2
Simple cycles: the $(\mathbb{C},\mathbb{C})$ case

We consider a diffeomorphism $f\colon M \to M$ having heterodimensional cycle of co-index two associated with a pair of saddles P and Q of indices $s+2$ and s, respectively, that is central separated. Let $s+2+u = d = \dim(M)$, where $s, u \geq 1$. This means that if $\alpha_1, \dots, \alpha_d$ are the eigenvalues of $Df_P^{\pi(P)}$ ordered in increasing modulus then $|\alpha_s| < |\alpha_{s+1}|$. Similarly, if $\beta_1, \dots, \beta_d$ are the eigenvalues of $Df_Q^{\pi(Q)}$ ordered in increasing modulus then $|\beta_{s+2}| < |\beta_{s+3}|$.

There are four possibilities according to the central eigenvalues of the cycle: (A) all central eigenvalues of the cycle are non-real; (B) either the central eigenvalues associated with P are real and the central eigenvalues associated with Q are non-real or vice-versa; (C) central eigenvalues of the cycle are real and equal in modulus; and (D) all central eigenvalues of the cycle are real and different in modulus.

We say that a diffeomorphism f has a $(\mathbb{C},\mathbb{C})$-*cycle* if it has a heterodimensional cycle of co-index two associated with saddles P and Q which is central separated, such that the central eigenvalues of Q are equal in modulus and the central eigenvalues of P are also equal in modulus (cases (A) and (C)). Analogously we say that a diffeomorphism f has a $(\mathbb{R},\mathbb{C})$-*cycle* if it has a heterodimensional cycle of co-index two associated with saddles P and Q which is central separated, such that the central eigenvalues of Q are real and different in modulus and the central eigenvalues of P are non-real (case (B)). We will study $(\mathbb{C},\mathbb{C})$-cycles in this chapter and Chapter 3, and $(\mathbb{R},\mathbb{C})$-cycles in Chapter 4.

Following closely [5], we prove that arbitrarily C^1-close to these heterodimensional cycles there are new cycles (associated with the same saddles) such that the dynamics in a neighborhood of these cycles is "affine" and partially hyperbolic (with bidimensional central direction). This new cycle is called *simple,* see Definition 2.1. The key point is that the dynamics of simple cycles can be essentially reduced to the analysis of a bidimensional iterated function system, where the details will be given in the next chapter.

2.1 Partially hyperbolic dynamics

We start defining partial hyperbolicity. Given a diffeomorphism $f \in \text{Diff}^1(M)$ and an f-invariant set Λ, a Df-invariant splitting with two bundles $E \oplus F$ of TM over Λ is *dominated* if there are constants $m > 0$ and $k < 1$ such that

$$\| Df_x^m |_E\| \cdot \| Df_x^{-m} |_F\| < k, \quad \text{for every } x \in \Lambda,$$

where $\| \cdot \|$ is the metric of M.

An Df-invariant splitting with three bundles $E \oplus F \oplus G$ is dominated if the bundles $(E \oplus F) \oplus G$ and $E \oplus (F \oplus G)$ are both dominated.

Assume that f has a heterodimensional cycle of co-index two associated with the saddles P and Q of indices $s+2$ and s as above. We define E_P^{ss} and E_P^c as the $Df_P^{\pi(P)}$-invariant spaces corresponding to the eigenvalues $(\alpha_1, \dots, \alpha_s)$ and $(\alpha_{s+1}, \alpha_{s+2})$, respectively. Since $|\alpha_s| < |\alpha_{s+1}| \le |\alpha_{s+2}| < 1 < |\alpha_{s+3}|$ these spaces are well defined and contained in the stable bundle of P. For a point A in the orbit $\mathcal{O}_P$ of P we let E_A^{ss} and E_A^c the corresponding iterates of E_P^{ss} and E_P^c by Df. Note that the stable bundle of $A \in \mathcal{O}_P$ is $E_A^s = E_A^{ss} \oplus E_A^c$. We proceed similarly with the point Q considering the $Df_Q^{\pi(Q)}$-invariant subespaces E_Q^{uu} and E_Q^c of the unstable bundle E_Q^u corresponding to the eigenvalues $(\beta_{s+2+1}, \dots, \beta_d)$ and $(\beta_{s+1}, \beta_{s+2})$ of $Df_Q^{\pi(Q)}$. We also consider the Df-invariant extensions of these bundles to the orbit of Q. In this way we obtain a Df-invariant dominated splitting defined over the orbits of P and Q. For notational convenience we write $E_B^{ss} = E_B^s$ if $B \in \mathcal{O}_Q$ and $E_A^{uu} = E_A^u$ if $A \in \mathcal{O}_P$. Then the splitting

$$T_A M = E_A^{ss} \oplus E_A^c \oplus E_A^{uu}, \quad \text{if } A \in \mathcal{O}_P \cup \mathcal{O}_Q$$

is well defined and dominated. Since the directions E^{ss} and E^{uu} are uniformly hyperbolic (contracting and expanding, respectively), we say that this splitting is partially hyperbolic.

2.2 $(\mathbb{C},\mathbb{C})$-Simple cycles

Let us start with an informal discussion about simple cycles. We will perform a series of perturbations of the initial cycle to get a new diffeomorphism with a heterodimensional cycle associated with the same saddles and such that the dynamics in the cycle is “affine”.

Fix heteroclinic points $X \in W^s(\mathcal{O}_P) \cap W^u(\mathcal{O}_Q)$ and $Y \in W^u(\mathcal{O}_P) \cap W^s(\mathcal{O}_Q)$. After an arbitrarily small perturbation we can assume that X is a transverse intersection and Y is a quasi-transverse one. We also can assume that there are small neighbourhoods $\mathcal{U}_P$ and $\mathcal{U}_Q$ of the orbits of P and Q, respectively, where f is linear. After replacing X by some backward iterate and Y by some forward iterate, and after a new perturbation, we will see that there are small neighbourhoods $\mathcal{U}_X \subset \mathcal{U}_Q$ of X and $\mathcal{U}_Y \subset \mathcal{U}_P$ of Y and large natural numbers n and m such that $f^n(\mathcal{U}_X) \subset \mathcal{U}_P$, $f^m(\mathcal{U}_Y) \subset \mathcal{U}_Q$, and f^n and f^m are affine maps (in local coordinates).

We fix the "neighbourhood of the cycle"

$$\mathcal{V} = \mathcal{U}_P \cup \mathcal{U}_Q \cup \Big(\bigcup_{i=-n}^{n} f^i(\mathcal{U}_X)\Big) \cup \Big(\bigcup_{i=-m}^{m} f^i(\mathcal{U}_Y)\Big)$$

and study the dynamics of f in this neighborhood. Using that this dynamics is affine and partially hyperbolic (with a partially hyperbolic splitting of the form $E^{ss} \oplus E^c \oplus E^{uu}$ where E^c is bidimensional), considering the quotient by the strong stable E^{ss} and strong unstable E^{uu} directions we will reduce this analysis to the study of a bidimensional iterated function system. We now go to the details of these constructions.

Given a complex number $\tau = \delta\, e^{2\,\pi\, i\, \psi}$, we consider the matrix

$$C_\tau = \delta \begin{pmatrix} \cos 2\pi\psi & -\sin 2\pi\psi \\ \sin 2\pi\psi & \cos 2\pi\psi \end{pmatrix}, \quad \delta > 0,\ \psi \in [0,1).$$

We now define linear maps $C_\alpha, C_\beta \colon \mathbb{R}^2 \to \mathbb{R}^2$ whose eigenvalues are

$$(\alpha \overset{\text{def}}{=} \alpha_{s+1} = \rho\, e^{2\,\pi\, i\, \phi}, \alpha_{s+2}) \quad \text{and} \quad (\beta \overset{\text{def}}{=} \beta_{s+1} = \varrho\, e^{2\,\pi\, i\, \varphi}, \beta_{s+2}), \tag{2.1}$$

respectively, where $0 < \rho < 1 < \varrho$ and $\phi, \varphi \in [0,1)$.

We also define the linear reflection along the $\mathbb{X}$-axis by $E_{\mathbb{X}}$.

Definition 2.1 ($(\mathbb{C},\mathbb{C})$-Simple cycle)**.** *A diffeomorphism f has a* $(\mathbb{C},\mathbb{C})$-simple cycle *of co-index two associated with P and Q and this cycle is unfolded in a* simple way *by the family $(f_t)_{t\in[-\epsilon,\epsilon]^2}$, $f_0 = f$, if the following conditions hold:*

i) There are local charts $\mathcal{U}_P$ and $\mathcal{U}_Q$ around P and Q

$$\mathcal{U}_P, \mathcal{U}_Q \backsimeq [-1,1]^s \times [-1,1]^2 \times [-1,1]^u,$$

where $f_t^{\pi(P)} \overset{\text{def}}{=} \mathcal{A}_t = \mathcal{A}$ and $f_t^{\pi(Q)} \overset{\text{def}}{=} \mathcal{B}_t = \mathcal{B}$ are linear maps of the form

$$\mathcal{A}(x^s,x^c,x^u) = \big(A^s(x^s), C_\alpha(x^c), A^u(x^u)\big) \quad \textit{and}$$
$$\mathcal{B}(x^s,x^c,x^u) = \big(B^s(x^s), C_\beta(x^c), B^u(x^u)\big),$$

where $A^s, B^s\colon \mathbb{R}^s \to \mathbb{R}^s$ are contractions, corresponding to the contracting eigenvalues $(\alpha_1,\dots,\alpha_s)$ and $(\beta_1,\dots,\beta_s)$, and $A^u, B^u\colon \mathbb{R}^u \to \mathbb{R}^u$ are expansions, corresponding to the expanding eigenvalues $(\alpha_{s+3},\dots,\alpha_d)$ and $(\beta_{s+3},\dots,\beta_d)$.

ii) There is a partially hyperbolic splitting $E^{ss}\oplus E^c\oplus E^{uu}$, defined over the orbits of P and Q, such that in these local charts they are of the form

$$E^{ss}=\mathbb{R}^s\times\{0^2\}\times\{0^u\},\quad E^c=\{0^s\}\times\mathbb{R}^2\times\{0^u\},\quad E^{uu}=\{0^s\}\times\{0^2\}\times\mathbb{R}^u.$$

iii) There are a quasi-transverse[1] *heteroclinic point $Y_P\in W^u(\mathcal{O}_P)\cap W^s(\mathcal{O}_Q)$ in the neighborhood $\mathcal{U}_P$, a natural number $\ell>0$, and a neighborhood $\mathcal{U}_{Y_P}$ of Y_P in $\mathcal{U}_P$, such that, in these local coordinates:*

- $Y_P=(0^s,0^2,y^u_P)$, *where* $y^u_P\in[-1,1]^u$;
- $Y_Q=f^\ell_t(Y_P)\in\mathcal{U}_Q$ *and* $Y_Q=(y^s_Q,0^2,0^u)$, *where* $y^s_Q\in[-1,1]^s$;
- $f^\ell_t(\mathcal{U}_{Y_P})\subset\mathcal{U}_Q$ *and*

$$f^\ell_t\stackrel{\text{def}}{=}T_{PQ,t}:\mathcal{U}_{Y_P}\to f^\ell_t(\mathcal{U}_{Y_P})$$

is an affine map of the form

$$T_{PQ,t}(x^s,x^c,x^u)=\big(T^s_{PQ}(x^s)+y^s_Q,T^c_{PQ}(x^c)+t,T^u_{PQ}(x^u-y^u_P)\big),$$

where $T^s_{PQ}\colon\mathbb{R}^s\to\mathbb{R}^s$ is a linear contraction (independent of t), $T^u_{PQ}\colon\mathbb{R}^u\to\mathbb{R}^u$ is a linear expansion (which also does not depend on t) and $T^c_{PQ}\colon\mathbb{R}^2\to\mathbb{R}^2$ is either $\pm\mathrm{Id}$ or the reflection $E_{\mathbb{X}}$.

iv) There are a transverse heteroclinic point $X_Q\in W^u(\mathcal{O}_Q)\pitchfork W^s(\mathcal{O}_P)$ in the neighborhood $\mathcal{U}_Q$, a natural number $r>0$, and a neighborhood $\mathcal{U}_{X_Q}$ of X_Q in $\mathcal{U}_Q$ such that, in these local coordinates:

- $X_Q=(0^s,x^c_Q,0^u)$, *where* $x_Q\in\mathbb{R}^2$;
- $X_P=f^r_t(X_Q)\in\mathcal{U}_P$ *and* $X_P=(0^s,x^c_P,0^u)$, *where* $x_P\in\mathbb{R}^2$;
- $f^r_t(\mathcal{U}_{X_Q})\subset\mathcal{U}_P$ *and*

$$f^r_t\stackrel{\text{def}}{=}T_{QP,t}=T_{QP}:\mathcal{U}_{X_Q}\to f^r_t(\mathcal{U}_{X_Q})$$

is an affine map of the form

$$T_{QP}(x^s,x^c,x^u)=\big(T^s_{QP}(x^s),T^c_{QP}(x^c)-x^c_Q+x^c_P,T^u_{QP}(x^u)\big),$$

[1] $\dim\big(T_{Y_P}W^s(\mathcal{O}_Q)\big)+\dim\big(T_{Y_P}W^u(\mathcal{O}_P)\big)=d-2=\dim(M)-2.$

where $T^s_{QP}\colon \mathbb{R}^s \to \mathbb{R}^s$ is a linear contraction, $T^u_{QP}\colon \mathbb{R}^u \to \mathbb{R}^u$ is a linear expansion and $T^c_{QP}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is either $\pm\mathrm{Id}$ or the reflection $E_{\mathbb{X}}$. Note that here the maps f_t do not depend on t.

We say that $\mathcal{A}$ and $\mathcal{B}$ are the linear parts *of the cycle, that X_Q and Y_P are the* heteroclinic points, *and T_{QP} and $T_{PQ,t}$ are the* transitions *of the cycle.*

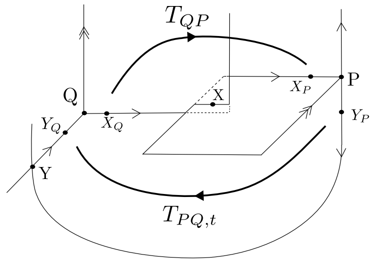

Figure 2.1: Transitions of the cycle

We have the next result about the approximation of cycles by simple ones:

Proposition 2.2. *Let f be a diffeomorphism with a heterodimensional cycle of co-index two associated with saddles P and Q which is central separated. Assume that the central eigenvalues satisfy*

$$|\alpha_{s+1}| = |\alpha_{s+2}| \quad \text{and} \quad |\beta_{s+1}| = |\beta_{s+2}|$$

Then any neighbourhood $\mathcal{U}$ of f contains diffeomorphisms having simple cycles associated with P and Q which are unfolded in a simple way.

Proof. We start with some preparations and fix some notation. For simplicity let us assume that Q and P are fixed points of f. By a small perturbation of f we can assume that there are small neighbourhoods of P and Q, say $\mathcal{U}_P$ and $\mathcal{U}_Q$, where f is linear.

Consider $W^{uu}(Q)$ the strong unstable manifold of Q (the unique f-invariant manifold tangent to E^{uu}_Q). Using local coordinates around Q define the following local manifolds of Q

$$\begin{aligned} W^s_{loc}(Q) &\overset{\text{def}}{=} \{(x^s, 0^c, 0^u)\} \subset W^s(Q) \cap \mathcal{U}_Q, \\ W^u_{loc}(Q) &\overset{\text{def}}{=} \{(0^s, x^c, x^u)\} \subset W^u(Q) \cap \mathcal{U}_Q, \\ W^{cu}_{loc}(Q) &\overset{\text{def}}{=} \{(0^s, x^c, 0^u)\} \subset W^u(Q) \cap \mathcal{U}_Q, \quad \text{and} \\ W^{uu}_{loc}(Q) &\overset{\text{def}}{=} \{(0^s, 0^2, x^u)\} \subset W^{uu}(Q) \cap \mathcal{U}_Q. \end{aligned}$$

Similarly, let $W^{ss}(P)$ be the strong stable manifold of P (the unique f-invariant manifold tangent to E^{ss}_P), using local coordinates we define the following local manifolds of P

$$\begin{aligned}
W^u_{loc}(P) &\stackrel{\text{def}}{=} \{(0^s,0^c,x^u)\} \subset W^u(P)\cap \mathcal{U}_P,\\
W^s_{loc}(P) &\stackrel{\text{def}}{=} \{(x^s,x^c,0^u)\} \subset W^s(P)\cap \mathcal{U}_P,\\
W^{cs}_{loc}(P) &\stackrel{\text{def}}{=} \{(0^s,x^c,0^u)\} \subset W^s(P)\cap \mathcal{U}_P, \quad \text{and}\\
W^{ss}_{loc}(P) &\stackrel{\text{def}}{=} \{(x^s,0^2,0^u)\} \subset W^{ss}(P)\cap \mathcal{U}_P.
\end{aligned}$$

We now choose heteroclinic points of the cycle. Take heteroclinic points $X \in W^u(Q)\cap W^s(P)$ and $Y \in W^s(Q)\cap W^u(P)$. After an arbitrarily small perturbation of f, we can assume that the first intersection is transverse and the second one quasi-transverse. Moreover, we can also suppose that $X \notin W^{uu}(Q)$ and $X \notin W^{ss}(P)$. Replacing X by some negative iterate we can assume that $X \in W^u_{loc}(Q)$. Write $X = (0^s, x^c, x^u)$ and $f^{-n}(X) = (0^s, x^c_n, x^u_n)$. Since $X \notin W^{uu}(Q)$ we have $x^c \neq 0^2$ and

$$\frac{||x^u_n||}{||x^c_n||} \leq \frac{|\beta_{s+3}|^{-n}}{|\beta_{s+1}|^{-n}} \cdot \frac{||x^u||}{||x^c||}.$$

As $|\beta_{s+3}| > |\beta_{s+1}|$ this implies that $f^{-n}(X)$ is much closer to $W^{cu}_{loc}(Q)$ than to $W^{uu}_{loc}(Q)$ for a sufficiently big n. Analogously, replacing X by some positive iterate we can assume that $X \in W^s_{loc}(P)$ and since $|\alpha_s| < |\alpha_{s+2}|$ we have that $f^m(X)$ is much closer to $W^{cs}_{loc}(P)$ than $W^{ss}_{loc}(P)$ for a sufficiently big m. Thus after arbitrarily small perturbations we can assume that there are backward iterate $\bar{X}_Q$ of X that is in $W^{cu}_{loc}(Q)$, and forward iterate $\bar{X}_P$ of X that is in $W^{cs}_{loc}(P)$. The points $\bar{X}_Q$ and $\bar{X}_P$ are depicted in Figure 2.2.

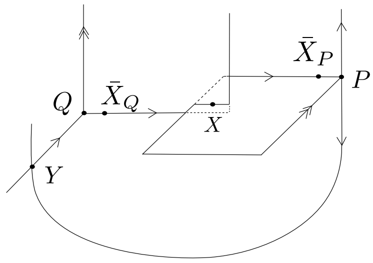

Figure 2.2: The heteroclinic points $\bar{X}_Q$ and $\bar{X}_P$

Now take a quasi-transverse heteroclinic point $Y \in W^s(Q)\cap W^u(P)$ and we fix iterates (backward) $\bar{Y}_P$ and (forward) $\bar{Y}_Q$ of it such that $\bar{Y}_P \in W^u_{loc}(P)$ and $\bar{Y}_Q \in W^s_{loc}(Q)$.

Claim 2.3. *After an arbitrarily small perturbation of f, we can assume that there are large $r_0, \ell_0 > 0$, negative iterates $\tilde{X}_Q$ of $\bar{X}_Q$ and $\tilde{Y}_P$ of $\bar{Y}_P$, and small neighborhoods $\mathcal{U}_{\tilde{X}_Q}$ of $\tilde{X}_Q$ and $\mathcal{U}_{\tilde{Y}_P}$ of $\tilde{Y}_P$ such that the restrictions of f^{r_0} to $\mathcal{U}_{\tilde{X}_Q}$ and of f^{ℓ_0} to $\mathcal{U}_{\tilde{Y}_P}$ are linear maps preserving the splitting $E^{ss}\oplus E^c \oplus E^{uu}$.*

Proof. In the neighborbood $\mathcal{U}_Q$ of Q there are f-invariant foliations $\mathcal{F}^u_Q$, $\mathcal{F}^{uu}_Q$, $\mathcal{F}^c_Q$, $\mathcal{F}^{ss}_Q$ and $\mathcal{F}^s_Q$ that are tangent to the bundles $E^{uu}\oplus E^c, E^{uu}, E^c, E^{ss}$ and $E^c\oplus E^{ss}$, respectively. Using the linearizing coordinates of f in $\mathcal{U}_Q\simeq[-1,1]^d$ we consider the following locally f-invariant foliations:

- $\mathcal{F}^u_Q$ the foliation by $(u+2)$-planes parallel to $\{0^s\}\times[-1,1]^2\times[-1,1]^u$,
- $\mathcal{F}^{uu}_Q$ the foliation by u-planes parallel to $\{0^s\}\times\{0^2\}\times[-1,1]^u$,
- $\mathcal{F}^c_Q$ the foliation by 2-planes parallel to $\{0^s\}\times[-1,1]^2\times\{0^u\}$,
- $\mathcal{F}^{ss}_Q$ the foliation by s-planes parallel to $[-1,1]^s\times\{0^2\}\times\{0^u\}$,
- $\mathcal{F}^s_Q$ the foliation by $(s+2)$-planes parallel to $[-1,1]^s\times[-1,1]^2\times\{0^u\}$.

Analogously, in the neighborbood $\mathcal{U}_P$ of P there are foliations $\mathcal{F}^u_P$, $\mathcal{F}^{uu}_P$, $\mathcal{F}^c_P$, $\mathcal{F}^{ss}_P$ and $\mathcal{F}^s_P$ that are tangent to the bundles $E^{uu}\oplus E^c, E^{uu}, E^c, E^{ss}$ and $E^c\oplus E^{ss}$, respectively. As these foliations have the same local expression, for simplicity, let us omit the subscript P and Q and consider the foliations $\mathcal{F}^u,\mathcal{F}^{uu},\mathcal{F}^c,\mathcal{F}^{ss}$ and $\mathcal{F}^s$ defined on $\mathcal{U}_Q\cup\mathcal{U}_P$ and denote by $\mathcal{F}^\sigma(X)$ the leaf of $\mathcal{F}^\sigma$ containing X, for $\sigma=u,uu,c,ss,s$.

By construction there is $r_1>0$ such that $f^{r_1}(\bar X_Q)=\bar X_P$. Let us consider images of these foliations by f^{r_1}. After an arbitrarily small perturbation of f we can assume that the following transversality conditions hold:

$$f^{r_1}\big(\mathcal{F}^u(\bar X_Q)\big)\pitchfork_{\bar X_P} E^{ss}.$$

Given a set A and a point $X\in A$ denote by $\mathcal{C}(A,X)$ the connected component of A containing X. By domination the images of the leaves of $\mathcal{F}^u$ are close to the leaves in $\mathcal{F}^u$ in $\mathcal{U}_P$. Replacing $\bar X_P$ by some forward iterate of it, say $f^{r_1+r_2}(\bar X_Q)=f^{r_2}(\bar X_P)$, we can assume that after an arbitrarily small perturbation we have

$$\mathcal{C}\big(f^{r_1+r_1}(\mathcal{F}^u(\bar X_Q))\cap\mathcal{U}_P, f^{r_2}(\bar X_P)\big)=\mathcal{F}^u(f^{r_2}(\bar X_P)),$$

then we have the invariance of the foliation $\mathcal{F}^u$. Consider now negative iterates of the foliations in $\mathcal{U}_P$ by $f^{r_1+r_2}$. Since the foliation $\mathcal{F}^u$ is $f^{r_1+r_2}$-invariant, we have the following transversality:

$$f^{-(r_1+r_2)}\big(\mathcal{F}^{ss}(f^{r_2}(\bar X_P))\big)\pitchfork_{\bar X_Q} E^u.$$

By domination the backward iterates of the leaves of $\mathcal{F}^{ss}$ are close to the leaves in $\mathcal{F}^{ss}$ in $\mathcal{U}_Q$. Then replacing $\bar X_Q$ by some backward iterate of it, say $f^{-r_3}(\bar X_Q)$, we can assume that after an arbitrarily small perturbation we have

$$\mathcal{C}\big(f^{-(r_1+r_2+r_3)}(\mathcal{F}^{ss}(f^{r_2}(\bar{X}_P)))\cap \mathcal{U}_Q, f^{-r_3}(\bar{X}_Q)\big) = \mathcal{F}^{ss}(f^{-r_3}(\bar{X}_Q)),$$

then we have the invariance of the foliations $\mathcal{F}^{ss}$ and $\mathcal{F}^{u}$. Similarly, now we consider the image of the foliations in $\mathcal{U}_P$ by $f^{r_1+r_2+r_3}$. After an arbitrarily small perturbation we can assume that:

$$f^{r_1+r_2+r_3}\big(\mathcal{F}^{uu}(f^{-r_3}(\bar{X}_Q))\big) \pitchfork_{\bar{X}_P} E^s.$$

By domination the images of the leaves of $\mathcal{F}^{uu}$ are close to the leaves in $\mathcal{F}^{uu}$ in $\mathcal{U}_P$. Replacing $f^{r_2}(\bar{X}_P)$ by some forward iterate of it, say $f^{r_2+r_4}(\bar{X}_P)$, we can assume that after an arbitrarily small perturbation we have

$$\mathcal{C}\big(f^{r_1+r_2+r_3+r_4}(\mathcal{F}^{uu}(f^{-r_3}(\bar{X}_Q)))\cap \mathcal{U}_P, f^{r_2+r_4}(\bar{X}_P)\big) = \mathcal{F}^{uu}(f^{r_2+r_4}(\bar{X}_P)),$$

then we have the invariance of the foliations $\mathcal{F}^{ss}, \mathcal{F}^{u}$ and $\mathcal{F}^{uu}$. Following analogously we have that there are $r_5, r_6 > 0$ such that for f^{r_0}, where $r_0 = r_1 + \cdots + r_6$, we get the invariance of all foliations.

Consider the $\tilde{X}_Q \stackrel{\text{def}}{=} f^{-(r_3+r_5)}(\bar{X}_Q)$ and $\tilde{X}_P \stackrel{\text{def}}{=} f^{r_1+r_2+r_4+r_6}(\bar{X}_P)$. This implies that (after a new arbitrarily small perturbation if necessary) there are small neighborhoods $\mathcal{U}_{X_Q}$ of $\tilde{X}_Q$ and $\mathcal{U}_{X_P}$ of $\tilde{X}_P$ such that f^{r_0} (or some positive iterate of it) preserves the foliations

$$f^{r_0}(\mathcal{F}^{\sigma}(Z)\cap \mathcal{U}_{X_Q}) = \mathcal{F}^{\sigma}(f^{r_0}(Z)) \subset \mathcal{U}_{X_P},$$

for $\sigma = u, uu, c, ss, s$, and the restriction of f^{r_0} to $\mathcal{U}_{X_Q}$ is linear.

Arguing analogously, we get ℓ_0, $\tilde{Y}_P$ and an small neighborhood of $\tilde{Y}_P$ such that $f^{\ell_0}(\tilde{Y}_P) = \tilde{Y}_Q$, the local foliations are f^{ℓ_0} invariant, and the restriction of f^{ℓ_0} to $\mathcal{U}_{Y_P}$ is linear. This completes the proof of the claim. □

In the local coordinates in $\mathcal{U}_Q$ and $\mathcal{U}_P$, write

$$\tilde{X}_Q = (0^s, \tilde{x}^c_Q, 0^u) \in \mathcal{U}_Q, \qquad \tilde{X}_P = f^{r_0}(\tilde{X}_Q) = (0^s, \tilde{x}^c_P, 0^u) \in \mathcal{U}_P,$$
$$\tilde{Y}_P = (0^s, 0^c, \tilde{y}^u_P) \in \mathcal{U}_P, \qquad \tilde{Y}_Q = f^{\ell_0}(\tilde{Y}_P) = (\tilde{y}^s_Q, 0^c, 0^u) \in \mathcal{U}_Q.$$

By the previous claim, in the local coordinates (around Q and P) the restriction of f^{r_0} to the neighborhood $\mathcal{U}_{\tilde{X}_Q}$ is of the form

$$f^{r_0}(x^s, x^c + \tilde{x}^c_Q, x^u) = \big(\tilde{T}^s_{QP}(x^s), \tilde{x}^c_P + \tilde{T}^c_{QP}(x^c), \tilde{T}^u_{QP}(x^u)\big),$$

where $\tilde{T}^s_{QP}$ is a linear contraction, $\tilde{T}^u_{QP}$ a linear expansion, and $\tilde{T}^c_{QP}$ linear.

Similarly, the restriction of f^{ℓ_0} to the neighborhood $\mathcal{U}_{\tilde{Y}_P}$ is of the form

$$f^{\ell_0}(x^s, x^c, x^u + \tilde{y}^u_P) = \big(\tilde{T}^s_{PQ}(x^s) + \tilde{y}^s_Q, \tilde{T}^c_{PQ}(x^c), \tilde{T}^u_{PQ}(x^u)\big),$$

where $\tilde{T}^s_{PQ}$ is a linear contraction, $\tilde{T}^u_{PQ}$ a linear expansion, and $\tilde{T}^c_{PQ}$ linear.

It remains to prove that (after a new perturbation and after replacing $\tilde{X}_Q$ and $\tilde{Y}_P$ by some backward iterates and $\tilde{X}_P$ and $\tilde{Y}_Q$ by some forward iterates) we have identities or reflections in the central coordinates.

We fix k_1 and $k_2 > 0$ (the choice of these numbers is explained below) and replace $\tilde{X}_Q$ and $\tilde{X}_P$, by $X_Q = f^{-k_1}(\tilde{X}_Q) = (0^s, x^c_Q, 0^u)$ and $X_P = f^{k_2}(\tilde{X}_P) = (0^s, x^c_P, 0^u)$. Let $r \stackrel{\text{def}}{=} k_1 + r_0 + k_2$, then the restriction of the map f^r to a small neighborhood of X_Q is of the form $f^r(x^s, x^c + x^c_Q, x^u) = (\bar{x}^s, \bar{x}^c, \bar{x}^u)$, where

$$\begin{aligned}
\bar{x}^s &= (A^s)^{k_2} \circ \tilde{T}^s_{QP} \circ (B^s)^{k_1}(x^s),\\
\bar{x}^c &= x^c_P + (C_\alpha)^{k_2} \circ \tilde{T}^c_{QP} \circ (C_\beta)^{k_1}(x^c),\\
\bar{x}^u &= (A^u)^{k_2} \circ \tilde{T}^u_{QP} \circ (B^u)^{k_1}(x^u).
\end{aligned} \tag{2.2}$$

Clearly, the action of this map in the s-coordinate is a linear contraction and its action in the u-coordinate is a linear expansion. Therefore we consider

$$T^s_{QP} = (A^s)^{k_2} \circ \tilde{T}^s_{QP} \circ (B^s)^{k_1} \quad \text{and} \quad T^u_{QP} = (A^u)^{k_2} \circ \tilde{T}^u_{QP} \circ (B^u)^{k_1}.$$

It remains to check that, for appropriate choices of large k_1 and k_2 and after a small perturbation, the central part $T^c_{QP} = (C_\alpha)^{k_2} \circ \tilde{T}^c_{QP} \circ (C_\beta)^{k_1}$ can be done as identity or reflection maps. Recall that $|\alpha_{s+1}| = |\alpha_{s+2}| < 1$ and $|\beta_{s+1}| = |\beta_{s+2}| > 1$ and also the notation

$$\alpha_{s+1} = \rho\, e^{2\pi\, i\, \phi},\ \phi \in [0,1),\ \rho < 1 \quad \text{and} \quad \beta_{s+1} = \varrho\, e^{2\pi\, i\, \varphi},\ \varphi \in [0,1),\ \varrho > 1.$$

We can assume, after a small perturbation, that $\rho^n\, \varrho^m = 1$ for some large n and m. In particular, $\rho^{n\,k}\, \varrho^{m\,k} = 1$ for all $k \geq 1$. We also can assume that $\phi, \varphi \in \mathbb{Q}$. In particular, $(C_\alpha)^{n\,j} = \rho^{n\,j}\, R^j_{n\,\phi}$, and $(C_\beta)^{m\,j} = \varrho^{m\,j}\, R^j_{m\,\varphi}$, where R_θ denotes the rotation of angle θ. As $R_{n\,\phi}$ and $R_{m\,\varphi}$ are rational rotation there is large k such that

$$R^k_{n\,\phi} = R^k_{m\,\varphi} = \text{Id}.$$

Fix $k_2 = n\,k$ and $k_1 = m\,k$, then $(C_\alpha)^{k_2} = \rho^{n\,k}\,\text{Id}$ and $(C_\beta)^{k_1} = \rho^{m\,k}\,\text{Id}$. Thus

$$(C_\alpha)^{k_2} \circ \tilde{T}^c_{QP} \circ (C_\beta)^{k_1} = \rho^{n\,k}\, \varrho^{m\,k}\, \tilde{T}^c_{QP} = \tilde{T}^c_{QP}.$$

As the segment of orbit going from X_Q to X_P can be chosen arbitrarily large (it is enough to take large k) we can modify the action of f in the central direction (without modifying the other directions) along the orbit $X_Q, f(X_Q), \dots, f^r(X_Q) = X_P$ to transform $\tilde{T}^c_{QP}$ in one of the maps Id, $-$Id, $E_{\mathbb{X}}$, depending on the eigenvalues of the transition $\tilde{T}_{QP}$. This concludes the construction of the transition map T_{QP} (this map does not depend on t). The construction of the transition T_{PQ} for the diffeomorphism f with a cycle is done arguing exactly as above.

Finally, we consider an unfolding $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ of $f = f_0$ as follows. Outside of a small neighborhood of $f^{-1}(Y_Q) = f^{\ell-1}(Y_P)$ we consider $f_t = f$ and we modify f in a neighborhood of $f^{-1}(Y_Q)$ in such a way the map f^ℓ_t is of the form

$$f^\ell_t(x^s, x^c, x^u) = \big(T^s_{PQ}(x^s), T^c_{PQ}(x^c) + t, T^u_{PQ}(x^u)\big).$$

This concludes the proof of the proposition. □