured that generation of cycles (heterodimensional cycles and homoclinic tangencies) is the only obstruction for hyperbolicity: every diffeomorphism may be approximated either by a hyperbolic one or by one with a cycle. This conjecture was proved for surface C^1-diffeomorphisms in [22]. In higher dimensions (and for C^1-diffeomorphisms) there are some partial results, see [11] where essential hyperbolicity is considered and [5] which deals with the so called tame diffeomorphisms. This illustrates the importance of the generation of cycles in dynamics.

On the other hand, the dynamical properties which exhibit some "persistence" or "robustness" play an important role in dynamics. A property of a diffeomorphism is robust if there is a neighborhood of it consisting of diffeomorphisms satisfying such a property. This leads to the notion of a robust cycle (see the precise definition below). Bearing in mind this concept, in the C^1-case Bonatti established in [2] a strong version of the "hyperbolicity versus cycles" conjecture above: the set of hyperbolic diffeomorphisms and the set of diffeomorphisms with robust cycles are two open sets whose union is dense in the space of C^1-diffeomorphisms. This conjecture was proved in [5] for tame systems. For surface diffeomorphisms, in [18] it was proved that there are no C^1-robust tangencies.

One of the main topics of this work is the study of *heterodimensional cycles* of diffeomorphisms f, that is cycles involving a pair of hyperbolic transitive sets Ω_1 and Ω_2 for f of different *s-indices* (dimension of their stable bundles) such that

$$W^s(\Omega_1; f) \cap W^u(\Omega_2; f) \neq \emptyset \quad \text{and} \quad W^u(\Omega_1; f) \cap W^s(\Omega_2; f) \neq \emptyset.$$

The *co-index* of the cycle is the absolute value of the difference of the s-indices of Ω_1 and Ω_2. In [5], it is proven that heterodimensional cycles of co-index one yield *C^1-robust heterodimensional cycles* (Definition 1.1). A stronger version of this result concerning the so-called stabilization of the initial cycle was stated in [8] (see the definition below).

In the first part of this work, motivated by [5], we study the generation of C^1-robust heterodimensional cycles of co-index two. Considering the so-called central eigenvalues of the cycle there are three different types of such a cycles - called $(\mathbb{R},\mathbb{R})$, $(\mathbb{R},\mathbb{C})$, and $(\mathbb{C},\mathbb{C})$ cycles (see the discussion in Section 2). In the cases $(\mathbb{R},\mathbb{C})$ and $(\mathbb{C},\mathbb{C})$, Theorem A states the generation of robust cycles as well as the stabilization of these cycles (see Definition 1.2).

The main technical step of the proof of Theorem A is to see that these cycles yield *strong homoclinic points*, that is, the periodic points whose *strong invariant manifolds* have non trivial intersections, see Equation (1.1). Using this intersection we get diffeomorphisms with *blenders* which generate robust heterodimensional cycles. To prove the existence of strong homoclinic points we analyse the dynamics of two-dimensional iterated function systems associated to the central part of the cycle (see Sections 3 and 4).

Blenders (Definition 1.8) are the most important ingredient of this thesis. In the second part of this thesis we study a special class of blenders called symbolic ones (Definition 1.10). These blenders are a generalization of the ones with one-dimensional "central contracting" direction in [4, 5, 6]. Here, following the ideas introduced by Nassiri and Pujals in [19], we study blenders with "central contracting" direction of any dimension.

Blenders are hyperbolic sets that are (in some sense) similar to the thick horseshoes introduced by Newhouse in [20]. Roughly, a blender is a mechanism that guarantees that the dimension of the closure of an invariant unstable manifold of a hyperbolic set is greater than the dimension of its unstable bundle. Besides playing a key role in the generation of robust cycles [5], blenders are also important in other settings such as the construction of non-hyperbolic robustly transitive sets [4] (also in the symplectic and Hamiltonian settings [19]), discontinuity of the dimension of hyperbolic sets [9], and stable ergodicity [23], for example.

In Theorem D we give a sufficient condition (the so-called covering property) for the existence of symbolic blenders with central contracting direction of any dimension. This result enables us to consider blenders in partially hyperbolic contexts with higher dimensional central direction.

Now we give precise definitions and state the main results.

1.1 Heterodimensional cycles of co-index two

In the first part of this work we study heterodimensional cycles. Consider a compact manifold M of dimension $d \geq 4$ and denote by $\mathrm{Diff}^1(M)$ the space of C^1-diffeomorphisms endowed with the C^1 topology.

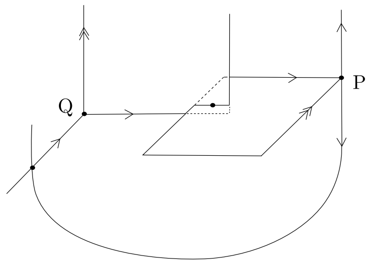

Figure 1.1: Heterodimensional cycle

A *heterodimensional cycle* of a diffeomorphism $f \in \mathrm{Diff}^1(M)$ is a cycle involving a pair of hyperbolic transitive sets Ω_1 and Ω_2 of f of different *s-indices*[1] (dimension of their stable bundles) such that

$$W^s(\Omega_1, f) \cap W^u(\Omega_2, f) \neq \emptyset \quad \text{and} \quad W^u(\Omega_1, f) \cap W^s(\Omega_2, f) \neq \emptyset.$$

A special case of heterodimensional cycle occurs when the sets Ω_i are periodic orbits. We are interested in cycles that cannot be destroyed by perturbations:

Definition 1.1 (Robust heterodimensional cycle)**.** *A diffeomorphism f has a* robust heterodimensional cycle of co-index c ($c \in \mathbb{N}$) *associated with its transitive hyperbolic sets Ω_1 and Ω_2 if the difference of the s-indices of Ω_1 and Ω_2 is $\pm c$ and there is a C^1-neighbourhood $\mathcal{U}$ of f such that every $g \in \mathcal{U}$ has a heterodimensional cycle associated with the hyperbolic continuations*[2] *Ω_1^g and Ω_2^g of Ω_1 and Ω_2, respectively.*

We consider heterodimensional cycles of co-index $c \geq 2$ associated with a pair of saddles P and Q of different s-indices. Recall that by the Kupka-Smale theorem the invariant manifolds of hyperbolic periodic points of generic diffeomorphisms are in general position (i.e., either they meet transversally or are disjoint). This immediately implies that any robust heterodimensional cycle (shortly, robust cycle) involves at least one non trivial hyperbolic set. This leads to the following definition introduced in [8].

Definition 1.2 (Stabilization)**.** *A diffeomorphism f having a heterodimensional cycle associated with the saddles P and Q can be C^1-*stabilized *if there is a diffeomorphism g arbitrarily C^1-close to f having two transitive hyperbolic sets $\Omega_1 \ni P_g$ and $\Omega_2 \ni Q_g$ with a robust cycle (here P_g and Q_g are the continuations of P and Q for g).*

*The cycle can be C^1-*semi-stabilized *if either $\Omega_1 \ni P_g$ or $\Omega_2 \ni Q_g$.*

[1] Analogously, the u-index is the dimension of the unstable bundle.

[2] Ω_i^g is a continuation of Ω_i for g if Ω_i^g is a hyperbolic set, Ω_i^g is close to Ω_i and $f|_{\Omega_i}$ and $g|_{\Omega_i^g}$ are conjugate, $i = 1, 2$.

The analysis of the dynamics of the cycles of co-index c associated with saddles is done in terms of the so-called central eigenvalues of the saddles in the cycle. Assume that the cycle is associated to P and Q of s-indices $s+c$ and s, respectively. Let $\pi(P)$ (resp. $\pi(Q)$) be the period of P (resp. Q) and $\alpha_1, \dots, \alpha_d$ (resp. $\beta_1, \dots, \beta_d$) be the eigenvalues of $Df_P^{\pi(P)}$ (resp. $Df_Q^{\pi(Q)}$), enumerated with multiplicity and ordered in increasing modulus,

$$|\sigma_i| \leq |\sigma_{i+1}|, \quad i = 1, \dots, d-1, \quad \sigma = \alpha, \beta.$$

The eigenvalues $\alpha_{s+1}, \dots, \alpha_{s+c}$ are the *central eigenvalues of the cycle associated with* P. Similarly, $\beta_{s+1}, \dots, \beta_{s+c}$ are the *central eigenvalues of the cycle associated with* Q. Following [5], we say that $\alpha_{s+1}, \dots, \alpha_{s+c}, \beta_{s+1}, \dots, \beta_{s+c}$ are the *central eigenvalues of the cycle* [3]. The cycle is *central separated* if $|\alpha_s| < |\alpha_{s+1}|$ and $|\beta_{s+c}| < |\beta_{s+c+1}|$. Note that, by definition, it follows that $|\alpha_{s+c}| < 1 < |\alpha_{s+c+1}|$ and $|\beta_s| < 1 < |\beta_{s+1}|$. Finally, we say that $\alpha_1, \dots, \alpha_s$ are the *strong stable eigenvalues* of P and $\beta_{s+c+1}, \dots, \beta_d$ are the *strong unstable eigenvalues* of Q.

In [5] it is proved that heterodimensional cycles of co-index one yield C^1-robust heterodimensional cycles. In [8] this result is generalized by proving that "most of these cycles" can be stabilized (see [7] for "pathological" examples of non-stablizable cycles). We prove a version of this result for some co-index two cycles.

Theorem A. *Let M be a manifold of dimension $d \geq 4$ and f a diffeomorphism in $\mathrm{Diff}^1(M)$ having a heterodimensional cycle associated with a pair of saddles P and Q of co-index two. Then*

- *If only one of the pairs of the central eigenvalues of the cycle is non-real, then there is diffeomorphism g arbitrarily C^1-close to f having a C^1-robust heterodimensional cycle of co-index one associated with the continuation Q_g of Q and a transitive hyperbolic set Γ_g, that is, the cycle can be semi-stabilized.*
- *If both pairs of the central eigenvalues of the cycle are non-real, then there is diffeomorphism g arbitrarily C^1-close to f having a C^1-robust heterodimensional cycle of co-index two associated with hyperbolic sets Γ_g and Ω_g such that $Q_g \in \Omega_g$ and $P_g \in \Gamma_g$ (where Q_g and P_g are the continuations of Q and P, respectively), that is, the cycle can be stabilized.*

Let us illustrate an application of the stabilization part of this theorem. Assume that there are an open set $\mathcal{U}$ in $\mathrm{Diff}^1(M)$ and a dense subset $\mathcal{D}$ of $\mathcal{U}$

[3] These eigenvalues are called connexion eigenvalues in [12] and central multipliers in [8].

such that every f in $\mathcal{U}$ has a pair of saddles P_f and Q_f (depending continuously on f) with s-indices $s+2$ and s and (both) central non-real eigenvalues. Moreover, assume that for every $f \in \mathcal{D}$ these two saddles are in the same chain recurrent set. Then Theorem A implies that there is an open and dense subset $\mathcal{V}$ of $\mathcal{U}$ such that for every f in $\mathcal{V}$ these saddles are in the same chain recurrent set. That is: the persistent property of being in the same chain recurrent class is indeed a robust property.

This observation follows by applying twice the connecting lemma for chain recurrent class (also known as connecting lemma for pseudo orbits) in [3]. This result implies that there is a dense subset $\mathcal{D}'$ of $\mathcal{U}$ consisting of diffeomorphisms f with heterodimensional cycles associated to P_f and Q_f: a first application of the connecting lemma provides a transverse intersection between $W^s(P;f)$ and $W^u(Q;f)$, a new application gives an intersection between $W^u(P;f)$ and $W^s(Q;f)$, since the first intersection persists one gets a heterodimensional cycle.

Theorem A provides an open and dense subset $\mathcal{V}$ of $\mathcal{U}$ such that there are transitive hyperbolic transitive sets $\Lambda_f \ni P_f$ and $\Sigma_f \ni Q_f$ involved in a robust cycle. This immediately implies that for every $f \in \mathcal{V}$ the chain recurrent classes of P_f and Q_f coincide (note that two hyperbolic transitive sets involved in a cycle are in the same chain recurrent class).

1.1.1 Ingredients of the proof of Theorem A

The proof of Theorem A is divided in three parts. The first one involves strong homoclinic intersection points. More precisely, let A be a partially hyperbolic periodic point of f of period $\pi(A)$ such that the derivative $Df_A^{\pi(A)}$ has some eigenvalue of modulus one, and such that there is a Df-invariant partially hyperbolic splitting with three non trivial bundles over the orbit $\mathcal{O}(A)$ of A of the form $E^{ss} \oplus E^c \oplus E^{uu}$, where E^{ss} and E^{uu} are the strong stable (corresponding to the eigenvalues α with $|\alpha| < 1$) and strong unstable bundles (corresponding to the eigenvalues β with $|\beta| > 1$), respectively, and E^c corresponds to the eigenvalues of modulus one. The diffeomorphism f has a *strong homoclinic intersection* associated to A if there is a point $R \notin \mathcal{O}(A)$ with

$$R \in W^{ss}\big(\mathcal{O}(A)\big) \cap W^{uu}\big(\mathcal{O}(A)\big), \tag{1.1}$$

where $W^{ss}(\mathcal{O}(A))$ and $W^{uu}(\mathcal{O}(A))$ are the strong stable and strong unstable manifolds of the orbit of A. These manifolds are the only f-invariant manifolds of the same dimension of E^{ss} and E^{uu} which are tangent to E^{ss} and E^{uu}, respectively, throughout the orbit of A. The point R is called a *strong* *homoclinic point* of A.

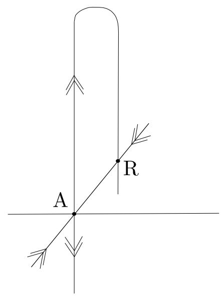

Figure 1.2: Strong homoclinic intersection point

We prove that under the hypothesis of Theorem A, there are diffeomorphisms arbitrarily C^1-close to f having strong homoclinic intersections:

Theorem 1.3 (Strong homoclinic intersections)**.** *Let f be a diffeomorphism in* $\mathrm{Diff}^1(M)$ *having a heterodimensional cycle associated with a pair of saddles P and Q of co-index two. Assume that at least one pair of the central eigenvalues of the cycle is non-real. Then there are diffeomorphisms arbitrarily C^1-close to f having strong homoclinic intersections associated to non-hyperbolic periodic points.*

This theorem is proved throughout Chapters 3 and 4.

In the second part of the proof of Theorem A, we see that diffeomorphisms with a periodic point (non-hyperbolic and with two dimensional center) having a strong homoclinic intersection yield diffeomorphisms with a robust heterodimensional cycle. More precisely:

Theorem 1.4 (Robust cycles of co-index two)**.** *Let f be diffeomorphism with a (non-hyperbolic) periodic point with bidimensional central direction which has a strong homoclinic intersection. Then every C^1-neighborhood $\mathcal{U}$ of f contains diffeomorphisms with C^1-robust heterodimensional cycle of co-index two.*

Theorem 1.4 is our version of the following result:

Theorem 1.5 (Robust cycles of co-index one, Theorem 2.4 in [5])**.** *Let f be a diffeomorphism with a strong homoclinic intersection associated with a saddle-node or a flip (one-dimensional central direction). Then every C^1-neighborhood of f contains diffeomorphisms with C^1-robust heterodimensional cycles of co-index one.*

Finally, note that with Theorems 1.3 and 1.5 we get the following:

Corollary 1.6. *Let f be a diffeomorphism in* $\mathrm{Diff}^1(M)$ *having a heterodimensional cycle associated with a pair of saddles P and Q of co-index two. Assume that the central eigenvalues of Q are real and of P are non-real. Then every C^1-neighborhood of f contains diffeomorphisms with C^1-robust heterodimensional cycles of co-index one.*

Let us observe that blenders are the main tool used in the proof of both theorems above. For further details see Chapter 5.

In the third part of the proof of Theorem A, we study the stabilization of the cycle.

1.2 Skew product maps

Let G be a compact manifold of dimension c, Σ_k the space of bi-infinite sequences of k symbols (endowed with the usual metric), and τ the Bernoulli shift map. We consider symbolic skew product maps Φ defined as follows

$$\Phi : \Sigma_k \times G \to \Sigma_k \times G, \qquad \Phi(\xi, x) = \big(\tau(\xi), \phi_\xi(x)\big), \tag{1.2}$$

where $\phi_\xi : G \to G$ are diffeomorphisms depending continuously on the point ξ. The space Σ_k is called the *base* and the second factor is the *fiber*. To emphasize the role of the fiber maps we write $\Phi = \tau \ltimes \phi_\xi$.

In what follows we fix an open set D of G and a map $\Phi = \tau \ltimes \phi_\xi$. We let $\mathrm{Per}(\Phi)$ be the set of periodic points of Φ and

$$K_\Phi = K_\Phi(D) \stackrel{\text{def}}{=} \overline{\mathscr{P}(\mathrm{Per}(\Phi)) \cap D},$$

where $\mathscr{P} : \Sigma_k \times G \to G$ is the projection on the fiber space.

Under the hypothesis of contracting fiber maps and Hölder dependence of the maps ϕ_ξ on ξ (see the precise definitions below) Theorem B claims that the projection of the maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is the set K_Φ in G and that these sets K_Φ depend continuously on Φ. Let us observe that there are somewhat related results of this theorem in [15].

We also note that in many cases the study of the dynamics of partially hyperbolic sets can be done by means of skew product maps as above. In this setting, under a suitable "domination" hypothesis partially hyperbolic sets are conjugate to skew-product maps with Hölder fiber maps, see [13, 17]. Thus the Hölder dependence hypothesis is quite natural.

To state precisely Theorem B we need some definitions.

Given constants $0 < \lambda < \beta$, we say that a map $\phi\colon \overline{D} \to D$ is (λ, β)-*Lipschitz*[4] on $\overline{D}$ if

$$\lambda \|x-y\| < \|\phi(x)-\phi(y)\| < \beta \|x-y\|, \qquad \text{for all } x, y \in \overline{D}, \tag{1.3}$$

where $\|\cdot\|$ is the metric of G and $\|x-y\|$ denotes the distance between x and y in G.

For a fixed $0 \leq \alpha \leq 1$, we say that $\Phi = \tau \ltimes \phi_\xi$ is *locally α-Hölder continuous* or that its fiber maps ϕ_ξ depend *locally α-Hölder continuously* on $\overline{D}$ with respect to the base point ξ if there is a constant $C \geq 0$ such that

$$\|\phi_\xi^{\pm 1}(x) - \phi_\zeta^{\pm 1}(x)\| \leq C\, d_{\Sigma_k}(\xi, \zeta)^\alpha \tag{1.4}$$

for every $x \in \overline{D}$ and $\xi, \zeta \in \Sigma_k$ with $\xi_0 = \zeta_0$, where d_{Σ_k} is the metric defined in $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ by

$$d_{\Sigma_k}(\xi, \zeta) \stackrel{\text{def}}{=} \nu^\ell, \quad \ell = \min\{i \in \mathbb{Z}^+ : \xi_i \neq \zeta_i \text{ or } \xi_{-i} \neq \zeta_{-i}\}, \quad 0 < \nu < 1. \tag{1.5}$$

We denote by C_Φ the smallest non-negative constant satisfying (1.4) and call it (local) *Hölder constant* of Φ on $\overline{D}$.

Definition 1.7 (Sets of symbolic skew products)**.** *Let $D \subset G$ be an open set, $r \geq 0$, $0 < \lambda < \beta$, $0 \leq \alpha \leq 1$, and $k > 1$. We define $\mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$ as the* set of symbolic skew product maps $\Phi = \tau \ltimes \phi_\xi$ *as in* (1.2) *such that*

- *ϕ_ξ is C^r-(λ, β)-Lipschitz on $\overline{D}$ for all $\xi \in \Sigma_k$,*
- *ϕ_ξ depends locally α-Hölder continuously on $\overline{D}$ with respect to ξ,*
- *if $\beta < 1$ then $\phi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$, and*
- *if $1 < \lambda$ then $\overline{D} \subset \phi_\xi(D)$ for all $\xi \in \Sigma_k$.*

The set $\mathcal{S} = \mathcal{S}^{r,\alpha}_{k,\lambda,\beta}(D)$ is endowed with the distance

$$d_{\mathcal{S}}(\Phi, \Psi) = \sup_{\xi \in \Sigma_k} d_{C^r}(\phi_\xi, \psi_\xi) + |C_\Phi - C_\Psi|, \tag{1.6}$$

where $\Phi = \tau \ltimes \phi_\xi$ and $\Psi = \tau \ltimes \psi_\xi$.

Finally, $\mathcal{K}(\overline{D})$ denotes the set whose elements are the non-empty compact subsets of $\overline{D}$ endowed with the Hausdorff metric and

$$W^u\big((\xi, x); \Phi\big) \stackrel{\text{def}}{=} \big\{(\zeta, y) \in \Sigma_k \times G : \lim_{n\to\infty} d(\Phi^{-n}(\zeta, y), \Phi^{-n}(\xi, x)) = 0\big\}$$

is the unstable set of (ξ, x) for Φ.

Theorem B. *Given $\Phi \in \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$, $\beta < 1$ and $\alpha > 0$, the following holds:*

[4] λ is called *contraction bound* in [19].

i) $\Gamma_\Phi \overset{\text{def}}{=} \bigcap_{n\in\mathbb{Z}} \Phi^n\big(\Sigma_k \times \overline{D}\big) = \bigcap_{n\in\mathbb{N}} \Phi^n\big(\Sigma_k \times \overline{D}\big)$,

ii) the restriction of Φ *to* Γ_Φ *is conjugate to the full shift* τ *of* k *symbols,*

iii) $W^u((\xi,x);\Phi) \subset \Gamma_\Phi$ *for all* $(\xi,x) \in \Gamma_\Phi$,

iv) there exists a unique continuous function $g_\Phi : \Sigma_k \to \overline{D}$ *such that for every periodic point* (ϑ,p) *of* Φ *one has that,*

- $\Gamma_\Phi = \overline{W^u((\vartheta,p);\Phi))} = \{(\xi, g_\Phi(\xi)) : \xi \in \Sigma_k\}$ *and*
- $\mathscr{P}(\Gamma_\Phi) = K_\Phi \in \mathcal{K}(D)$,

v) the map $\mathscr{L} : \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D) \to \mathcal{K}(\overline{D})$ *given by* $\mathscr{L}(\Phi) = K_\Phi$ *is continuous.*

An inmediate consequence of this theorem is the following:

$$W^u(\Gamma_\Phi;\Phi) \overset{\text{def}}{=} \big\{(\xi,x) \in \Sigma_k \times G : \lim_{n\to\infty} d(\Phi^{-n}(\xi,x),\Gamma_\Phi) = 0\big\} = \Gamma_\Phi. \tag{1.7}$$

1.3 Symbolic blender-horseshoes

Before introducing symbolic blender-horseshoes let us recall the definition of a blender (with one-dimensional center contracting direction):

Definition 1.8 (Blenders, [6])**.** *Let* f *be a* C^1*-diffeomorphism of a compact manifold* M *and* $\Gamma \subset M$ *a transitive hyperbolic set of* f *with a dominated splitting of the form* $E^{ss} \oplus E^{cs} \oplus E^u$*, such that the stable bundle of* Γ *is* $E^s = E^{ss} \oplus E^{cs}$ *and has dimension equal to* $k \geq 2$ *and* E^{cs} *is one-dimensional.*

The set Γ *is a* cs-blender *if it has a* C^1-robust superposition region $\mathcal{H}$*: There are a* C^1*-neighborhood* $\mathcal{V}$ *of* f *and a* C^1*-open set* $\mathcal{H}$ *of embeddings of* $(k-1)$*-dimensional disks* H *into* M *such that for every diffeomorphism* $g \in \mathcal{V}$*, every disk* $H \in \mathcal{H}$ *intersects the local unstable manifold* $W^u_{loc}(\Gamma_g)$ *of the continuation* Γ_g *of* Γ *for* g*.*

To adapt the definition of a blender to the symbolic context we first define a family of almost horizontal disks, which will provide the superposition region of the symbolic blender. We define the local stable set of $\zeta \in \Sigma_k$ for the shift map τ by

$$W^s_{loc}(\zeta;\tau) = \{\xi \in \Sigma_k : \xi_i = \zeta_i,\, i \geq 0\}.^5$$

Definition 1.9 (Almost horizontal disks)**.** *Consider* $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$*. Given* $\delta > 0$ *and an open subset* B *of* D*, we say that* $H^s \subset \Sigma_k \times G$ *is an* almost δ-horizontal

[5] Similarly we define the local unstable set of $\zeta \in \Sigma_k$ for τ by $W^u_{loc}(\zeta;\tau) = \{\xi \in \Sigma_k : \xi_i = \zeta_i,\, i \leq 0\}$.

disk *in* $\Sigma_k \times B$ *if there are* $(\zeta, z) \in \Sigma_k \times B$ *and a* (α, C)*-Hölder function* $h : W^s_{loc}(\zeta; \tau) \to B$ *such that* $C\nu^\alpha < \delta$ *and* $\|z - h(\xi)\| < \delta$ *for all* $\xi \in W^s_{loc}(\zeta; \tau)$, *and*

$$H^s = \{(\xi, h(\xi)) : \xi \in W^s_{loc}(\zeta; \tau)\}.$$

We say that H^s *is associated to* $W^s_{loc}(\zeta; \tau) \times \{z\}$ *and the graph map* h.

We are now ready to formulate the definition of a symbolic blender.

Definition 1.10 (Symbolic cs-blender-horseshoes)**.** *Consider a skew product map* $\Phi \in \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$, $\beta < 1$ *and* $\alpha > 0$. *The maximal invariant set* Γ_Φ *of* Φ *in* $\Sigma_k \times \overline{D}$ *is a* symbolic cs-blender-horseshoe *if there are* $\delta > 0$, *a non-empty open set* $B \subset D$, *and a neighborhood* $\mathcal{V}$ *of* Φ *in* $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ *such that for every* $\Psi \in \mathcal{V}$ *and every almost* δ*-horizontal disk* H^s *in* $\Sigma_k \times B$ *one has that*

$$W^u_{loc}(\Gamma_\Psi; \Psi) \cap H^s \neq \emptyset, \tag{1.8}$$

where Γ_Ψ *is the maximal invariant set of* Ψ *in* $\Sigma_k \times \overline{D}$.

The family of almost δ*-horizontal disks throughout* $\Sigma_k \times B$ *is the* superposition region *of the symbolic* cs*-blender-horseshoe. The open set* B *is the* superposition domain *of the blender.*

We say that the set (blender) Γ_Ψ is the continuation of Γ_Φ for Ψ.

Since by (1.7) $W^u(\Gamma_\Phi; \Phi) = \Gamma_\Phi$ holds, and then condition (1.8) can be written as follows:

$$\Gamma_\Psi \cap H^s \neq \emptyset, \quad \text{where } \Gamma_\Psi \text{ is the continuation of } \Gamma_\Phi \text{ for } \Psi.$$

1.4 One-step setting, iterated function systems, and symbolic blenders

A special case of skew product maps are the *one-step* ones where the fiber maps ϕ_ξ only depend on the coordinate ξ_0 of $\xi = (\xi_i)_{i \in \mathbb{Z}} \in \Sigma_k$. These maps are called *locally constant* in [19] and *step skew product* in [13, 17, 14]. In this case, we have $\phi_\xi = \phi_i$ if $\xi_0 = i$, and write $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k)$.

Let us recall the definition in [19] for one-step symbolic blenders. Consider the subset $\mathcal{Q}^0_{k,\lambda,\beta}(D)$ of $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ consisting of one-step skew product maps.

Definition 1.11 (One-step symbolic blender)**.** *Let* $\Phi \in \mathcal{Q}^0_{k,\lambda,\beta}(D)$, $\beta < 1$, *be a one-step skew product map. The maximal invariant set* Γ_Φ *of* Φ *in* $\Sigma_k \times \overline{D}$ *is an* one-step symbolic cs-blender-horseshoe *if there are a non-empty open set* $B \subset D$, *a fixed point* $(\vartheta, p) \in \Sigma_k \times D$ *of* Φ, *and a neighborhood* $\mathcal{V}$ *of* Φ *in* $\mathcal{Q}^0_{k,\lambda,\beta}(D)$ *such that for every* $\Psi \in \mathcal{V}$, *one has that*

$$W^u((\vartheta, p_\Psi); \Psi) \cap \left(W^s_{loc}(\xi;\tau) \times U\right) \neq \emptyset, \tag{1.9}$$

for every $\xi \in \Sigma_k$ and every non-empty open subset U in B[6]*. Here (ϑ, p_Ψ) is the continuation of the fixed point (ϑ, p) for Ψ.*

Observe that Definition 1.10 "implies" Definition 1.11. Let (ϑ, p) be a fixed point of $\Phi \in \mathcal{Q}^0_{k,\lambda,\beta}(D)$. Note that for $(\xi, x) \in \Sigma_k \times B$ and for any $\delta > 0$, the set $W^s_{loc}(\xi;\tau) \times \{x\}$ is a δ-almost horizontal disk. Then by Definition 1.10 the closure of $W^u((\vartheta, p_\Psi); \Psi)$ meets $W^s_{loc}(\xi;\tau) \times \{x\}$ for every perturbation Ψ of Φ in $\mathcal{Q}^0_{k,\lambda,\beta}(D)$. Hence the set $W^u((\vartheta, p_\Psi); \Psi)$ meets $W^s_{loc}(\xi;\tau) \times U$, for every open set $x \in U$.

In this work, the *cs*-blender-horseshoes are obtained by small perturbations of one-step skew products. Note that the set $W^s_{loc}(\theta;\tau) \times \{x\}$ is a δ-horizontal disk, for any δ and in the one-step case it is a local strong stable set. For Hölder perturbations of one-step maps the local strong stable sets will be almost horizontal disks.

In what follows we consider a one-step map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$. In the construction of blenders in Proposition 3.6 in [19] the maps $\phi_1, \dots, \phi_k$ satisfy the *covering* and *well-distribution of periodic points*[7] properties. Our construction of one-step symbolic blender-horseshoes does not use the well-distribution property. Our approach involves the so-called Hutchinson operator of a contracting iterated function system (IFS), see [16]. Given a family of maps $\phi_1, \dots, \phi_k \colon \overline{D} \to D$ its Hutchinson operator $\mathcal{G}_{\phi_1,\dots,\phi_k}$ associates to each subset B of $\overline{D}$ the set

$$\mathcal{G}_{\phi_1,\dots,\phi_k}(B) \stackrel{\text{def}}{=} \bigcup_{i=1}^{k} \phi_i(B).$$

The map $\mathcal{G}_{\phi_1,\dots,\phi_k}$ has the *covering property* if there is an open set $B \subset D$ such that $\overline{B} \subset \mathcal{G}_{\phi_1,\dots,\phi_k}(B)$. In this case we say that B has the covering property.

Note that $\mathcal{G}_{\phi_1,\dots,\phi_k}$ acts continuously in $\mathcal{K}(\overline{D})$ and if the maps ϕ_i are contractions then $\mathcal{G}_{\phi_1,\dots,\phi_k}$ is also contracting.

For the next result recall Equation (1.5), Definition 1.7 and consider the set $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ with $\lambda < 1$ and such that β has no restriction.

Theorem C. *Consider a one-step map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ with $\nu^\alpha < \lambda < 1$, $\alpha > 0$ and let B be an open set in D. Then B has the covering property for $\mathcal{G}_{\phi_1,\dots,\phi_k}$ if, and only if, there are $\delta > 0$ and a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ such that for every $\Psi \in \mathcal{V}$ it holds*

[6] The set $W^s_{loc}(\xi;\tau) \times U$ is called *s-strip* in [19]

[7] The fixed points z_i's of ϕ_i's satisfy the well-distribution property if any open ball of diameter d and centered in B contains some z_i, where $d \geq \max\{r;\, \forall\, x \in B,\, \exists\, i, B_r(x) \subset \phi_i(B)\}$.

$$\Gamma^+_\Psi(\Sigma_k \times B) \cap H^s \neq \emptyset, \tag{1.10}$$

for every almost δ-horizontal disk H^s in $\Sigma_k \times B$, where

$$\Gamma^+_\Psi(\Sigma_k \times B) \stackrel{\text{def}}{=} \bigcap_{n\geq 0} \Psi^n(\Sigma_k \times B)$$

is the forward maximal invariant set of Ψ in $\Sigma_k \times B$.

Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$ be a one-step skew product map. If $\beta < 1$ then $\phi_i(\overline{D}) \subset D$ for all $i = 1, \dots, k$. If $\Psi = \tau \ltimes \psi_\xi$ is close enough to Φ then $\psi_\xi(\overline{D}) \subset D$ for all $\xi \in \Sigma_k$. Thus

$$\Gamma^+_\Psi(\Sigma_k \times B) \stackrel{\text{def}}{=} \bigcap_{n\geq 0} \Psi^n(\Sigma_k \times B) \subset \bigcap_{n\in\mathbb{Z}} \Psi^n(\Sigma_k \times \overline{D}) \stackrel{\text{def}}{=} \Gamma_\Psi. \tag{1.11}$$

Theorem B, Theorem C and Definition 1.10 imply the following result:

Theorem D. *Consider a one-step skew product map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}(D)$, with $\nu^\alpha < \lambda < \beta < 1$ and $\alpha > 0$. Assume that there exists an open set B in D satisfying the covering property for $\mathcal{G}_{\phi_1,\dots,\phi_k}$. Then the maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}$ is a symbolic cs-blender-horseshoe for Φ whose superposition domain contains B.*

1.5 Partial hyperbolicity: an application

As a consequence of the results about symbolic blenders (Theorem D) we get the following results about perturbations of partially hyperbolic maps of the form $F \times \mathrm{Id}$.

Theorem E. *Consider compact manifolds N and G, a diffeomorphism $F : N \to N$ with a Smale horseshoe Λ, and the identity map $\mathrm{Id} : G \to G$. Then there is $g \in \mathrm{Diff}^1(N \times G)$ arbitrarily C^1-close to $f = F \times \mathrm{Id}$ such that g has a C^1-robust heterodimensional cycle of co-index c, where $c \geq 1$ is the dimension of the manifold G.*

Let us remark that this result was proved in [5] for the case $c = 1$.

1.6 Organization of this thesis

This work is organized as follows. In Chapters 2, 3, 4 we give preliminary results and prove Theorem 1.3. More precisely, in Chapter 2 we introduce the so-called simple cycles (an affine model for heterodimensional cycles). These simple cycles have associated affine quotient dynamics which are studied in

Chapter 3 and 4. Using these families we obtain strong homoclinic intersections.

In Chapter 5 we study blenders with bidimensional central direction. In Chapter 6 we prove Theorem 1.4 and the stabilization of the cycles (which all central eigenvalues are non-real), ending the proof of Theorem A.

In Chapter 7 we start to study symbolic blender-horseshoes and prove Theorem B. The study of symbolic blender-horseshoes in the one-step setting is done Chapter 8. The proof of Theorem C is completed in Chapter 9.

Finally, in Chapter 10 we prove Theorem E.

1.7 Colaboration

The results related to symbolic blender-horseshoes (Theorems B, C, D and E) are joint work with Pablo G. Barrientos (Pontifícia Universidade Católica do Rio de Janeiro) and Artem Raibekas (Universidade Federal Fluminense).