3
Strong homoclinic intersections: the $(\mathbb{C}, \mathbb{C})$ case

Our goal in this chapter is prove the following result:

Proposition 3.1. *Let f be a diffeomorphism having a co-index two $(\mathbb{C}, \mathbb{C})$-cycle associated with a pair of saddles P and Q. Then there is a diffeomorphism g arbitrarily C^1-close to f having a saddle A with strong homoclinic intersection with bidimensional central direction: $W^{ss}(A; g) \cap W^{uu}(A; g) \neq \emptyset$. Moreover,*

$$
\begin{aligned}
&W^{ss}(A; g) \cap W^{u}(Q_g; g) \neq \emptyset, \\
&W^{uu}(A; g) \cap W^{s}(Q_g; g) \neq \emptyset \quad \textit{and} \\
&W^{uu}(A; g) \cap W^{s}(P_g; g) \neq \emptyset
\end{aligned}
$$

where Q_g and P_g are the continuations of the saddles Q and P, respectively, for g.

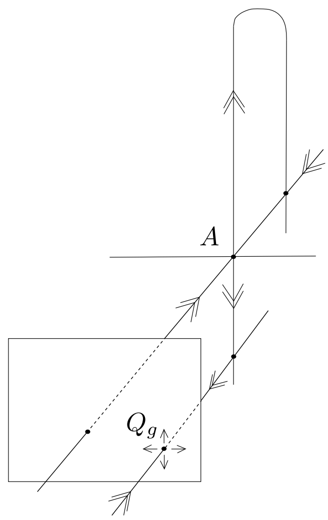

Figure 3.1: Strong homoclinic intersection A and the point Q_g

This result is obtained by using the model and quotient families (Definitions 3.2 and 3.4) in this chapter.

Note first that by Proposition 2.2 we can assume (and we do) that the cycle associated to f is a $(\mathbb{C}, \mathbb{C})$-simple cycle. We prove that there is a sequence of diffeomorphisms $f_k \to f$ such that, for every $k \geq 0$, each f_k has a periodic point A_k such that

- the orbit of A_k has a partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$, where E^{ss} is (uniformly) contracting, E^{uu} is (uniformly) expanding, and E^c is a bidimensional bundle;
- the modulus of the central eigenvalues of $Df_k^{\pi(A_k)}(A_k)$ (corresponding to the bundle E^c) is one;
- $W^{uu}(A_k; f_k) \cap W^{ss}(A_k; f_k)$ is non-trivial (that is, it contains points that do not belong to the orbit of A_k).

To prove the proposition we study the central dynamics associated to the unfolding family $(f_t)_t$ of the diffeomorphism $f_0 = f$ in a neighborhood of this simple cycle. This dynamics depends essentially on the central eigenvalues of the cycle. This family keeps invariant the foliation generated by the sum of the strong stable and strong unstable bundles of the dominated splitting. Considering the quotient of the family by this foliation, we will obtain a bidimensional family (Definition 3.4). Some properties of the global dynamics of this family (as the existence of a strong homoclinic intersection) can be written in terms of this bidimensional dynamics. In the next sections we explain that in detail.

3.1 Model families

We will study the family of bidimensional iterated function systems associated to the central dynamics of a simple cycle f. We see how properties of this system can be translated to properties of the family of diffeomorphisms.

We now define *model families*. Consider a diffeomorphism f with a simple cycle unfolded in a simple way by the family $(f_t)_t$, $f_0 = f$. In what follows we use the notation and terminology in Definition 2.1. Consider the heteroclinic points of the cycle $X_Q = (0^s, x_Q^c, 0^u)$ and $Y_P = (0^s, 0^c, y_P^u)$ and their neighborhoods $\mathcal{U}_{X_Q}$ and $\mathcal{U}_{Y_P}$ (in local coordinates)

$$\mathcal{U}_{X_Q} = [-1,1]^s \times B_\delta(x_Q^c) \times [-1,1]^u \quad \text{and} \quad \mathcal{U}_{Y_P} = [-1,1]^s \times B_\delta(0^2) \times B_\delta^u(y_P^u),$$

where $B_\delta(x_Q^c)$, $B_\delta(0^2)$, and $B_\delta^u(y_P^u)$ are δ-neighborhoods of x_Q^c, 0^2, and y_P^u.

Definition 3.2 (Model unfolding families)**.** *Consider a family of diffeomorphisms* $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ *unfolding a simple cycle in a simple way at* $t = (0,0)$.

Suppose that this cycle is associated with the saddles P *and* Q *with linear parts* $\mathcal{A}$ *and* $\mathcal{B}$*, heteroclinic points* X_Q *and* Y_P*, and transition maps*

T_{PQ} and T_{QP}. The model unfolding family $(F_t)_{t\in[-\epsilon,\epsilon]^2}$ *associated to the family $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ is defined as follows:*

$$F_t : \mathcal{U}_Q \cup \mathcal{U}_P \to M, \qquad F_t(x) = \begin{cases} T_{QP}(x) & \text{if } x \in \mathcal{U}_{X_Q}, \\ \mathcal{A}(x) & \text{if } x \in \mathcal{U}_P \backslash \mathcal{U}_{Y_P}, \\ T_{PQ,t}(x) & \text{if } x \in \mathcal{U}_{Y_P}, \\ \mathcal{B}(x) & \text{if } x \in \mathcal{U}_Q \backslash \mathcal{U}_{X_Q}, \end{cases}$$

where $\mathcal{U}_{X_Q} \subset \mathcal{U}_Q$ and $\mathcal{U}_{Y_P} \subset \mathcal{U}_P$ are neighborhoods of X_Q and Y_P, respectively.

Note that if f is a diffeomorphism with a co-index two simple cycle, then there is a model map F_0 such that f and F_0 have the same dynamics in the neighborhood of the cycle. More precisely, using local coordinates, consider $P = (0^s, 0^2, 0^u) \in \mathcal{U}_P$ and $Q = (0^s, 0^2, 0^u) \in \mathcal{U}_Q$. Note that these points are periodic points of the model map F_0 of s-indices (dimension of their stable bundles) $s+2$ and s, respectively. Moreover P and Q are associated with F_0 by a co-index two heterodimensional cycle. Then we have that the following result is a reformulation of Proposition 3.1 in terms of model unfolding families.

Proposition 3.3. *Consider a diffeomorphism f with a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with the saddles P and Q. Then there is a diffeomorphism $\tilde{f}$ arbitrarily C^1-close to f with a simple cycle associated with P and Q whose associated model map F has a periodic point A of arbitrarily large period $\pi = \pi(A)$ such that*

i) there is a constant $C > 0$ such that the central eigenvalues of DF^{π}_A satisfy $\lambda_{s+1}, \lambda_{s+2} \in [C^{-1}, C]$,

ii) the intersection $W^{uu}(A;F) \cap W^{ss}(A;F)$ is non-trivial,

iii) $W^{uu}(A;F) \cap W^{s}(Q;F) \neq \emptyset$ and $W^{ss}(A;F) \cap W^{u}(Q;F) \neq \emptyset$,

iv) $W^{uu}(A;F) \cap W^{s}(P;F) \neq \emptyset$.

Observing that the periodic point A is a periodic point of $\tilde{f}$ satisfying the same intersection properties we obtain Proposition 3.1.

The proof of Proposition 3.3 is reduced to the study of a bidimensional (central) family associated to the model family $(F_t)_t$. We give precise definitions in next section.

3.2 Bidimensional quotient families

By construction, the model family keeps invariant the foliation generated by the sum of the strong stable and strong unstable bundles of the dominated splitting. Considering the quotient of the model family by these foliations, we obtain a bidimensional family as follows.

Write $x^c = (x_1, x_2)$, $x^c_Q = (x_{q_1}, x_{q_2})$, $x^c_P = (x_{p_1}, x_{p_2})$, and $t = (t_1, t_2)$. Let θ_{QP} be the restriction of T^c_{QP} to the central direction. The possibilities for the map $\theta_{QP}\colon B_\delta(x^c_Q) \to B_\delta(x^c_P)$ are the following:

$$(x_1, x_2) + (x_{q_1}, x_{q_2}) \overset{\theta_{QP}}{\longmapsto} \begin{cases} (x_1, x_2) + (x_{p_1}, x_{p_2}) \\ (-x_1, -x_2) + (x_{p_1}, x_{p_2}) \\ (x_1, -x_2) + (x_{p_1}, x_{p_2}). \end{cases}$$

Similarly for the restriction of $T^c_{PQ,t}$ to the central direction we have the map $\theta_{PQ,t}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$(x_1, x_2) \overset{\theta_{PQ,t}}{\longmapsto} \begin{cases} (x_1, x_2) + (t_1, t_2) \\ (-x_1, -x_2) + (t_1, t_2) \\ (x_1, -x_2) + (t_1, t_2). \end{cases}$$

We use the notation θ^i_{QP} and $\theta^j_{PQ,t}$, where $i, j = +, -, \pm$, for the different cases above, respectively.

The relevant part of the quotient of the model family are the linear maps C_α and C_β and also the maps θ^i_{QP} and $\theta^j_{PQ,t}$, which give the definition of quotient family below:

Definition 3.4 (Quotient families)**.** *For each $m, \ell \in \mathbb{N}$ we consider the composition*

$$\mathcal{Q}^{\alpha,\beta,i,j}_{m,\ell,t} \overset{\text{def}}{=} C^\ell_\beta \circ \theta^j_{PQ,t} \circ C^m_\alpha \circ \theta^i_{QP} : \mathbf{B}^{\alpha,\beta,i,j}_{m,\ell,t}(x^c_Q) \to B_\delta(x^c_Q), \tag{3.1}$$

for $i, j = +, -, \pm$ and where $\mathbf{B}^{\alpha,\beta,i,j}_{m,\ell,t}(x^c_Q)$ is the maximal subset of $B_\delta(x^c_Q)$ where the map $\mathcal{Q}^{\alpha,\beta,i,j}_{m,\ell,t}$ is defined.

To each model unfolding family $(F_t)_{t\in[-\epsilon,\epsilon]^2}$ we associate its quotient family *of maps $(\mathcal{Q}^{\alpha,\beta,i,j}_{m,\ell,t})$, where $m, \ell \in \mathbb{N}$, $t \in [-\epsilon, \epsilon]^2$, and α, β, i, j are chosen according to the form of the central part of the model map F_t.*

When the central part is fixed (i.e., the values α, β and i, j are fixed) we omit them and simply write $\mathcal{Q}_{m,\ell,t}$ or $\mathcal{Q}^{\alpha,\beta}_{m,\ell,t}$, according to the case.

Next proposition claims that given an initial quotient family there are numbers $\hat\alpha, \hat\beta$ arbitrarily close to the initial central eigenvalues α, β of the cycle,

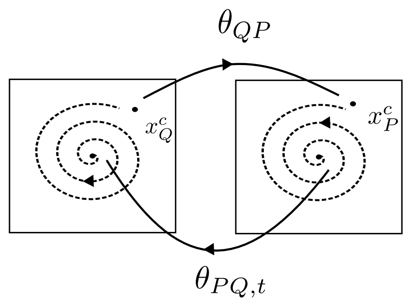

Figure 3.2: Quotient map

large natural numbers $m, \ell, \tilde{m}, \tilde{\ell}$, and k, and small $|t|$ such that the quotient maps $\mathcal{Q}^{\hat{\alpha},\hat{\beta}}_{m,\ell,t}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}}_{\tilde{m},\tilde{\ell},t}$ have the same fixed point x_Q^c and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}}_{k,0,t}(x_Q^c) = 0^2$. More precisely:

Proposition 3.5. *Given a quotient family* $(\mathcal{Q}^{\alpha,\beta,i,j}_{m,\ell,t})$ *and numbers* $\mu_0, \epsilon_0 > 0$, *and* $N_0 \in \mathbb{N}$, *there are natural numbers* $k, \ell, \tilde{\ell}, m, \tilde{m} > N_0$, *with* $(m, \ell) \neq (\tilde{m}, \tilde{\ell})$, *and numbers* $\hat{\alpha}, \hat{\beta}, t$ *with*

$$|\alpha - \hat{\alpha}| < \mu_0, \quad |\beta - \hat{\beta}| < \mu_0, \quad |t| < \epsilon_0,$$

such that

i) x_Q^c *is a common fixed point of* $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{m,\ell,t}$ *and* $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{\tilde{m},\tilde{\ell},t}$,

ii) let λ_1 *and* λ_2 *be the eigenvalues of the derivative of* $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{m,\ell,t}$ *at* x_Q^c. *Then* $\lambda_1, \lambda_2 \in (0, 2]$.

iii) $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{k,0,t}(x_Q^c) = 0^2$.

Proof. Without lost of generality, we can assume that in the local coordinates $x_P^c = (1, 0)$ and $x_Q^c = (1, 0)$. Recall that $\alpha = \rho\, e^{2\pi i \phi}$ and $\beta = \varrho\, e^{2\pi i \varphi}$, where $0 < \rho < 1 < \varrho$ and $\phi, \varphi \in [0, 1)$.

We need to consider the different cases according to the choices of $(i, j) \in \{+, -, \pm\}^2$. Consider first the case $(+, +)$ (the other cases follows similarly). For $t = (t_1, t_2)$ the bidimensional quotient map $\mathcal{Q}^{\alpha,\beta,+,+}_{m,\ell,t}(x, y)$ is of the form:

$$\varrho^\ell \begin{pmatrix} \cos \ell\, 2\pi\varphi & -\sin \ell\, 2\pi\varphi \\ \sin \ell\, 2\pi\varphi & \cos \ell\, 2\pi\varphi \end{pmatrix} \left[\rho^m \begin{pmatrix} \cos m\, 2\pi\phi & -\sin m\, 2\pi\phi \\ \sin m\, 2\pi\phi & \cos m\, 2\pi\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right].$$

We choose $k > 0$ (we will explain this choice later) and consider

$$\tilde{t} = (\tilde{t}_1, \tilde{t}_2) = (-\rho^k \cos k\, 2\pi\phi, -\rho^k \sin k\, 2\pi\phi).$$

We want to prove that

$$\mathcal{Q}^{\alpha,\beta,+,+}_{m,\ell,\tilde{t}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using trigonometric formulae and considering $x^c_Q = (1,0)$ and $\tilde{t}$ above this equality can be read as follows

$$\rho^m \varrho^\ell \begin{pmatrix} \cos(\ell\, 2\pi\varphi + m\, 2\pi\phi) \\ \sin(\ell\, 2\pi\varphi + m\, 2\pi\phi) \end{pmatrix} - \varrho^\ell \rho^k \begin{pmatrix} \cos(\ell\, 2\pi\varphi + m\, 2\pi\phi) \\ \sin(\ell\, 2\pi\varphi + m\, 2\pi\phi) \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Which is equivalent to the following system:

$$\begin{cases} \rho^m \varrho^\ell \cos(\ell\, 2\pi\varphi + m\, 2\pi\phi) - \rho^k \varrho^\ell \cos(\ell\, 2\pi\varphi + k\, 2\pi\phi) = 1 \\ \rho^m \varrho^\ell \sin(\ell\, 2\pi\varphi + m\, 2\pi\phi) - \rho^k \varrho^\ell \sin(\ell\, 2\pi\varphi + k\, 2\pi\phi) = 0. \end{cases} \tag{3.2}$$

Multiplying second equation by i and adding first equation we get

$$\begin{aligned} &\rho^m \varrho^\ell\, e^{i\,(\ell\, 2\pi\varphi + m\, 2\pi\phi)} - \rho^k \varrho^\ell\, e^{i\,(\ell\, 2\pi\varphi + k\, 2\pi\phi)} = 1 \\ &\varrho^\ell\, e^{i\,\ell\, 2\pi\varphi} \left(\rho^m\, e^{i\,m\,2\pi\phi} - \rho^k\, e^{i\,k\,2\pi\phi}\right) = 1 \\ &\big(\underbrace{\rho\, e^{i\,2\pi\phi}}_{\alpha}\big)^m - \big(\underbrace{\rho\, e^{i\,2\pi\phi}}_{\alpha}\big)^k = \big(\underbrace{\varrho^{-1}\, e^{-i\,2\pi\varphi}}_{\beta^{-1}}\big)^\ell. \end{aligned} \tag{3.3}$$

Therefore we need to find a pair of natural numbers m and ℓ satisfying the previous equality.

Fact 3.6. *There are* $r, s \in \mathbb{R}^+$ *such that* $\alpha^r = \beta^{-s}$.

Using that the rational numbers are dense in the real line, we get the following corollary:

Corollary 3.7. *There are rational numbers* $\frac{p_r}{q_r}$ *and* $\frac{p_s}{q_s}$ *close to* r *and* s*, respectively, and* $\breve{\alpha}$ *and* $\breve{\beta}$ *close to* α *and* β *with*

$$\breve{\alpha}^{\frac{p_r}{q_r}} = \breve{\beta}^{-\frac{p_s}{q_s}}.$$

Let $N \stackrel{\text{def}}{=} q_r \cdot q_s$, then by the corollary above we get

$$\begin{aligned} \left(\breve{\alpha}^{\frac{p_r}{q_r}}\right)^N &= \left(\breve{\beta}^{-\frac{p_s}{q_s}}\right)^N \\ \breve{\alpha}^{p_r \cdot q_s} &= \breve{\beta}^{-p_s \cdot q_r}. \end{aligned}$$

We now choose large $k > 0$ such that $(\breve{\varrho})^{p_s \cdot q_r} (\breve{\rho})^k < 1$, and $\tilde{\alpha}^k$ is close to zero for every $\tilde{\alpha}$ close to $\breve{\alpha}$. Take $\hat{\alpha}, \hat{\beta}$ close to $\breve{\alpha}, \breve{\beta}$ such that

$$(\hat{\alpha})^{p_r \cdot q_s} - (\hat{\alpha})^k = (\hat{\beta})^{-p_s \cdot q_r}.$$

Therefore Equation (3.3) holds for $m = p_r \cdot q_s$, $\ell = p_s \cdot q_r$, and

$$t = (-\hat{\rho}^k \cos k\, 2\pi\hat{\phi}, -\hat{\rho}^k \sin k\, 2\pi\hat{\phi}). \tag{3.4}$$

Indeed we need to obtain Equation (3.3) for two different pair of itineraries. This follows using an analogous argument. By Fact 3.6 there are four real positive numbers (actually infinitely many) $r, s, \hat{r}$ and $\hat{s}$ with $(r, s) \neq (\hat{r}, \hat{s})$ such that hold $\alpha^r = \beta^{-s}$ and $\alpha^{\hat{r}} = \beta^{-\hat{s}}$. Thus by Corollary 3.7 there are rational numbers $\frac{p_r}{q_r}$, $\frac{p_s}{q_s}$, $\frac{\hat{p}_r}{\hat{q}_r}$ and $\frac{\hat{p}_s}{\hat{q}_s}$ close to $r, s, \hat{r}$ and $\hat{s}$, respectively, and $\breve{\alpha}$ and $\breve{\beta}$ close to α and β, respectively, such that

$$(\breve{\alpha})^{\frac{p_r}{q_r}} = (\breve{\beta})^{-\frac{p_s}{q_s}} \qquad \text{and} \qquad (\breve{\alpha})^{\frac{\hat{p}_r}{\hat{q}_r}} = (\breve{\beta})^{-\frac{\hat{p}_s}{\hat{q}_s}}.$$

Now consider $M \stackrel{\text{def}}{=} q_r \cdot q_s \cdot \hat{q}_r \cdot \hat{q}_s$ and following analogously as above, we get that $x_Q^c = (1, 0)$ is a common fixed point of $\mathcal{Q}^{\hat{\alpha},\hat{\beta},+,+}_{m,\ell,t}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta},+,+}_{\tilde{m},\tilde{\ell},t}$, for natural numbers

$$m = p_r \cdot q_s \cdot \hat{q}_r \cdot \hat{q}_s, \qquad \ell = p_s \cdot q_r \cdot \hat{q}_r \cdot \hat{q}_s,$$
$$\tilde{m} = \hat{p}_r \cdot q_r \cdot q_s \cdot \hat{q}_s, \qquad \tilde{\ell} = \hat{p}_s \cdot q_r \cdot q_s \cdot \hat{q}_r,$$

$\hat{\alpha}$ and $\hat{\beta}$ close to $\breve{\alpha}$ and $\breve{\beta}$, respectively.

Note that by the choice of t in (3.4), we also have that $\mathcal{Q}^{\hat{\alpha},\hat{\beta},+,+}_{k,0,t}(x_Q^c) = 0^2$.

It remains to prove that the eigenvalues of $\mathcal{Q}^{\hat{\alpha},\hat{\beta},+,+}_{m,\ell,t}$ are uniformly bounded independently of $m, \ell, t, \hat{\alpha}$, and $\hat{\beta}$. First note that the derivative of $\mathcal{Q}^{\hat{\alpha},\hat{\beta},+,+}_{m,\ell,t}$ at $x_Q^c = (1, 0)$ is given by

$$\begin{pmatrix} \hat{\varrho}^\ell \hat{\rho}^m \cos(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi}) & -\hat{\varrho}^\ell \hat{\rho}^m \sin(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi}) \\ \hat{\varrho}^\ell \hat{\rho}^m \sin(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi}) & \hat{\varrho}^\ell \hat{\rho}^m \cos(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi}) \end{pmatrix}.$$

Therefore the eigenvalues are

$$\lambda_1 = \hat{\varrho}^\ell \hat{\rho}^m [\cos(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi}) + \sin(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi})] \quad \text{and}$$
$$\lambda_2 = \hat{\varrho}^\ell \hat{\rho}^m [\cos(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi}) - \sin(\ell\, 2\pi\hat{\varphi} + m\, 2\pi\hat{\phi})].$$

Using system (A), we have that these eigenvalues are

$$\lambda_1 = 1 - \hat{\varrho}^\ell \hat{\rho}^k [\cos(\ell\, 2\pi\hat{\varphi}) + \sin(\ell\, 2\pi\hat{\varphi})] \quad \text{and}$$
$$\lambda_2 = 1 - \hat{\varrho}^\ell \hat{\rho}^k [\cos(\ell\, 2\pi\hat{\varphi}) - \sin(\ell\, 2\pi\hat{\varphi})].$$

Note that $\cos(\ell\, 2\pi\hat{\varphi}) \pm \sin(\ell\, 2\pi\hat{\varphi}) \in [-2, 2]$. Since we chose k sufficiently large such that $\breve{\varrho}^\ell \breve{\rho}^k < 1$, by continuity, it holds that $\hat{\varrho}^\ell \hat{\rho}^k < 1$, and therefore we have that $\hat{\varrho}^\ell \hat{\rho}^k [\cos(\ell\, 2\pi\hat{\varphi}) \pm \sin(\ell\, 2\pi\hat{\varphi})] \in (-1, 1]$. Thus $\lambda_1, \lambda_2 \in (0, 2]$. The proof of this first case is now complete.

The other cases follow analogously and the proof will be omitted. $\square$

3.3 Proof of Proposition 3.3

The next result is a technical reformulation of Proposition 3.3. It allows us to translate dynamical properties of the quotient family to the model map. Indeed, this result is a version of [5, Proposition 3.8] for bidimensional central direction, by completeness we include the proof here.

Note that to each model map F_t of the diffeomorphism f there is associated a family of quotient dynamics $(\mathcal{Q}^{\alpha,\beta,i,j}_{m,\ell,t^2})_{m,\ell}$ for an appropriate choice of numbers α,β and $(i,j)\in\{+,-,\pm\}^2$. As the role of α,β,i,j is not important in this part, we omit these superscripts and just denote these maps by $\mathcal{Q}_{m,\ell}$.

Proposition 3.8. *Let* $F_t \stackrel{\text{def}}{=} F$ *be a model map associated to a family of diffeomorphisms unfolding a co-index two* $(\mathbb{C},\mathbb{C})$*-cycle (associated to saddles* P *and* Q*). Then there is* $N>0$ *with the following property.*

Consider $m,\ell>N$ *and a fixed point* a^c *of* $\mathcal{Q}_{m,\ell}$*, with* $a^c\in\mathbf{B}^{\alpha,\beta,i,j}_{m,\ell,t}(x^c_Q)\subset B_\delta(x^c_Q)$ *as in Proposition 3.5. Then the following holds:*

i) There is a periodic point $A=(a^s,a^c,a^u)\in\mathcal{U}_{X_Q}$ *of the model map* F *of period* $m+2+\ell$ *such that the modulus of the central eigenvalues of* $DF^{m+2+\ell}_A$ *are* $|\alpha|^m\,|\beta|^\ell$*.*

ii) Suppose that there are $\tilde m,\tilde\ell>N$*, with* $(m,\ell)\neq(\tilde m,\tilde\ell)$*, such that* a^c *is also a fixed point of* $\mathcal{Q}_{\tilde m,\tilde\ell}$*. Then* $W^{uu}(A;F)\cap W^{ss}(A;F)\neq\emptyset$*.*

iii) Suppose that there is $k>0$ *such that* $\mathcal{Q}_{k,0,t}(a^c)=0^2$*. Then* $W^{uu}(A;F)\cap W^s(Q;F)\neq\emptyset$*.*

Observe that Proposition 3.5 and 3.8 imply Proposition 3.3. Indeed, let f be a diffeomorphism having a co-index two cycle satisfying conditions of Proposition 3.3. By Proposition 3.5, there are sequences $t_k\to 0^2$, $\alpha_k\to\alpha$, $\beta_k\to\beta$, $m_k,\ell_k,\tilde m_k$ and $\tilde\ell_k$, with $(m_k,\ell_k)\neq(\tilde m_k,\tilde\ell_k)$, such that $\mathcal{Q}^{\alpha_k,\beta_k,i,j}_{m_k,\ell_k,t_k}$ and $\mathcal{Q}^{\alpha_k,\beta_k,i,j}_{\tilde m_k,\tilde\ell_k,t_k}$ have a common fixed point. Then by Proposition 3.8, for every k, each model map F_k has a periodic point A_k satisfying conditions (i) and (ii) of proposition above. For each k, let f_k be a diffeomorphism with a simple cycle associated to the model map F_k, in this way we get the diffeomorphism arbitrarily close to f, proving Proposition 3.3.

Proof of Proposition 3.8. Consider the following norm of a linear map A

$$\| A \| = \sup_{v\neq 0}\frac{|A\,v|}{|v|},$$

and note that since $A^s, (A^u)^{-1}, B^s$ and $(B^u)^{-1}$ are linear contractions there is $N \in \mathbb{N}$ such that for all $m, \ell > N$ it holds

$$\| (A^s)^m \| < \delta, \quad \| (A^u)^{-m} \| < \delta, \quad \| (B^s)^\ell \| < \delta, \quad \text{and} \quad \| (B^u)^{-\ell} \| < \delta.$$

Fix some $m, \ell > N$ such that $\mathcal{Q}_{m,\ell}$ has a fixed point $a^c \in \mathbf{B}_{m,\ell} \subset B_\delta(x_Q)$.

We say that a subset V of $\mathcal{U}_Q \cup \mathcal{U}_P$ is a *vertical rectangle* at a point $X = (x^s, x^c, x^u) \in V$ if there is a closed ball $K^s \subset [-1,1]^s$ such that

$$V = K^s \times \{x_1, x_2\} \times [-1,1]^u.$$

Analogously, a set H is a *horizontal rectangle* at $X = (x^s, x^c, x^u) \in H$ if there is a closed ball $K^u \subset [-1,1]^u$ such that $H = [-1,1]^s \times \{x_1, x_2\} \times K^u$.

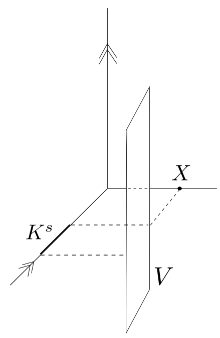

Figure 3.3: Vertical rectangle

We will prove that there are points $A = (a^s, a^c, a^u)$ and $\tilde{A} = (\tilde{a}^s, a^c, \tilde{a}^u)$, for some $a^s, \tilde{a}^s \in \mathbb{R}^s$ and $a^u, \tilde{a}^u \in \mathbb{R}^u$, a horizontal rectangle $\mathbf{H}$ at the point A, and a vertical rectangle $\mathbf{V}$ at the point $\tilde{A}$ such that $F^r(\mathbf{H}) \supset \mathbf{V}$ for $r = m+2+\ell$.

Let $D \overset{\text{def}}{=} [-1,1]^s \times \{a^c\} \times [-1,1]^u \subset \mathcal{U}_{X_Q}$. We will iterate D by F,

$$\begin{aligned} F(D) &= T_{QP}(D) \\ F^2(D) &= \mathcal{A} \circ T_{QP}(D) \\ &\vdots \\ F^{m+1}(D) &= \mathcal{A}^m \circ T_{QP}(D). \end{aligned}$$

Since in the s-coordinate the map F^{m+1} is a linear contraction and in the u-coordinate the map F^{m+1} is a linear expansion there is a horizontal rectangle $H_{m+1} \subset D$ that is mapped by F^{m+1} into a vertical rectangle V_{m+1}, $F^{m+1}(H_{m+1}) = V_{m+1}$.

Recall that $\mathcal{U}_{Y_P} = Y_P + ([-1,1]^2 \times [-\delta,\delta]^c \times [-\delta,\delta]^u)$ is a neighborhood of the heteroclinic point $Y_P = (0^s, 0^c, y_P^u)$. Since N is big and $m > N$, we have

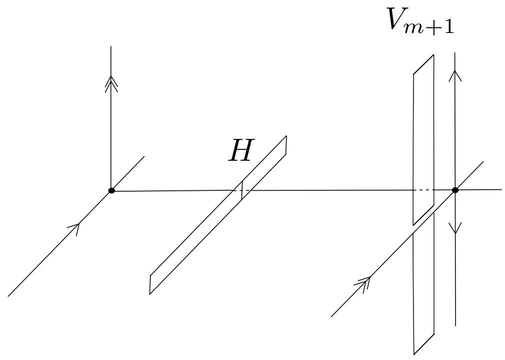

Figure 3.4: H and V_{m+1}

that

$$\Delta_{m+1} \stackrel{\text{def}}{=} V_{m+1} \cap \mathcal{U}_{Y_P} = Y_P + \big(\Delta^s_{m+1} \times \{a^c_{m+1}\} \times [-\delta, \delta]^u\big).$$

Note that there is a horizontal rectangle H'_{m+1} contained in H_{m+1} such that $F^{m+1}(H'_{m+1}) \subset \Delta_{m+1}$.

Consider now the set $\Delta_{m+2} \stackrel{\text{def}}{=} F(\Delta_{m+1}) = T_{PQ}(\Delta_{m+1})$. Note that the set Δ_{m+2} contains a set of the form $\Delta^s_{m+2} \times \{b^c\} \times [-\delta, \delta]^u$, where Δ^s_{m+2} is a ball and b^c is the central coordinate of Δ_{m+2}. Arguing as above, as ℓ is big, we have that $F^\ell(\Delta_{m+2}) = \mathcal{B}^\ell(\Delta_{m+2})$ contains a vertical rectangle $V_{m+2+\ell}$.

Putting together these relations we get a horizontal rectangle $H''_{m+1} \subset H'_{m+1} \subset H_{m+1}$ that is mapped by $F^{m+2+\ell}$ into a vertical rectangle $V_{m+\ell+2}$.

Note that the central coordinate of the rectangle $V_{m+2+\ell}$ is $\mathcal{Q}_{m,\ell}(a^c)$. By hypothesis, $\mathcal{Q}_{m,\ell}(a^c) = a^c$. Note that a^c is also the central coordinate of H''_{m+1}. Since the restriction of $F^{m+2+\ell}$ to H''_{m+1} is a linear contraction in the s-coordinate and a linear expansion in the u-coordinate we get a^s, a^u such that $A = (a^s, a^c, a^u) \in H''_{m+1} \cap V_{m+2+\ell}$ and $F^{m+2+\ell}(A) = A$.

To conclude the proof of the first item of the proposition just note that the central eigenvalues of $DF^{m+2+\ell}_A$ are the ones of map $\mathcal{Q}_{m,\ell}$, which have modulus $|\alpha|^m\,|\beta|^\ell$ (recall Equation (3.1)).

To prove item (ii), suppose that there are $\tilde{m}, \tilde{\ell} > N$ such that a^c is also a fixed point of $\mathcal{Q}_{\tilde{m},\tilde{\ell}}$. The previous construction provides a periodic point $\tilde{A} = (\tilde{a}^s, a^c, \tilde{a}^u)$ of F of period $\tilde{m} + 2 + \tilde{\ell}$. Note that since $(m, \ell) \neq (\tilde{m}, \tilde{\ell})$ we have that A and $\tilde{A}$ are different points.

By construction there is a point $B = (\tilde{a}^s, a^c, b^u)$ such that $F^{\tilde{m}+2+\tilde{\ell}}(B) = (\tilde{a}^s, a^c, a^u)$. Consider the point $B' = (a^s, a^c, b^u)$. By the expression of F this implies that $F^{-(\tilde{m}+2+\tilde{\ell})}(B') = (\bar{a}^s, a^c, a^u) = B''$. Noting that $B' \in W^{uu}(A; F)$ and $B'' \in W^{ss}(A; F)$ we have that $W^{uu}(A; F) \cap W^{uu}(A; F) \neq \emptyset$.

It remains to prove item (iii). Consider the following set in local charts

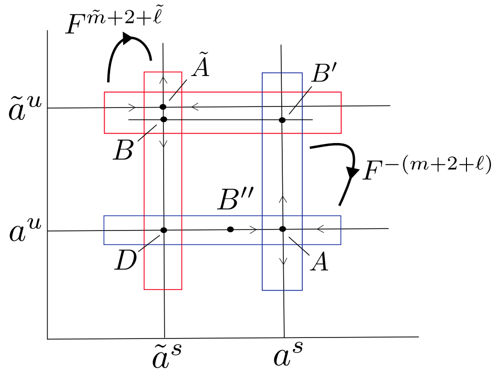

Figure 3.5: Intersections

around Q:

$$\triangle \stackrel{\text{def}}{=} \{(a^s, a^c)\} \times [-1,1]^u.$$

Note that $\triangle \subset W^{ss}(A; F(\triangle))$ and $F^{1+k+1}(\triangle) \supset \{(\tilde{a}^s, \mathcal{Q}_{k,0,t}(a^c))\} \times [-1,1]^u$, for some $\tilde{a}^s \in [-1,1]^s$. By hypothesis $\mathcal{Q}_{k,0,t}(a^c) = 0^2$, then

$$\{(\tilde{a}^s, 0^2)\} \times [-1,1]^u \subset F^{1+k+1}(\triangle).$$

Note also that $[-1,1]^s \times \{(\tilde{a}^s, 0^2)\} \subset W^s(Q; F)$ and

$$\{(\tilde{a}^s, 0^2)\} \times [-1,1]^u \cap [-1,1]^s \times \{(\tilde{a}^s, 0^2)\}.$$

Then we have $W^s(Q; F) \cap W^{ss}(A; F) \neq \emptyset$. This completes the proof of the proposition. $\square$

3.4
$(\mathbb{C},\mathbb{C})$-cycles with several quasi-transverse heteroclinic points

In this section we prove the following proposition that will be used for the stabilization of the cycles.

Let f be a diffeomorphism with a co-index two cycle associated with the saddles P and Q, where the s-index of Q is smaller than the one of P. In this case we write $Q <_s P$. We say that a heteroclinic point X of the cycle is of *type* $\vec{PQ}$ if $X \in W^u(P; f) \cap W^s(Q; f)$. Note that this intersection point cannot be a transverse one.

Recall that the *homoclinic class* of a saddle P, denoted by $H(P; f)$, is the closure of the transverse intersections of the invariant manifolds of P, or the closure of the set of points R that are *homoclinic related* to P, that is,

$$W^s(P; f) \pitchfork W^u(R; f) \neq \emptyset \quad \text{and} \quad W^u(P; f) \pitchfork W^s(R; f) \neq \emptyset. \tag{3.5}$$

Now we can state the main result of this section:

Proposition 3.9. *Let f be a diffeomorphism having a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with the saddles P and Q where $Q <_s P$. Then there is a diffeomorphism g arbitrarily C^1-close to f with a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with the saddles P and Q and having (at least) three different heteroclinic orbits of type $\vec{PQ}$.*

This proposition is a consequence of the following two lemmas:

Lemma 3.10. *Let f be a diffeomorphism with a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with the saddles P and Q where $Q <_s P$. Then there is a diffeomorphism g arbitrarily C^1-close to f with a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with P and Q such that the homoclinic class of P is non-trivial.*

Lemma 3.11. *Let f be a diffeomorphism with a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with the saddles P and Q where $Q <_s P$. Assume that the homoclinic class of P is non-trivial. Then there is a diffeomorphism g arbitrarily C^1-close to f having a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with P and Q such that*

i) the homoclinic class of P is non-trivial,

ii) the cycle has (at least) two different heteroclinic orbits of type $\vec{PQ}$.

These two lemmas immediately imply the proposition. To see why this is so, note that by Lemma 3.10 we can assume that, after an arbitrarily small perturbation, the saddle P in the cycle has a non-trivial homoclinic class. Applying now Lemma 3.11 we can assume, after a new arbitrarily small perturbation, that the saddle P has a non-trivial homoclinic class and the cycle has two heteroclinic points X_1 and X_2 with different orbits.

It is now enough to observe that we can select one of these two heteroclinic orbits, say X_1, and perform the perturbation of Lemma 3.11 preserving such a heteroclinic orbit. More precisely, we apply this lemma to the part of the cycle associated with the heteroclinic orbit X_2, obtaining a new cycle (associated to P and Q) with two different heteroclinic orbits Y_1 and Y_2. As the heteroclinic orbit X_1 was preserved this new cycle has at least three heteroclinic orbits of type $\vec{PQ}$. This implies the proposition.

We now give the proof of the two lemmas above.

Proof of Lemma 3.10. Let f be a diffeomorphism with a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with the saddles P and Q where $Q <_s P$. Considering the associated model unfolding family and the bidimentional quotient family, to prove the lemma it is enough to prove Proposition 3.5, replacing item (iii) $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{k,0,t}(x^c_Q) = 0^2$ by

$$\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{0,\tilde{k},\tilde{t}}(0^2) = x^c_Q. \tag{3.6}$$

Let us remark that to prove Equation (3.6) the procedure is analogous to the proof of Proposition 3.5, see Appendix A.

Using the same argument of item (iii) in Proposition 3.8, we have that equation above implies that there is a diffeomorphism h arbitrarily close to f having a strong homoclinic point A satisfying the intersection:

$$W^{ss}(A;h) \cap W^u(P;h) \neq \emptyset.$$

Thus we have proved the following result (which is an analogous to Proposition 3.1):

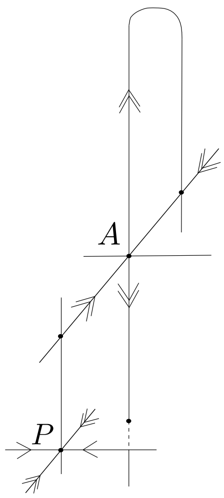

Figure 3.6: Claim 3.12

Claim 3.12. *There is a diffeomorphism h arbitrarily close to f having a strong homoclinic point A satisfying*

$$W^{ss}(A;h) \pitchfork W^u(Q;h) \neq \emptyset,$$
$$W^{ss}(A;h) \cap W^u(P;h) \neq \emptyset \quad \textit{and}$$
$$W^{uu}(A;h) \pitchfork W^s(P;h) \neq \emptyset$$

After a small perturbation[1] in the neighborhood of the point A we have that the intersections above are transverse. Moreover, these perturbations can be done preserving the intersections of the cycle, that is,

$$W^u(Q;h) \pitchfork W^s(P;h) \neq \emptyset \quad \text{and} \quad W^s(Q;h) \cap W^u(P;h) \neq \emptyset.$$

[1]Analogous to the map ϕ in the central coordinate in Section 6.1, that is, a small contraction in the central coordinate

Therefore we have that h has a heterodimensional cycle associated with Q and P and the homoclinic class of P is non-trivial (contains the point A that is homoclinically related to P). $\square$

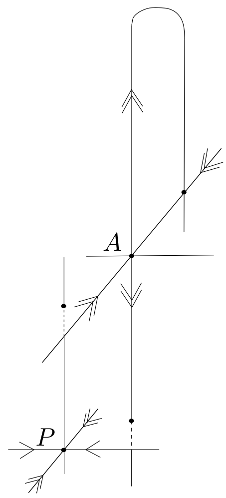

Figure 3.7: Perturbation

Remark 3.13. *Lemma 3.10 also holds for* $(\mathbb{C},\mathbb{R})$*-cycle. It is enough to have a result equivalent to Proposition 3.1 (just replacing item* (iii)*), and we have it in Proposition 4.11. See Appendix B.*

Proof of Lemma 3.11. Recall the terminology and the notation in the definition of simple cycles in Definition 2.1. After a small perturbation (Proposition 2.2), we can assume that f has a $(\mathbb{C},\mathbb{C})$-simple cycle associated with P and Q which are unfolded in a simple way by the family $(f_t)_t$ and such that the homoclinic class $H(P;f)$ is still non-trivial. In the coordinates around Q, assume that there is a homoclinic point A of P of the form $(a^s,1,0,0^u)$ such that the u-disk $\Delta^u=\{(a^s,1,0)\}\times[-1,1]^u$ is contained in $W^u(P;f_t)$ (this holds for every f_t, small $|t|$).

Let Y be the heteroclinic point of the cycle of type $\vec{PQ}$. Fix $\vec{t_0}\overset{\text{def}}{=}(t_0,0)$ for small $t_0>0$ and consider the diffeomorphism $f_{\vec{t_0}}$. In local coordinates (around Q) the unfolded heteroclinic point Y_1 of Y is of the form:

$$Y_1=(y_1^s,t_0,0,0^u)$$

and note that there is a u-disk $\tilde{\Delta}^u=\{(y_1^s,t_0,0)\}\times[-1,1]^u$ contained in $W^u(P;f_{\vec{t_0}})$.

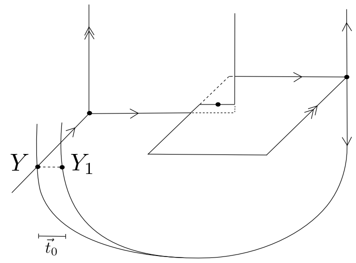

Figure 3.8: The points Y and Y_1

Note that the pre-image by $(T_{PQ,t_0})^{-1}\big(W^s_{loc}(Q; f_{\vec{t_0}})\big)$ contains a disk of the form (in the local coordinates around P)

$$\Delta^s = [-1,1]^s \times \{(\pm t_0, 0, y_1^u)\},$$

where the symbol $\pm$ is choosing according to the type of transition T_{PQ}. Let us assume that $\pm = +$ (the other cases are analogous).

Recall that ℓ is the number of iterations of f associated to T_{PQ}, see item (iii) in Definition 2.1. Write the map $f_{\vec{t_0}} = (f^s_{\vec{t_0}}, f^c_{\vec{t_0}}, f^u_{\vec{t_0}})$.

Claim 3.14. *Consider (in the local coordinates around Q) a u-disk Υ^u of the form*

$$\Upsilon^u = \{(v^s, a, b)\} \times [-1,1]^u.$$

Suppose that there is some $k \geq 0$ such that

$$(f^c_{\vec{t_0}})^k(a,b) = (t_0, 0).$$

Then $(f_{\vec{t_0}})^{k+\ell}(\Upsilon^u)$ intersects $W^s_{loc}(Q, f_{\vec{t_0}})$.

This claim immediately follows observing that $(f_{\vec{t_0}})^r(\Upsilon^u)$ intersects the disk $\Delta^s \subset f_{\vec{t_0}}\big(W^s_{loc}(Q; f_{\vec{t_0}})\big)$, where r is the number of iterations of $f_{\vec{t_0}}$ associated to T_{QP}, see item (iv) in Definition 2.1.

Note that (in the local coordinates around P)

$$\hat{\Delta}^u = \{(\hat{a}^s, \pm 1, 0)\} \times [-1,1]^u \subset T_{QP}(\Delta^u).$$

As above let us assume that $\pm = +$. Since t_0 is arbitrarily small we can modify $f_{\vec{t_0}}$ in the central coordinate in such a way that there is some (large) $k > 0$ with

$$(f^c_{\vec{t_0}})^k(1,0) = (t_0, 0). \tag{3.7}$$

By Claim 3.14 this implies that

$$(f_{\vec{t_0}})^{r+k+\ell}(\Delta^u) \cap W^s_{loc}(Q; f_{\vec{t_0}}) \neq \emptyset.$$

This provides the first heteroclinic point of type $\vec{PQ}$ for $f_{\vec{t_0}}$.

To get the second heteroclinic point we argue as above. As before, as t_0 is arbitrarily small we can modify $f_{\vec{t_0}}$ in the central coordinate (around Q) in such a way that there is some (large) $n > 0$ with

$$(f^c_{\vec{t_0}})^n(t_0, 0) = (1, 0).$$

This implies that the there is some y^s_3 such that

$$\{(y^s_3, 1, 0) \times [-1, 1]^u \subset (f_{\vec{t_0}})^n(\overline{\Delta}^u),$$

where $\overline{\Delta}^u$ is a u-disk of the form $\{(y^s_2, t_0, 0)\} \times [-1, 1]^u$. As above, there is some y^s_4 with

$$\{(y^s_4, 1, 0) \times [-1, 1]^u \subset (f_{\vec{t_0}})^{n+r}(\overline{\Delta}^u).$$

Finally, by the choice of k in Equation 3.7 this implies that there is y^s_5 with

$$\{(y^s_5, t_0, 0) \times [-1, 1]^u \subset (f_{\vec{t_0}})^{n+r+k}(\overline{\Delta}^u).$$

As above using Claim 3.14 we get a new heteroclinic intersection of type $\vec{PQ}$. Ending the proof the of the lemma. □