10
Robust cycles from symbolic blender-horseshoes

In this chapter we obtain robust heterodimensional cycles of any co-index (Definition 1.1) using symbolic blender-horseshoes (Definition 1.10). Our goal is prove Theorem E.

10.1
Robust symbolic cycles

To define symbolic cu-blender-horseshoes, we introduce the inverse symbolic skew product maps. Given $\Phi = \tau \ltimes \phi_\xi \in \mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$, for a bounded open subset D of a compact manifold G, constants $1 < \lambda < \beta$, $\alpha > 0$, and $k > 1$ (recall Definition 1.7), the symbolic skew product map

$$\Phi^* = \tau \ltimes \phi^*_\xi \in \mathcal{S}^{\alpha}_{k,\beta^{-1},\lambda^{-1}}(D),$$

where $\phi^*_\xi : D \to D$ is given by $\phi^*_\xi(x) = \phi^{-1}_{\xi^*}(x)$, is called *associated inverse skew product map for* Φ. Here ξ and ξ^* are points of Σ_k of the form: $\xi = (\dots\xi_{-1};\xi_0,\xi_1,\dots)$ and $\xi^* = (\dots\xi_1;\xi_0,\xi_{-1},\dots)$ (the conjugate sequence of ξ). Since $\tau(\xi)^* = \tau^{-1}(\xi^*)$, iterates of Φ^* correspond to iterates of Φ^{-1}. A *symbolic cu-blender-horseshoes* for Φ in $\mathcal{S}^{\alpha}_{k,\lambda,\beta}(D)$ is a symbolic cs-blender-horseshoe for Φ^*.

There is a similar result for symbolic cu-blender-horseshoes of Corollary D.

Corollary 10.1. *Let $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{S}^{0,\alpha}_{k,\lambda_u,\beta_u}(D_u)$ with $1 < \lambda_u < \beta_u < \nu^{-\alpha}$. Assume that there exists an open set B_u in D_u satisfying the covering property for $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$. Then the maximal invariant set Γ_Φ of Φ in $\Sigma_k \times \overline{D}_u$ is a symbolic cu-blender-horseshoe for Φ whose superposition domain contains B_u.*

Remark 10.2 (Continuations of blenders)**.** *Let Γ^s_Φ be a symbolic cs-blender-horseshoe (resp. cu-blender-horseshoe) for a map $\Phi \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$ (resp. $\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u)$). We consider Γ^s_Ψ (resp. Γ^u_Ψ) the continuation of Γ^s_Φ (resp. Γ^u_Φ) for $\Psi \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$ (resp. $\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u)$).*

Let Γ_Φ be a symbolic blender-horseshoe of Φ. Recall that

$$W^s(\Gamma_\Phi;\Phi) = \{(\xi,x)\in\Sigma_k\times G : \lim_{n\to\infty} d(\Phi^n(\xi,x),\Gamma_\Phi)=0\} \quad \text{and}$$
$$W^u(\Gamma_\Phi;\Phi) = \{(\xi,x)\in\Sigma_k\times G : \lim_{n\to\infty} d(\Phi^{-n}(\xi,x),\Gamma_\Phi)=0\}.$$

Definition 10.3 (Cycles associated to symbolic blenders)**.** *Let Φ be a skew-product map defined on $\Sigma_k\times G$. Assume that there are open bounded subsets D_s and D_u of G such that*

- $\Phi_s = \Phi_{|D_s} \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$ *and* $\Phi_u = \Phi_{|D_u} \in \mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_s)$*, here* $\Phi_{|D_i}$ *denotes the restriction of* Φ *to* D_i*,* $i=s,u$.
- Φ_s *has a symbolic cs blender-horseshoe* Γ^s_Φ *contained in* $\Sigma_k\times D_s$ *and* Φ_u *has a symbolic cu blender-horseshoe* Γ^u_Φ *contained in* $\Sigma_k\times D_u$.

We say that Φ has a symbolic cycle *associated to Γ^s_Φ and Γ^u_Φ if their stable and unstable sets meet cyclically:*

$$W^s(\Gamma^s_\Phi;\Phi)\cap W^u(\Gamma^u_\Phi;\Phi)\neq\emptyset \quad \textit{and} \quad W^u(\Gamma^s_\Phi;\Phi)\cap W^s(\Gamma^u_\Phi;\Phi)\neq\emptyset.$$

We say that this symbolic cycle is $\mathcal{S}$-robust *if there is a neighborhood $\mathcal{V}$ of Φ in $\mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)\cap\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u)$ such that for every $\Psi\in\mathcal{V}$ it holds $\Psi_{|D_s}\in\mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)$, $\Psi_{|D_u}\in\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_s)$, and Ψ has a symbolic cycle associated to the cs blender Γ^s_Ψ and the cu blender Γ^u_Ψ, where Γ^s_Ψ and Γ^s_Ψ are the continuations of Γ^s_Φ and Γ^u_Φ, respectively.*

The main technical step of the proof of Theorem E is the following:

Proposition 10.4. *Let $\phi_1,\dots,\phi_k,\varphi$ be $(\gamma,\hat\gamma)$-Lipschitz C^1-diffeomorphisms on a compact Riemannian manifold G, recall Equation* (1.3)*. Consider a one-step skew product map $\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)$ defined on $\Sigma_k\times G$. Assume that there are disjoint open and bounded subsets D_s,D_u of G such that*

$$\Phi=\tau\ltimes(\phi_1,\dots,\phi_k)\in\mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s)\cap\mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u),$$

where $\alpha\in(0,1]$, $\nu\in(0,1)$, and

$$0<\nu^\alpha<\gamma\le\lambda_s<\beta_s<1<\lambda_u<\beta_u\le\hat\gamma<\nu^{-\alpha}.$$

Assume also that there are open subsets $B_s\subset D_s$ and $B_u\subset D_u$ such that the following two properties hold.

i) Covering properties: $\overline{B}_s\subset\bigcup_{i=1}^k\phi_i(B_s)$ *and* $\overline{B}_u\subset\bigcup_{i=1}^k(\phi_i)^{-1}(B_u)$.

ii) Cyclic intersections: *there are a subset $\tilde{B}_s$ of B_s, points $x \in \tilde{B}_s$ and $y \in B_u$ such that*

$$\varphi^n(x) \in B_u, \quad \phi_j^m(y) \in D_s, \quad \textit{and} \quad (\phi_\ell)^{-1}(y) \in D_u$$

for some $n, m > 0$ and $j, \ell \in \{1, \dots, k\}$.

Then the one-step map $\hat{\Phi} = \tau \ltimes (\phi_1, \dots, \phi_k, \varphi)$[1] *defined on $\Sigma_{k+1} \times G$ has a robust symbolic cycle associated to a cs-blender-horseshoe Γ^s_Φ and to a cu-blender-horseshoe Γ^u_Φ contained in $\Sigma_k \times D_s$ and $\Sigma_k \times D_u$, respectively.*

To prove this proposition we need the following lemma:

Lemma 10.5. *Let $\phi_1, \dots, \phi_k, \varphi$ be $(\gamma, \hat{\gamma})$-Lipschitz be C^1-diffeomorphisms on a compact Riemannian manifold G. Consider a one-step skew product $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ defined on $\Sigma_k \times G$ with a symbolic cs-blender-horseshoe Γ_Φ contained in $\Sigma_k \times D$, $D \subset G$. Assume that superposition domain of the blender contains an open subset B of D satisfying the covering property:*

$$\overline{B} \subset \phi_1(B) \cup \cdots \cup \phi_k(B).$$

Let $\hat{\Phi} = \tau \ltimes (\phi_1, \dots, \phi_k, \varphi)$ be a skew product map defined on $\Sigma_{k+1} \times G$ such that

$$\Phi = \hat{\Phi}|_{\Sigma_k \times G}.$$

There are a small $\delta > 0$ and a subset $\tilde{B}$ of B such that every small perturbation $\hat{\Psi}$ of $\hat{\Phi}$ satisfies

$$W^{uu}(\Gamma^s_\Psi; \hat{\Psi}) \cap \hat{H}^s \neq \emptyset,$$

for every almost δ-horizontal disk $\hat{H}^s$ in $\Sigma_{k+1} \times B$, recall Definition 1.9, where Γ^s_Ψ is the continuation of Γ^s_Φ for $\Psi = \hat{\Psi}|_{\Sigma_k \times G}$.

Proof. First observe that for each $(\xi, x) \in \Sigma_{k+1} \times G$, we have that

$$W^{uu}_{loc}((\xi, x); \hat{\Phi}) = W^u_{loc}(\xi; \tau) \times \{x\},$$

for the one-step map $\hat{\Phi}$. Then, by shrinking slightly the superposition domain B we can assume that there is an open set $\tilde{B} \subset B$ such that for every $\hat{\Psi}$ close enough to $\hat{\Phi}$, it holds

$$W^{uu}_{loc}((\xi, x); \hat{\Psi}) \subset \Sigma_{k+1} \times B \quad \text{for all } (\xi, x) \in \Sigma_{k+1} \times \tilde{B}. \tag{10.1}$$

We can also assume that $\tilde{B}$ satisfies the covering property for IFS(Φ).

[1]Note that the maps ϕ_i correspond to the symbol i, for $i \in \{1, \dots, k\}$, and the map φ correspond to the symbol $k + 1$.

A slight modification on the proof of Theorem C (using that $\nu^\alpha < \gamma$, instead of $\nu^\alpha < \lambda$) shows that there is a small $\delta > 0$ such that for every small perturbation $\hat{\Psi}$ of $\hat{\Phi}$ we have

$$\Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B}) \cap \hat{H}^s \neq \emptyset \tag{10.2}$$

for every δ-horizontal disk $\hat{H}^s$ in $\Sigma_{k+1} \times \tilde{B}$, where

$$\Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B}) \stackrel{\text{def}}{=} \bigcap_{n\geq 0} \hat{\Psi}^n(\Sigma^-_{k,k+1} \times \tilde{B})$$

and

$$\Sigma^-_{k,k+1} \stackrel{\text{def}}{=} \{\xi = (\xi_i)_{i\in\mathbb{Z}} \in \Sigma_{k+1} : \xi_i \in \{1, \dots, k\} \text{ for every } i < 0\},$$

this means that the symbol $k+1$ corresponding to iterations by φ is not involved.

Consider a skew-product map $\hat{\Psi}$ close to $\hat{\Phi}$ satisfying (10.1) and (10.2), and let $\Gamma^s_\Psi \subset \Sigma_k \times G$ be the continuation of Γ^s_Φ for $\Psi = \hat{\Psi}|_{\Sigma_k\times G}$.

Claim 10.6. $\Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B}) \subset W^{uu}_{loc}(\Gamma^s_\Psi; \hat{\Psi})$.

Proof. For a given $(\xi, x) \in \Gamma^+_{\hat{\Psi}}(\Sigma^-_{k,k+1} \times \tilde{B})$ we will show that

$$W^{uu}_{loc}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times B) \subset \Gamma^s_\Psi. \tag{10.3}$$

Note that this inclusion implies the claim: if $(\zeta, z) \in W^{uu}_{loc}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times B)$ then by equation above, $(\zeta, z) \in \Gamma^s_\Psi$. Thus

$$W^{uu}_{loc}((\zeta, z); \hat{\Psi}) \subset W^{uu}_{loc}(\Gamma^s_\Psi; \hat{\Psi}).$$

Observe that $(\xi, x) \in W^{uu}_{loc}((\zeta, z); \hat{\Psi})$, then inclusion above implies that $(\xi, x) \in W^{uu}_{loc}(\Gamma^s_\Psi; \hat{\Psi})$, ending the proof of the claim.

To prove inclusion (10.3), recall that Γ^s_Ψ is a symbolic blender-horseshoe and thus, by Equation (1.11), we have that

$$\Gamma^+_\Psi(\Sigma_k \times B) \stackrel{\text{def}}{=} \bigcap_{n\geq 0} \Psi^n(\Sigma_k \times B) \subset \Gamma^s_\Psi \stackrel{\text{def}}{=} \bigcap_{n\in\mathbb{Z}} \Psi^n(\Sigma_k \times \overline{D}).$$

Hence, given $(\zeta, z) \in W^{uu}_{loc}((\xi, x); \hat{\Psi}) \cap (\Sigma_k \times B)$ it is enough to show that

$$\hat{\Psi}^{-n}(\zeta, z) = \Psi^{-n}(\zeta, z) \in \Sigma_k \times B, \quad \text{for all } n \geq 0.$$

Observe that from the invariance of the local strong unstable set we get

$$\hat{\Psi}^{-n}(\zeta, z) \in \hat{\Psi}^{-n}\big(W^{uu}_{loc}((\xi, x); \hat{\Psi})\big) \subset W^{uu}_{loc}(\hat{\Psi}^{-n}(\xi, x); \hat{\Psi}) \tag{10.4}$$

for all $n \geq 0$. Since $(\xi, x) \in \Gamma^{+}_{\hat{\Psi}}(\Sigma^{-}_{k,k+1} \times \tilde{B})$ we have that $\hat{\Psi}^{-n}(\xi, x) \in \Sigma_{k+1} \times \tilde{B}$ for all $n \geq 0$, thus by (10.1) it follows that

$$W^{uu}_{loc}(\hat{\Psi}^{-n}(\xi, x); \hat{\Psi}) \subset \Sigma_{k+1} \times B, \quad \text{for all } n \geq 0. \tag{10.5}$$

Therefore $\zeta \in \Sigma_k$, (10.4) and (10.5) imply that $\hat{\Psi}^{-n}(\zeta, z) \in \Sigma_k \times B$ for all $n \geq 0$. Thus $(\zeta, z) \in \Gamma^{+}_{\Psi}(\Sigma_k \times B) \subset \Gamma^{s}_{\Psi}$ proving (10.3), and thus the claim. □

The lemma follows noting that Claim 10.6 and Equation (10.2) imply that $W^{uu}_{loc}(\Gamma^{s}_{\Psi}; \hat{\Psi}) \cap \hat{H}^{s} \neq \emptyset$ for every δ-horizontal disk $\hat{H}^{s}$ in $\Sigma_{k+1} \times \tilde{B}$. □

Proof of Proposition 10.4. By Corollaries D and 10.1, the maximal invariant sets Γ^{s}_{Φ} and Γ^{u}_{Φ} of Φ in $\Sigma_k \times \overline{D}_s$ and $\Sigma_k \times \overline{D}_u$, respectively, are a *cs* and a *cu*-blenders horseshoes whose superpositions domains contain B_s and B_u, respectively.

We split the proof of the proposition into several steps. First, we prove that $\hat{\Phi}$ has a symbolic cycle associated to Γ^{s}_{Φ} and Γ^{u}_{Φ} (see Claims 10.7, 10.8, and 10.9).

Claim 10.7. *Let* $\vartheta = (\vartheta_i)_{i \in \mathbb{Z}} \in \Sigma_{k+1}$ *with* $\vartheta_i = \ell \in \{1, \ldots, k\}$ *for all* $i \in \mathbb{Z}$. *Then* $W^{s}_{loc}(\vartheta; \tau) \times D_s \subset W^{s}(\Gamma^{s}_{\Phi}; \hat{\Phi})$ *and* $W^{u}_{loc}(\vartheta; \tau) \times D_u \subset W^{u}(\Gamma^{u}_{\Phi}; \hat{\Phi})$.

Proof. We see that $W^{s}_{loc}(\vartheta; \tau) \times D_s \subset W^{s}(\Gamma^{s}_{\Phi}; \hat{\Phi})$, the other inclusion is analogous. Let p_ℓ be the attracting fixed point of ϕ_ℓ in D_s. Note that $(\vartheta, p_\ell) \in \Gamma^{s}_{\Phi}$ and that for every $(\xi, x) \in W^{s}_{loc}(\vartheta; \tau) \times D_s$ one has that $d(\hat{\Phi}^{n}(\xi, x), (\vartheta, p_\ell)) \to 0$ as $n \to \infty$. Thus $(\xi, x) \in W^{s}(\Gamma^{s}_{\Phi}; \hat{\Phi})$, ending the proof of the claim. □

To get a symbolic cycle for $\hat{\Phi}$ associated to Γ^{s}_{Φ} and Γ^{u}_{Φ} we will see that the invariant sets meet cyclically.

Claim 10.8. $W^{s}(\Gamma^{s}_{\Phi}; \hat{\Phi}) \cap W^{u}(\Gamma^{u}_{\Phi}; \hat{\Phi}) \neq \emptyset$.

Proof. By hypothesis, there are $y \in D_u$ and $m > 0$ such that $\phi^{m}_{j}(y) \in D_s$ and $\phi^{-1}_{\ell}(y) \in D_u$ for some $j, \ell \in \{1, \ldots, k\}$. Let

$$\zeta = (\ldots, \ell, \ell; j, \overset{m}{\ldots}, j, \ell, \ell, \ldots).$$

Note that $\tau^{-1}(\zeta) \in W^{u}_{loc}(\vartheta; \tau)$, where ϑ is the constant sequence with $\vartheta_i = \ell \in \{1, \ldots, k\}$ for all i. By Claim 10.7,

$$\hat{\Phi}^{-1}(\zeta, y) = (\tau^{-1}(\zeta), \phi^{-1}_{\ell}(y)) \in W^{u}_{loc}(\vartheta; \tau) \times D_s \subset W^{u}(\Gamma^{u}_{\Phi}; \hat{\Phi}).$$

Thus, $(\zeta, y) \in W^u(\Gamma^u_\Phi; \hat{\Phi})$. Analogously we have,

$$\hat{\Phi}^m(\zeta, y) = \Phi^m(\zeta, y) = (\tau^m(\zeta), \phi^m_j(y)) \in W^s_{loc}(\vartheta; \tau) \times D_s \subset W^s(\Gamma^s_\Phi; \hat{\Phi}),$$

obtaining that $(\zeta, y) \in W^s(\Gamma^s_\Phi; \hat{\Phi})$. Thus $(\zeta, y) \in W^s(\Gamma^s_\Phi; \hat{\Phi}) \cap W^u(\Gamma^u_\Phi; \hat{\Phi})$. □

Claim 10.9. $W^u(\Gamma^s_\Phi; \hat{\Phi}) \cap W^s(\Gamma^u_\Phi; \hat{\Phi}) \neq \emptyset$.

Proof. By hypothesis, there are $x \in B_s$, $n > 0$ such that $\varphi^n(x) \in B_u$. Since Γ^u_Φ is a *cu*-blender-horseshoe we have that $B_u \subset \mathscr{P}(\Gamma^u_\Phi)$. Hence there is $\xi \in \Sigma_k$ such that $(\xi, \varphi^n(x)) \in \Gamma^u_\Phi$. Consider the sequence

$$\zeta = (\dots, 1, 1; k+1, \overset{n}{\dots}, k+1, \xi_0, \dots, \xi_n, \dots) \in \Sigma_{k+1}.$$

As the strong stable set of Γ^u_Φ is contained in its stable set we get

$$\begin{aligned}\hat{\Phi}^n\big(W^s_{loc}(\zeta; \tau) \times \{x\}\big) &\subset W^s_{loc}(\xi; \tau) \times \{\varphi^n(x)\} = W^{ss}\big((\xi, \varphi^n(x)); \hat{\Phi}\big) \\ &\subset W^s(\Gamma^u_\Phi; \hat{\Phi}).\end{aligned}$$

On the one hand, inclusion above implies that the horizontal disk $\hat{H}^s = W^s_{loc}(\zeta; \tau) \times \{x\}$ in $\Sigma_{k+1} \times B_s$ satisfies $\hat{H}^s \subset W^s(\Gamma^u_\Phi; \hat{\Phi})$. On the other hand, by Lemma 10.5 we have that $W^u(\Gamma^s_\Phi; \hat{\Phi}) \cap \hat{H}^s \neq \emptyset$. Therefore, $W^u(\Gamma^s_\Phi; \hat{\Phi}) \cap W^s(\Gamma^u_\Phi; \hat{\Phi}) \neq \emptyset$, proving the claim. □

Consider a neighborhood $\mathcal{V}$ of $\hat{\Phi}$ such that for every $\hat{\Psi} \in \mathcal{V}$ it holds Lemma 10.5, covering property and cyclic intersection, that is,

- there are neighborhoods $\mathcal{U}^s_i$ of ϕ_i and $\mathcal{U}^u_i$ of ϕ^{-1}_i such that for $i = 1, \dots, k$

  $$B_{s,i} = \operatorname{int}\big(\cap_{\psi \in \mathcal{U}^s_i} \psi(B_s)\big) \quad \text{and} \quad B_{u,i} = \operatorname{int}\big(\cap_{\psi \in \mathcal{U}^u_i} \psi(B_u)\big),$$

  are open covering of $\overline{B}_s$ and $\overline{B}_u$, respectively;
- fix $x \in \tilde{B}_s$ and $y \in B_u$, $n, m > 0$, and $j, \ell \in \{1, \dots, k\}$ in cyclic intersection condition. Then for every $\hat{\Psi} = \tau \ltimes \psi_\xi \in \mathcal{V}$ it holds

  $$\psi^n_\eta(x) \in B_u, \quad \psi^m_{\hat{\eta}}(y) \in D_s \quad \text{and} \quad \psi^{-1}_{\tilde{\eta}}(y) \in B_u$$

  for every $\eta, \hat{\eta}, \tilde{\eta} \in \Sigma_{k+1}$ such that $\eta_0 = \dots = \eta_{n-1} = k+1$ and $\eta_i \in \{1, \dots k\}$ for $i < 0$ and $i \geq n$, $\hat{\eta}_0 = \dots = \hat{\eta}_{m-1} = j$ and $\hat{\eta}_i \in \{1, \dots k\}$ for $i < 0$ and $i \geq m$, and $\tilde{\eta}_0 = \ell$.

We will see that for every map in the neighborhood $\mathcal{V}$ the cyclic conditions hold, obtaining a robust symbolic cycle (Definition 10.3).

Let $\hat{\Psi} \in \mathcal{V}$ and Γ^s_Ψ and Γ^u_Ψ be the continuation of Γ^s_Φ and Γ^u_Φ, respectively for $\Psi = \hat{\Psi}|_{\Sigma_k \times G}$. We have the assertions below which are analogous to Claims 10.7, 10.8 and 10.9.

Claim 10.10. *Let $\vartheta = (\vartheta_i)_{i\in\mathbb{Z}} \in \Sigma_{k+1}$ with $\vartheta_i = \ell \in \{1, \dots, k\}$ for all $i \in \mathbb{Z}$. Then $W^s_{loc}(\vartheta; \tau) \times D_s \subset W^s(\Gamma^s_\Psi; \hat{\Psi})$ and $W^u_{loc}(\vartheta; \tau) \times D_u \subset W^u(\Gamma^u_\Psi; \hat{\Psi})$.*

The proof is analogous to Claim 10.7 and thus it is omitted.

Claim 10.11. *$W^s(\Gamma^s_\Psi; \hat{\Psi}) \cap W^u(\Gamma^u_\Psi; \hat{\Psi}) \neq \emptyset$.*

Proof. Consider $\zeta_0 = \cdots = \zeta_{m-1} = j$ and the sequence

$$\zeta = (\dots, \ell, \ell; \zeta_0, \zeta_1, \dots, \zeta_{m-1}, \ell, \ell, \dots).$$

Note that $\tilde{\eta} \stackrel{\text{def}}{=} \tau^{-1}(\zeta) \in W^u_{loc}(\vartheta; \tau)$, where ϑ is the constant sequence with $\vartheta_i = \ell \in \{1, \dots, k\}$ for all i and, by hypothesis, $\psi^{-1}_{\tilde{\eta}}(y) \in B_u$. By Claim 10.10,

$$\hat{\Psi}^{-1}(\zeta, y) = \Psi^{-1}(\zeta, y) = (\tau^{-1}(\zeta), \psi^{-1}_{\tilde{\eta}}(y)) \in W^u(\Gamma^u_\Psi; \hat{\Psi}).$$

Thus, $(\zeta, y) \in W^u(\Gamma^u_\Psi; \hat{\Psi})$. Analogously we have $\tau^m(\zeta) \in W^s_{loc}(\vartheta; \tau)$ and $\psi^m_\zeta(y) \in D_s$. Then by Claim 10.10,

$$\hat{\Psi}^m(\zeta, y) = \Psi^m(\zeta, y) = (\tau^m(\zeta), \psi^m_\zeta(y)) \in W^s_{loc}(\vartheta; \tau) \times D_s \subset W^s(\Gamma^s_\Psi; \hat{\Psi}),$$

getting that $(\zeta, y) \in W^s(\Gamma^s_\Psi; \hat{\Psi})$. Thus $(\zeta, y) \in W^s(\Gamma^s_\Psi; \hat{\Psi}) \cap W^u(\Gamma^u_\Psi; \hat{\Psi})$. □

Lemma 10.12. *$W^u(\Gamma^s_\Psi; \hat{\Psi}) \cap W^s(\Gamma^u_\Psi; \hat{\Psi}) \neq \emptyset$.*

Proof. Let $\eta = (\eta_i)_i \in \Sigma_{k+1}$ such that $\eta_0 = \cdots = \eta_{n-1} = k+1$. By hypothesis $\psi^n_\eta(x) \in B_u$. Since Γ^u_Ψ is the continuation of the *cu*-blender-horseshoe Γ^u_Φ we have that $B_u \subset \mathscr{P}(\Gamma^u_\Psi)$. Hence there is $\xi \in \Sigma_k$ such that $(\xi, \psi^n_\eta(x)) \stackrel{\text{def}}{=} (\xi, w) \in \Gamma^u_\Psi$.

Claim 10.13. *Consider small $\delta > 0$ and the almost δ-horizontal disk $\hat{H}^s$ associated to $W^s_{loc}(\zeta; \tau) \times \{x\}$ (recall Definition 1.9) where*

$$\zeta \stackrel{\text{def}}{=} (\dots, \eta_{-1}; k+1, \overset{n}{\dots}, k+1, \xi_0, \dots, \xi_{n-1}, \xi_n \dots) \in \Sigma_{k+1}.$$

Then

$$\hat{H}^s \subset W^s((\xi, w); \hat{\Psi}) \subset W^s(\Gamma^u_\Psi; \hat{\Psi}).$$

Proof of the claim. Note that if $(\xi, w) \in \Gamma^u_\Psi$, then the second inclusion of the claim holds. Thus it is enough to see that for any given $(\mu, z) \in \hat{H}^s$ we have that $\hat{\Psi}^n(\mu, z) \in W^s((\xi, w); \hat{\Psi})$ (where n is as in hypothesis (ii)), that is,

$$d\big(\hat{\Psi}^m(\hat{\Psi}^n(\mu,z)),\hat{\Psi}^m(\xi,w)\big)\to 0 \quad \text{as } m\to\infty. \tag{10.6}$$

First note that by Definition 1.9, one has that $\mu\in W^s_{loc}(\zeta;\tau)$, thus $\tau^n(\mu)^+=\xi^+$. Then

$$\begin{aligned}\lim_{m\to\infty} d\big(\hat{\Psi}^m(\hat{\Psi}^n(\mu,z)),\hat{\Psi}^m(\xi,w)\big) &= \lim_{m\to\infty} d\big(\hat{\Psi}^m(\tau^n(\mu),\psi^n_\mu(z)),\hat{\Psi}^m(\xi,\psi^n_\eta(x))\big)\\ &= \lim_{m\to\infty} \|\psi^m_{\tau^n(\mu)}(\psi^n_\mu(z))-\psi^m_\xi(\psi^n_\eta(x))\|.\end{aligned}$$

Fact 10.14. $\|\psi^n_\mu(z)-\psi^n_\eta(x)\|$ *is bounded, say,* $\|\psi^n_\mu(z)-\psi^n_\eta(x)\|\le M$.

Let $A\stackrel{\text{def}}{=}\max\{\beta,\nu^\alpha\}$ and note that $A<1$. Let us prove that

$$\|\psi^m_{\tau^n(\mu)}(\psi^n_\mu(z))-\psi^m_\xi(\psi^n_\eta(x))\|\le\beta^m\,M+C_\Psi\,m\,A^m. \tag{10.7}$$

To prove inequality above, first recall that $\mu\in W^s_{loc}(\zeta;\tau)$, then by definition of ζ, we have that $\tau^n(\mu)^+=\xi^+$, and thus $d_{\Sigma_k}(\tau^n(\mu),\xi)\le\nu$.

Using the triangle inequality, β-contraction of ψ_μ, the Hölder property, Fact 10.14 and observation above, we have the following estimates for $m=1$:

$$\begin{aligned}\|\psi_{\tau^n(\mu)}\big(\psi^n_\mu(z)\big)-\psi_\xi\big(\psi^n_\eta(x)\big)\| &\le \|\psi_{\tau^n(\mu)}\big(\psi^n_\mu(z)\big)-\psi_{\tau^n(\mu)}\big(\psi^n_\eta(x)\big)\|+\\ &\quad+\|\psi_{\tau^n(\mu)}\big(\psi^n_\eta(x)\big)-\psi_\xi\big(\psi^n_\eta(x)\big)\|\\ &\le\beta\,\|\psi^n_\mu(z)-\psi^n_\eta(x)\|+C_\Psi\,d_{\Sigma_k}(\tau^n(\mu),\xi)^\alpha\\ &\le\beta\,M+C_\Psi\,\nu^\alpha.\end{aligned}$$

Using the same arguments it follows the estimate for $m=2$:

$$\begin{aligned}&\|\psi^2_{\tau^n(\mu)}\big(\psi^n_\mu(z)\big)-\psi^2_\xi\big(\psi^n_\eta(x)\big)\|=\\ &\quad=\|\psi_{\tau^{n+1}(\mu)}\circ\psi_{\tau^n(\mu)}\big(\psi^n_\mu(z)\big)-\psi_{\tau(\xi)}\circ\psi_\xi\big(\psi^n_\eta(x)\big)\|\\ &\quad\le\|\psi_{\tau^{n+1}(\mu)}\circ\psi_{\tau^n(\mu)}\big(\psi^n_\mu(z)\big)-\psi_{\tau^{n+1}(\mu)}\circ\psi_\xi\big(\psi^n_\eta(x)\big)\|+\\ &\qquad\qquad+\|\psi_{\tau^{n+1}(\mu)}\circ\psi_\xi\big(\psi^n_\eta(x)\big)-\psi_{\tau(\xi)}\circ\psi_\xi\big(\psi^n_\eta(x)\big)\|\\ &\quad\le\beta\,\|\psi_{\tau^n(\mu)}\big(\psi^n_\mu(z)\big)-\psi_\xi\big(\psi^n_\eta(x)\big)\|+C_\Psi\,d_{\Sigma_k}\big(\tau^{n+1}(\mu),\tau(\xi)\big)^\alpha\\ &\quad\le\beta\,(\beta\,M+C_\Psi\,\nu^\alpha)+C_\Psi\,\nu^{2\,\alpha}=\beta^2\,M+C_\Psi\,\beta\,\nu^\alpha+C_\Psi\,\nu^{2\,\alpha}.\end{aligned}$$

Using analogous estimates, we get that $\|\psi^m_{\tau^n(\mu)}(\psi^n_\mu(z))-\psi^m_\xi(\psi^n_\eta(x))\|$ is smaller than

$$\beta^m\,M+C_\Psi\,\beta^{m-1}\,\nu^\alpha+C_\Psi\,\beta^{m-2}\,\nu^{2\,\alpha}+\cdots+C_\Psi\,\beta\,\nu^{(m-1)\alpha}+C_\Psi\,\nu^{m\,\alpha}.$$

Since $A=\max\{\beta,\nu^\alpha\}<1$, we have that

$$\|\psi^m_{\tau^n(\mu)}(\psi^n_\mu(z))-\psi^m_\xi(\psi^n_\eta(x))\|\le\beta^m\,M+C_\Psi\,m\,A^m.$$

Then Equation (10.7) is proved. Taking $m \to \infty$ in the same equation, we get (10.6), and thus that the almost horizontal disk $\hat{H}^s \subset W^s(\Gamma^u_\Psi; \hat{\Psi})$, proving the claim. □

On the other hand, by Lemma 10.5, we have that $W^u(\Gamma^s_\Psi; \hat{\Psi}) \cap \hat{H}^s \neq \emptyset$. Therefore, $W^u(\Gamma^s_\Psi; \hat{\Psi}) \cap W^s(\Gamma^u_\Psi; \hat{\Psi}) \neq \emptyset$, proving the lemma. □

Claim 10.11 and Lemma 10.12 imply that the invariant sets Γ^s_Ψ and Γ^u_Ψ meet cyclically for $\hat{\Psi}$, proving the proposition. □

10.2 Heterodimensional cycles

In this section, we prove Theorem E. First, we need the following lemma which is a reformulation of a result in [19] using our terminology.

Lemma 10.15 (Proposition 2.3 in [19])**.** *Let D be a bounded subset of G, $\phi : \overline{D} \to D$ be a (λ, β)-Lipschitz map with $\nu^\alpha < \lambda < \beta < 1$ and p_ϕ its attracting fixed point. Then there are $k \in \mathbb{N}$, an open neighborhood B of p_ϕ, and translations (in local coordinates) $\phi_1 \stackrel{\text{def}}{=} \phi, \phi_2, \ldots, \phi_k$ of ϕ such that the maps $\phi_1, \ldots, \phi_k$ satisfy the covering property for the set B:*

$$\overline{B} \subset \phi_1(B) \cup \ldots \cup \phi_k(B).$$

We observe that the number k of translations of ϕ depends on the dimension of G and the constant λ.

From Lemma 10.15 and Corollary D we immediately obtain the following consequence.

Corollary 10.16. *Let $\phi_1, \phi_2, \ldots, \phi_k$ be as in Lemma 10.15. Then the skew-product map $\Phi = \tau \ltimes (\phi_1, \ldots, \phi_k) \in \mathcal{S}^\alpha_{k,\lambda,\beta}(D)$ has a symbolic cs-blender-horseshoe whose superposition domain contains B.*

Proof of Lemma 10.15. Consider the open ball $B_\varepsilon(p_\phi) \subset D$ of radius $\varepsilon > 0$ centered at p_ϕ. Note that there are $k > 0$ and points $d_1 = p_\phi, d_2, \ldots, d_k \in B_\varepsilon(p_\phi)$, such that

$$\overline{B}_\varepsilon(p_\phi) \subset B_{\frac{\lambda}{2}\varepsilon}(d_1) \cup B_{\frac{\lambda}{2}\varepsilon}(d_2) \cup \ldots \cup B_{\frac{\lambda}{2}\varepsilon}(d_k).$$

Consider (in local coordinates) translations ϕ_i of ϕ, such that $B_{\frac{\lambda}{2}\varepsilon}(d_i) \subset \phi_i\big(B_\varepsilon(p_\phi)\big)$, for all $i = 1, \ldots, k$. The choice of the points d_i and the inclusion above imply that

$$\overline{B}_\varepsilon(p_\phi) \subset \phi_1\big(B_\varepsilon(p_\phi)\big) \cup \ldots \cup \phi_k\big(B_\varepsilon(p_\phi)\big),$$

proving the lemma. □

Proof of Theorem E. By hypothesis, the map F has a horseshoe Λ in N. Suppose that the contraction rate of F is $0 < \nu < 1$. Fix positive constants $\gamma, \hat{\gamma}$ with $\nu < \gamma < 1 < \hat{\gamma} < \nu^{-1}$.

Consider $\phi\colon G \to G$ a small perturbation of the identity map such that ϕ has two hyperbolic fixed points p and q satisfying the properties:

i) p is a sink,

ii) q is a source, and

iii) $W^s(p,\phi) \pitchfork W^u(q,\phi) \neq \emptyset$.

There are disjoint neighborhoods D_s of p and D_u of q such that $\phi_{|D_s}$ and $\phi^{-1}_{|D_u}$ are (λ,β)-Lipschitz maps for some $\lambda < \beta < 1$, recall Definition 1.3, satisfying $\gamma < \lambda < \beta < 1 < \beta^{-1} < \lambda^{-1} < \hat{\gamma}$.

By Lemma 10.15 there are $k > 0$, open sets $B_s \subset D_s$ and $B_u \subset D_u$ containing p and q, respectively, and maps $\phi^s_1 = \phi, \dots, \phi^s_k$ and $\phi^u_1 = \phi, \dots, \phi^u_k$ such that

$$\overline{B}_s \subset \phi^s_1(B_s) \cup \ldots \cup \phi^s_k(B_s) \quad \text{and} \quad \overline{B}_u \subset (\phi^u_1)^{-1}(B_u) \cup \ldots \cup (\phi^u_k)^{-1}(B_u). \tag{10.8}$$

Consider the maps $\phi_i\colon G \to G$, $i = 1, \dots, k$, as follows:

$$\phi_i(x) = \begin{cases} \phi^s_i(x) & \text{if } x \in D_s \\ \phi^u_i(x) & \text{if } x \in D_u. \end{cases}$$

Note that these maps are $(\gamma,\hat{\gamma})$-Lipschitz C^1-diffeomorphisms. Then (10.8) is equivalent to

$$\overline{B}_s \subset \bigcup_{i=1}^{k} \phi_i(B_s) \quad \text{and} \quad \overline{B}_u \subset \bigcup_{i=1}^{k} \phi_i^{-1}(B_u).$$

By construction,

$$\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{S}^{1,\alpha}_{k,\lambda_s,\beta_s}(D_s) \cap \mathcal{S}^{1,\alpha}_{k,\lambda_u,\beta_u}(D_u),$$

where $\lambda_s = \lambda$, $\beta_s = \beta$, $\lambda_u = \beta^{-1}$ and $\beta_u = \lambda^{-1}$.

Let $\varphi\colon G \to G$ be a $(\gamma,\hat{\gamma})$-Lipschitz C^1-diffeomorphism such that $\varphi^n(x) \in B_u$ for some $x \in B_s$ and some $n \in \mathbb{N}$.

Claim 10.17. *The map $\hat{\Phi} = \tau \ltimes (\phi_1, \dots, \phi_k, \varphi)$ has a symbolic cycle associated to a cs-blender-horseshoe and cu-blender-horseshoe whose superposition domains contain B_s and B_u, respectively.*

Proof. By condition (iii) there is a point $y \in B_u \cap (W^s(p, \phi_1) \cap W^u(q, \phi_1))$ such that $\phi_1^m(y) \in B_s$, for some $m \in \mathbb{N}$. Since $y \in B_u \cap (W^s(p, \phi_1) \cap W^u(q, \phi_1))$ we have that $\phi_1^{-1}(y) \in D_u$. Then adding the choice of φ above, the hypothesis of Proposition 10.4 are satisfied. Ending the proof of the claim. $\square$

To continue the proof, we need the following Proposition[2]:

Proposition 10.18. *[15, 13] Let f be a skew product diffeomorphism given by*

$$f\colon N \times G \to N \times G, \quad f(z, x) = \big(F(z), \phi_z(x)\big),$$

where $F\colon N \to N$ is a C^1-diffeomorphism with a horseshoe $\Lambda \subset N$ of d "legs" and with contraction rate $\nu < 1$, and $\phi_z(\cdot)\colon G \to G$ is a C^1-diffeomorphism such that

$$\gamma < \|D\phi_z(x)\, v\| < \hat{\gamma}$$

for $v \in T_xG$ and $z \in \Lambda$, where $\nu < \gamma < \hat{\gamma} < \nu^{-1}$.

Then for every map g C^1-close to f has a locally maximal invariant set $\Delta \subset N \times G$ homoeomorphic to $\Lambda \times G$ such that $g|_\Delta$ is conjugated to a symbolic skew-product map $\Psi_g \in \mathcal{S}^{1,\alpha}_{d,\gamma,\hat{\gamma}}(G)$, for $\alpha > 0$, recall Definition 1.7.

Recall that F has a horseshoe Λ in N. There is $\ell \in \mathbb{N}$ such that $F^\ell|_\Lambda$ is conjugated to a full shift of d symbols, for some $d > k$. Consider "rectangles" $R_1, \ldots, R_d$ in N such that $\{R_1 \cap \Lambda, \ldots, R_d \cap \Lambda\}$ is a Markov partition for $F^\ell|_\Lambda$.

Let $f : N \times G \to N \times G$ be the C^1-diffeomorphism such that

$$f_{|_{(R_i \cap \Lambda)\times G}} = F^\ell \times \phi_i \quad \text{for } i = 1, \ldots, k$$
$$f_{|_{(R_i \cap \Lambda)\times G}} = F^\ell \times \varphi \quad \text{for } i = k+1, \ldots, d.$$

The map f restricted to $\Lambda \times G$ is conjugated to the symbolic one-step skew product map $\hat{\Phi} = \tau \ltimes (\phi_1, \ldots, \phi_k, \varphi, \ldots, \varphi) \in \mathcal{S}^{1,\alpha}_{d,\gamma,\hat{\gamma}}(G)$. To emphasize this conjugacy between f and $\hat{\Phi}$, we denote $f \stackrel{\text{def}}{=} f_{\hat{\Phi}}$.

Therefore, by Claim 10.17 and Proposition 10.18 we have that $f_{\hat{\Phi}}$ has a robust heterodimensional cycle associated with hyperbolic sets Γ^s_f and Γ^u_f, which came from the symbolic *cs*-blender-horseshoe and the *cu*-blender-horseshoe.

The proof of the theorem is now complete. $\square$

[2]This proposition is detailed in [1].