4
The $(\mathbb{R},\mathbb{C})$ case: simple cycles, model families, and strong homoclinic intersections

In this chapter we consider diffeomorphisms f having heterodimensional cycles of co-index two associated with saddles P and Q which are central separated and such that the central eigenvalues of Q are real and different in modulus and the central eigenvalues of P are non-real and conjugated. That is we consider $(\mathbb{R},\mathbb{C})$-cycles.

The strategy for $(\mathbb{R},\mathbb{C})$ cycles is similar to the one we followed for $(\mathbb{C},\mathbb{C})$ cycles. We first construct $(\mathbb{R},\mathbb{C})$-simple cycles (Proposition 4.2). Thereafter we define an associated model unfolding family (Definition 4.4) that has the same dynamics in the neighborhood of the cycle. We study a bidimensional family (Definition 4.5) that is the quotient of this model family by the sum of the strong stable and strong unstable bundles. Finally, we translate the properties of the bidimensional family to the initial diffeomorphism f obtaining strong homoclinic intersections (Theorem 1.3).

4.1
$(\mathbb{R},\mathbb{C})$-simple cycles

Let f be a diffeomorphism having a co-index two $(\mathbb{C},\mathbb{C})$-cycle associated with saddles P and Q which is central separated, that is, with central eigenvalues β_{s+1},β_{s+2} of Q real and different in modulus and central eigenvalues $\alpha_{s+1},\alpha_{s+2}$ of P non-real. For this sort of cycles we prove a result somewhat similar to Proposition 2.2: any neighborhood of f (as above) contains diffeomorphisms having cycles as the simple ones associated with P and Q which are unfolded in a simple way. The main difference here is that the transition maps T^c_{QP} and T^c_{PQ} are not isometries. In this case the restriction of these maps to restricted the central part are diagonal affine maps. We now give the precise definition and go to the details of this constructions.

Write $\alpha=\alpha_{s+1}=\rho\, e^{2\pi i\phi}$, $\phi\in[0,1)$, $\rho<1$ and note that $1<|\beta_{s+1}|<|\beta_{s+2}|$. Consider the linear maps $C_\alpha, D_\beta\colon \mathbb{R}^2\to\mathbb{R}^2$ as follows

$$C_\alpha=\rho\begin{pmatrix}\cos 2\pi\phi & -\sin 2\pi\phi\\ \sin 2\pi\phi & \cos 2\pi\phi\end{pmatrix}\quad\text{and}\quad D_\beta=\begin{pmatrix}\beta_{s+1} & 0\\ 0 & \beta_{s+2}\end{pmatrix}. \tag{4.1}$$

Definition 4.1 ($(\mathbb{R},\mathbb{C})$-Simple cycle)**.** *A diffeomorphism f has a* $(\mathbb{R},\mathbb{C})$-simple cycle *of co-index two associated with P and Q and this cycle is unfolded in a* simple way *by the family $(f_t)_{t\in[-\epsilon,\epsilon]^2}$, $f_0 = f$, if the following conditions hold:*

i) There are local charts $\mathcal{U}_P$ and $\mathcal{U}_Q$ around P and Q

$$\mathcal{U}_P, \mathcal{U}_Q \backsimeq [-1,1]^s \times [-1,1]^2 \times [-1,1]^u,$$

where $f_t^{\pi(P)} \overset{\text{def}}{=} \mathcal{A}_t = \mathcal{A}$ and $f_t^{\pi(Q)} \overset{\text{def}}{=} \mathcal{B}_t = \mathcal{B}$ are linear maps of the form

$$\mathcal{A}(x^s,x^c,x^u) = \big(A^s(x^s), C_\alpha(x^c), A^u(x^u)\big) \quad \text{and}$$
$$\mathcal{B}(x^s,x^c,x^u) = \big(B^s(x^s), D_\beta(x^c), B^u(x^u)\big),$$

where $A^s, B^s \colon \mathbb{R}^s \to \mathbb{R}^s$ are contractions, corresponding to the eigenvalues $(\alpha_1,\dots,\alpha_s)$ and $(\beta_1,\dots,\beta_s)$, and $A^u, B^u \colon \mathbb{R}^u \to \mathbb{R}^u$ are expansions, corresponding to the eigenvalues $(\alpha_{s+3},\dots,\alpha_d)$ and $(\beta_{s+3},\dots,\beta_d)$.

ii) There is a partially hyperbolic splitting $E^{ss} \oplus E^c \oplus E^{uu}$, defined over the orbits of P and Q, such that in these local charts they are of the form

$$E^{ss} = \mathbb{R}^s \times \{0^2\} \times \{0^u\}, \;\; E^c = \{0^s\} \times \mathbb{R}^2 \times \{0^u\}, \;\; E^{uu} = \{0^s\} \times \{0^2\} \times \mathbb{R}^u.$$

iii) There are a quasi-transverse *heteroclinic point $Y_P \in W^u(\mathcal{O}_P) \cap W^s(\mathcal{O}_Q)$ in the neighborhood $\mathcal{U}_P$, a natural number $\ell > 0$, and a neighborhood $\mathcal{U}_{Y_P}$ of Y_P in $\mathcal{U}_P$, such that, in these local coordinates:*

- $Y_P = (0^s, 0^2, y_P^u)$, *where* $y_P^u \in [-1,1]^u$;
- $Y_Q = (y_Q^s, 0^2, 0^u) = f_0^\ell(Y_P) \in \mathcal{U}_Q$, *where* $y_Q^s \in [-1,1]^s$;
- $f_t^\ell(\mathcal{U}_{Y_P}) \subset \mathcal{U}_Q$ *and*

$$f_t^\ell \overset{\text{def}}{=} T_{PQ,t} : \mathcal{U}_{Y_P} \to f_t^\ell(\mathcal{U}_{Y_P})$$

is an affine map of the form

$$T_{PQ,t}(x^s,x^c,x^u) = \big(T^s_{PQ}(x^s) + y_Q^s, T^c_{PQ}(x^c) + t, T^u_{PQ}(x^u - y_P^u)\big),$$

where $T^s_{PQ} \colon \mathbb{R}^s \to \mathbb{R}^s$ is a linear contraction, $T^u_{PQ} \colon \mathbb{R}^u \to \mathbb{R}^u$ is a linear expansion and $T^c_{PQ} \colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & M_1 \end{pmatrix}, \qquad \text{where } |M_1| > 1.$$

iv) There are a transverse heteroclinic point $X_Q \in W^u(\mathcal{O}_Q) \pitchfork W^s(\mathcal{O}_P)$ in the neighborhood $\mathcal{U}_Q$, a natural number $r > 0$, and a neighborhood $\mathcal{U}_{X_Q}$ of X_Q in $\mathcal{U}_Q$ such that, in these local coordinates:

- $X_Q = (0^s, x^c_Q, 0^u) = (0^s, x_{q_1}, 0, 0^u)$, *where* $x_{q_1} \neq 0$;
- $X_P = f^r_t(X_Q) \in \mathcal{U}_P$ *and* $X_P = (0^s, x^c_P, 0^u)$, *where* $x^c_P \in \mathbb{R}^2$;
- $f^r_t(\mathcal{U}_{X_Q}) \subset \mathcal{U}_P$ *and*

$$f^r_t \stackrel{\text{def}}{=} T_{QP,t} = T_{QP} : \mathcal{U}_{X_Q} \to f^r_t(\mathcal{U}_{X_Q})$$

is an affine map of the form

$$T_{QP}(x^s, x^c, x^u) = \big(T^s_{QP}(x^s), T^c_{QP}(x^c) - x^c_Q + x^c_P, T^u_{QP}(x^u)\big),$$

where $T^s_{QP}\colon \mathbb{R}^s \to \mathbb{R}^s$ is a linear contraction, $T^u_{QP}\colon \mathbb{R}^u \to \mathbb{R}^u$ is a linear expansion and $T^c_{QP}\colon \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map of the form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & M_2 \end{pmatrix}, \qquad \text{where } |M_2| > 1.$$

We say that $\mathcal{A}$ and $\mathcal{B}$ are the linear parts *of the cycle, that X_Q and Y_P are the* heteroclinic points, *and T_{QP} and $T_{PQ,t}$ are the* transitions *of the cycle.*

In this context we have a similar result of Proposition 2.2:

Proposition 4.2. *Let f be a diffeomorphism having a $(\mathbb{R},\mathbb{C})$-cycle associated with saddles Q and P. Then any neighborhood of f contains diffeomorphisms having $(\mathbb{R},\mathbb{C})$-simple cycles associated with P and Q which are unfolded in a simple way.*

Proof. The proof of this proposition follows arguing as in Proposition 2.2, thus we will omit some details of this construction and focus on the forms of the transitions T_{QP} and $T_{PQ,t}$ which are the main difference. We also use the same notation as in the proof of Proposition 2.2.

For simplicity let us assume that Q and P are fixed points of f. By a small perturbation of f we can assume that there are small neighbourhoods of P and Q, say $\mathcal{U}_P$ and $\mathcal{U}_Q$, where f is linear.

The construction starts with the choice of heteroclinic points of the cycle. After an arbitrarily small perturbation of f, we can assume that there is a transverse intersection point $X \in W^u(Q) \cap W^s(P)$ and a quasi-transverse intersection point $Y \in W^s(Q) \cap W^u(P)$. We can also assume that $X \notin W^{uu}(Q)$, $X \notin W^{ss}(P)$, and $f^{-n_1}(X) \in W^u_{loc}(Q)$, for some $n_1 > 0$. Analogously, replacing X by some positive iterate we can assume that $f^{m_1}(X) \in W^s_{loc}(P)$.

By domination, we have that $f^{-n_2-n_1}(X)$ is much closer to $W^{cu}_{loc}(Q)$ than to $W^{uu}_{loc}(Q)$ for a sufficiently big n_2, and also $f^{m_2+m_1}(X)$ is much closer to $W^{cs}_{loc}(P)$ than $W^{ss}_{loc}(P)$ for a sufficiently big m_2. Thus after arbitrarily small perturbations we can assume that there are backward iterate $\bar{X}_Q$ of X that is in $W^{cu}_{loc}(Q)$, and forward iterate $\bar{X}_P$ of X that is in $W^{cs}_{loc}(P)$. Moreover, since $|\beta_{s+1}| < |\beta_{s+2}|$, by domination we can assume that the central coordinate of the point $\bar{X}_Q$ is of the form

$$\bar{X}_Q = (0^s, \bar{x}^c_Q, 0^u) = (0^s, \bar{x}_{q_1}, 0, 0^u), \quad \text{with} \quad \bar{x}_{q_1} \neq 0.$$

Now take a quasi-transverse heteroclinic point $Y \in W^s(Q) \cap W^u(P)$ and we fix backward iterate $\bar{Y}_P$ and forward iterate $\bar{Y}_Q$ of it such that $\bar{Y}_P \in W^u_{loc}(P)$ and $\bar{Y}_Q \in W^s_{loc}(Q)$.

We have the following claim whose proof is exactly as Claim 2.3.

Claim 4.3. *After an arbitrarily small perturbation of f, we can assume that there are large $r_0, \ell_0 > 0$, negative iterates $\tilde{X}_Q$ of $\bar{X}_Q$ and $\tilde{Y}_P$ of $\bar{Y}_P$, and small neighborhoods $\mathcal{U}_{\tilde{X}_Q}$ of $\tilde{X}_Q$ and $\mathcal{U}_{\tilde{Y}_P}$ of $\tilde{Y}_P$ such that the restrictions of f^{r_0} to $\mathcal{U}_{\tilde{X}_Q}$ and of f^{ℓ_0} to $\mathcal{U}_{\tilde{Y}_P}$ are linear maps preserving the splitting $E^{ss} \oplus E^c \oplus E^{uu}$.*

In the local coordinates in the neighborhoods $\mathcal{U}_Q$ and $\mathcal{U}_P$, write

$$\begin{aligned}
\tilde{X}_Q &= (0^s, \tilde{x}^c_Q, 0^u) \in \mathcal{U}_Q, \qquad & \tilde{X}_P &= f^{r_0}(\tilde{X}_Q) = (0^s, \tilde{x}^c_P, 0^u) \in \mathcal{U}_P,\\
\tilde{Y}_P &= (0^s, 0^c, \tilde{y}^u_P) \in \mathcal{U}_P, \qquad & \tilde{Y}_Q &= f^{\ell_0}(\tilde{Y}_P) = (\tilde{y}^s_Q, 0^c, 0^u) \in \mathcal{U}_Q.
\end{aligned}$$

By the previous claim, in the local coordinates the restriction of f^{r_0} to the neighborhood $\mathcal{U}_{\tilde{X}_Q}$ is of the form

$$f^{r_0}(x^s, x^c + \tilde{x}^c_Q, x^u) = \big(\tilde{T}^s_{QP}(x^s), \tilde{x}^c_P + \tilde{T}^c_{QP}(x^c), \tilde{T}^u_{QP}(x^u)\big),$$

where $\tilde{T}^s_{QP}$ is a linear contraction, $\tilde{T}^u_{QP}$ a linear expansion, and $\tilde{T}^c_{QP}$ linear.

Similarly, the restriction of f^{ℓ_0} to the neighborhood $\mathcal{U}_{\tilde{Y}_P}$ is of the form

$$f^{\ell_0}(x^s, x^c, x^u + \tilde{y}^u_P) = \big(\tilde{T}^s_{PQ}(x^s) + \tilde{y}^s_Q, \tilde{T}^c_{PQ}(x^c), \tilde{T}^u_{PQ}(x^u)\big),$$

where $\tilde{T}^s_{PQ}$ is a linear contraction, $\tilde{T}^u_{PQ}$ a linear expansion, and $\tilde{T}^c_{PQ}$ linear.

In Proposition 2.2, we proved that considering some forward and backward iterates of f and some small perturbations, the central parts of these transitions were the identity or a reflection. Here the construction is somewhat different.

To get the heteroclinic points in Definition 4.1, we fix k_1 and k_2 then we choose an arbitrarily large number $k_3 > 0$ (we will explain these choices later) and let $r \stackrel{\text{def}}{=} (k_1 + r_0 + k_2) + k_3$. The heteroclinic point X_Q will be a backward iterate $f^{-k_1}(\tilde{X}_Q) = X_Q = (0^s, x^c_Q, 0^u)$ of $\tilde{X}_Q$ and $X_P = f^r(X_Q)$. Note that the restriction of f^r to a small neighborhood of X_Q is of the form $f^r(x^s, x^c + x^c_Q, x^u) = (\bar{x}^s, \bar{x}^c, \bar{x}^u)$, where

$$\begin{aligned}
\bar{x}^s &= (A^s)^{k_2} \circ \tilde{T}^s_{QP} \circ (B^s)^{k_1}(x^s),\\
\bar{x}^c &= x^c_P + (C_\alpha)^{k_3} \circ (C_\alpha)^{k_2} \circ \tilde{T}^c_{QP} \circ (C_\beta)^{k_1}(x^c),\\
\bar{x}^u &= (A^u)^{k_2} \circ \tilde{T}^u_{QP} \circ (B^u)^{k_1}(x^u).
\end{aligned}$$

Clearly the map $T^s_{QP} = (A^s)^{k_2} \circ \tilde{T}^s_{QP} \circ (B^s)^{k_1}$ is a linear contraction and the map $T^u_{QP} = (A^u)^{k_2} \circ \tilde{T}^u_{QP} \circ (B^u)^{k_1}$ is a linear expansion. The important part for us is the central one. For that we introduce some perturbations of the derivative and choose appropriately k_1, k_2 and k_3.

After an arbitrarily small perturbation we can assume that there are (large) n and (even) m such that

$$\rho^n\,(\beta_{s+1})^m = 1. \tag{4.2}$$

Note that $\rho^{n\,k}\,(\beta_{s+1})^{m\,k} = 1$ for all $k \geq 1$. We can also assume that ϕ is a rational number and thus there is large k with

$$C_\alpha^{n\,k} = \rho^{n\,k}\, R_{n\,k\,\phi} = \rho^{n\,k}\,\mathrm{Id}.$$

Note that since $|\beta_{s+1}| < |\beta_{s+2}|$ by (4.2), we have that

$$\tilde{M}_1 = \tilde{M}_1(n,m,k) \stackrel{\text{def}}{=} \rho^{n\,k}(\beta_{s+2})^{m\,k} >> 1.$$

Consider $k_1 = m\,k$ and $k_2 = n\,k$. By this choice we have that $(C_\alpha)^{k_2} \circ \tilde{T}^c_{QP} \circ (C_\beta)^{k_1}$ is a map of the form:

$$\rho^{n\,k}\,\mathrm{Id} \circ \tilde{T}^c_{QP} \circ (C_\beta)^{k_1} = \tilde{T}^c_{QP} \begin{pmatrix} \rho^{n\,k}(\beta_{s+1})^{m\,k} & 0 \\ 0 & \rho^{n\,k}(\beta_{s+2})^{m\,k} \end{pmatrix}.$$

Now take an arbitrarily large k_3 such that the central part of f^r keeps orthogonal vectors in orthogonal vectors. Then arguing analogously as in the proof of same proposition, (note that we are choosing arbitrarily large k and k_3) we can modify the action of f in the central direction, without modifying the others directions, along the orbit of $X_Q, f(X_Q), \ldots, f^r(X_Q) = X_P$ to transform T^c_{QP} in a linear map of diagonal form

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & M_1 \end{pmatrix}. \tag{4.3}$$

The construction of the transition map T_{PQ} is analogous to the one above and thus omitted. Similarly, the construction of the unfolding family is analogous to the one in Proposition 2.2.

The sketch of the proof of the proposition is now complete. □

4.2 Model and quotient families

Model families associated to $(\mathbb{R},\mathbb{C})$-simple cycles are defined as in Definition 3.2 with the natural changes. Recall also the notation in Definition 4.1 of a $(\mathbb{R},\mathbb{C})$-simple cycles.

Definition 4.4 (Model unfolding families)**.** *Let $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ be a family of diffeomorphisms unfolding a $(\mathbb{R},\mathbb{C})$-simple cycle in a simple way at $t=(0,0)$.*

Suppose that this cycle is associated to the saddles P and Q with linear parts $\mathcal{A}$ and $\mathcal{B}$, heteroclinic points X_Q and Y_P, and transition maps T_{PQ} and T_{QP}. The model unfolding family $(F_t)_{t\in[-\epsilon,\epsilon]^2}$ *associated to the family $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ is defined as follows:*

$$F_t\colon \mathcal{U}_Q\cup\mathcal{U}_P\to M,\qquad F_t(x)=\begin{cases} T_{QP}(x) & \text{if } x\in\mathcal{U}_{X_Q},\\ \mathcal{A}(x) & \text{if } x\in\mathcal{U}_P\backslash\mathcal{U}_{Y_P},\\ T_{PQ,t}(x) & \text{if } x\in\mathcal{U}_{Y_P},\\ \mathcal{B}(x) & \text{if } x\in\mathcal{U}_Q\backslash\mathcal{U}_{X_Q},\end{cases}$$

where $\mathcal{U}_P$ and $\mathcal{U}_Q$ are small (linearizing) neighborhoods of P and Q, and $\mathcal{U}_{X_Q}\subset\mathcal{U}_Q$ and $\mathcal{U}_{Y_P}\subset\mathcal{U}_P$ are neighborhoods of X_Q and Y_P, respectively.

As in Section 3.2, we consider the quotient of this model family by the sum of the strong stable and strong unstable bundles, obtaining the following family of bidimensional maps. Let θ_{QP} be the restriction of T^c_{QP} to the central direction (see items (iii) and (iv) in Definition 4.1).

Recall that $B_\delta(x^c_Q)$ and $B_\delta(x^c_P)$ are δ-neighborhoods of x^c_Q and x^c_P, respectively, and the definition of the numbers $M_1, M_2>0$ in itens (iii) and (iv).

The possibilities for $\theta_{QP}\colon B_\delta(x^c_Q)\to B_\delta(x^c_P)$ are

$$(x_1,x_2)+(x_{q_1},x_{q_2})\overset{\theta_{QP}}{\longmapsto}\begin{cases}(x_1,M_1\,x_2)+(x_{p_1},x_{p_2})\\(-x_1,M_1\,x_2)+(x_{p_1},x_{p_2}).\end{cases}\tag{4.4}$$

Similarly for the restriction $\theta_{PQ,t}$ of $T^c_{PQ,t}$ to the central direction we have the map $\theta_{PQ,t}\colon\mathbb{R}^2\to\mathbb{R}^2$ defined by

$$(x_1,x_2)\overset{\theta_{PQ,t}}{\longmapsto}\begin{cases}(x_1,M_2\,x_2)+(t_1,t_2)\\(-x_1,M_2\,x_2)+(t_1,t_2).\end{cases}\tag{4.5}$$

We use the notation θ^i_{QP} and $\theta^j_{PQ,t}$, where $(i,j)\in\{+,-\}^2$, for the different cases above.

Considering the linear maps C_α and D_β in Equation (4.1) and the maps θ^i_{QP} and $\theta^j_{PQ,t}$, we define the quotient family below:

Definition 4.5 (Quotient families)**.** *Consider small $|t|$. For each $m,\ell\in\mathbb{N}$ we consider the composition*

$$\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t} \stackrel{\text{def}}{=} D^\ell_\beta\circ\theta^j_{PQ,t}\circ C^m_\alpha\circ\theta^i_{QP} : \mathbf{B}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t}(x^c_Q)\to B_\delta(x^c_Q),$$

for $(i,j)\in\{+,-\}^2$, where $\mathbf{B}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t}(x^c_Q)$ is the maximal subset of $B_\delta(x^c_Q)$ where the map $\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t}$ is defined.

Associated to a model unfolding family $(F_t)_{t\in[-\epsilon,\epsilon]^2}$ there is defined its quotient family $(\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t})_{m,\ell,t}$*, where $m,\ell\in\mathbb{N}$, and $\alpha,\beta_{s+1},\beta_{s+2},i,j$ are chosen according to the form of the central part of F_0.*

Proposition 4.6. *Given a quotient family $(\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t})_{m,\ell\in\mathbb{N}}$ and positive numbers $\epsilon_0,\mu_0>0$, there are a parameter $t_0\in\mathbb{R}^2$ with $|t_0|<\epsilon_0$, arbitrarily large natural numbers $k,\ell,\tilde{\ell},m,\tilde{m}$ with $(m,\ell)\neq(\tilde{m},\tilde{\ell})$, and numbers $\hat{\alpha}\in\mathbb{C}$ and $\hat{\beta}_{s+1}>1$ with*

$$|\alpha-\hat{\alpha}|<\mu_0,\quad |\beta_{s+1}-\hat{\beta}_{s+1}|<\mu_0,$$

such that

i) x^c_Q is a common fixed point of $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{m,\ell,t_0}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{\tilde{m},\tilde{\ell},t_0}$.

ii) The derivative of $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{m,\ell,t_0}$ at x^c_Q is of the form

$$\begin{pmatrix}\pm(\hat{\beta}_{s+1})^\ell\,\rho^m & 0\\ 0 & (\beta_{s+2})^\ell\,\rho^m\,M_1\,M_2\end{pmatrix},$$

where
$$1-(\hat{\beta}_{s+1})^{-1}\le|(\hat{\beta}_{s+1})^\ell\,\rho^m|\le\frac{1}{1-\rho}.$$

iii) $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{k,0,t_0}(x^c_Q)=0^2$.

Proof. Without lost of generality, we can assume that in local coordinates $x^c_P=(1,0)$ and $x^c_Q=(1,0)$. After an arbitrarily small perturbation we can assume that α has a rational argument ϕ. Fix $n>0$ such that the map $C^n_\alpha=\rho^n\,R^n_\phi=\rho^n\,\mathrm{Id}$, where R_ϕ denotes the rotation of angle ϕ.

We need to consider the different cases according to the choices of $(i,j) \in \{+,-\}^2$. Recalling Equations (4.5) and (4.4), for the case $(+,+)$ we have

$$\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},+,+}_{n\,m,\ell,(t_1,t_2)}(x,y) = \big((\beta_{s+1})^{\ell}[(\rho^n)^m\,x + t_1],\,(\beta_{s+2})^{\ell}[(\rho^n)^m\,M_1\,M_2\,y + t_2]\big).$$

Let $t=(t_1,t_2)=(t_1,0)$ and consider a point $(x,0)$. Then

$$\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},+,+}_{n\,m,\ell,(t_1,0)}(x,0) = \big((\beta_{s+1})^{\ell}[(\rho^n)^m\,x + t_1],\,0\big). \tag{4.6}$$

We will choose pairs $(n\,m,\ell)$ and $(n\,(m+1),\tilde{\ell})$ and a parameter t_1 such that (after a small perturbation) the point $x^c_Q = (1,0)$ is a fixed point for these compositions.

After an arbitrarily small perturbation of β_{s+1} we can assume that there are arbitrarily large m and (even) ℓ such that

$$\rho^{n\,m}(1-\rho^n) = (\rho^n)^m - (\rho^n)^{m+1} = (\beta_{s+1})^{-\ell}.$$

Consider an even $\tilde{\ell} >> \ell$ such that $(\tilde{\beta}_{s+1})^{-\tilde{\ell}}$ is close to zero for all $\tilde{\beta}_{s+1}$ close to β_{s+1}. Take $k>0$ (close to $n(m+1)$), $\hat{\beta}_{s+1}$ close to β_{s+1} and $\hat{\rho}$ close to ρ such that

$$(\hat{\rho}^n)^m - (\hat{\rho}^n)^{m+1} = (\hat{\beta}_{s+1})^{-\ell} - (\hat{\beta}_{s+1})^{-\tilde{\ell}} \quad \text{and} \quad \hat{\rho}^k = (\hat{\rho}^n)^{m+1} - \hat{\beta}^{-\tilde{\ell}}_{s+1}. \tag{4.7}$$

Let

$$t_1 = -(\hat{\rho}^n)^m + (\hat{\beta}_{s+1})^{-\ell}. \tag{4.8}$$

With these choices we have the following:

Claim 4.7. *The point $(1,0)$ is fixed for $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{n\,m,\ell,(t_1,0)}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{n\,(m+1),\tilde{\ell},(t_1,0)}$.*

Proof. The first assertion follows from (4.6) and $(\hat{\beta}_{s+1})^{\ell}[(\hat{\rho}^n)^m + t_1] = 1$ (which is a consequence of (4.8)).

Similarly, for the second assertion it is enough to see that

$$(\hat{\beta}_{s+1})^{\tilde{\ell}}\,[(\hat{\rho}^n)^{m+1} + t_1] = 1.$$

By the definition of t_1 in (4.8) and (4.7), we have

$$\begin{aligned}(\hat{\beta}_{s+1})^{\tilde{\ell}}\,\big[(\hat{\rho}^n)^{m+1} + t_1\big] &= (\hat{\beta}_{s+1})^{\tilde{\ell}}\,\big[(\hat{\rho}^n)^{m+1} - (\hat{\rho}^n)^m + (\hat{\beta}_{s+1})^{-\ell}\,\big]\\ &= (\hat{\beta}_{s+1})^{\tilde{\ell}}\,\big[(\hat{\beta}_{s+1})^{-\tilde{\ell}} - (\hat{\beta}_{s+1})^{-\ell} + (\beta_{s+1})^{-\ell}\,\big] = 1,\end{aligned}$$

proving the claim. □

Claim 4.8. $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{k,0,(t_1,0)}(1,0) = (0,0)$.

Proof. By first equation in (4.7) we have that

$$t_1 = -(\hat{\rho}^n)^m + (\hat{\beta}_{s+1})^{-\ell} = -(\hat{\rho}^n)^{m+1} + (\hat{\beta}_{s+1})^{-\tilde{\ell}}.$$

Using Equation (4.6), note that by second equation in (4.7) and the choice of t_1 above we get

$$\hat{\rho}^k + t_1 = (\hat{\rho}^n)^{m+1} - \hat{\beta}_{s+1}^{-\tilde{\ell}} - (\hat{\rho}^n)^{m+1} + (\hat{\beta}_{s+1})^{-\tilde{\ell}} = 0,$$

ending the proof of the claim. □

Note that the case $(-,-)$ follows similarly. For the cases $(+,-)$ and $(-,+)$ it is enough to consider similar construction with $t_1 = (\hat{\rho}^n)^m + (\hat{\beta}_{s+1})^{-\tilde{\ell}}$ to have the point $(1,0)$ fixed point for $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,-}_{n\,m,\tilde{\ell},(t_1,0)}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,-}_{n\,(m+1),\ell,(t_1,0)}$. This completes the proof of the first item of the proposition.

The assertion about the derivative of $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{n\,m,\ell,(t_1,0)}$ is immediate. For the estimates on $|(\hat{\beta}_{s+1})^{\ell}\,\hat{\rho}^m|$ recall Equation (4.7) and note that

$$(\hat{\rho}^n)^m\,(\hat{\beta}_{s+1})^{\ell} = \frac{1-(\hat{\beta}_{s+1})^{-\tilde{\ell}+\ell}}{1-\hat{\rho}^n}.$$

Therefore

$$(\hat{\rho}^n)^m\,(\hat{\beta}_{s+1})^{\ell} \leq \frac{1}{1-\hat{\rho}}.$$

Since $1-(\hat{\beta}_{s+1})^{-1} \leq 1-(\hat{\beta}_{s+1})^{-\tilde{\ell}+\ell}$ we have that

$$1-(\hat{\beta}_{s+1})^{-1} \leq (\hat{\rho}^n)^m\,(\hat{\beta}_{s+1})^{\ell}.$$

These two estimates complete the proof of item (ii). □

Proposition 4.9. *Let $(F_t)_{t\in[-\epsilon,\epsilon]^2}$ be a model family whose quotient family $(\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2}}_{m,\ell,t})_{m,\ell\in\mathbb{N}}$ satisfies Proposition 4.6 for t_0, (m,ℓ) and $(\tilde{m},\tilde{\ell})$. Let*

$$C \stackrel{\text{def}}{=} \max\{[1-(\beta_{s+1})^{-1}]^{-1}, [1-\rho]^{-1}\}.$$

Then

i) There is periodic point $A=(a^s,a^c,a^u)\in\mathcal{U}_{X_Q}$ of F_{t_0} of period $m+2+\ell$ such that the central eigenvalues of $D(F_{t_0})^{m+2+\ell}_A$ are

$$\lambda_{s+1} = \rho^m\,(\beta_{s+1})^{\ell} \in [C^{-1}, C], \qquad \lambda_{s+2} = \rho^m\,M_1\,M_2\,(\beta_{s+2})^{\ell}.$$

ii) The intersection $W^{uu}(A;F_{t_0})\cap W^{ss}(A;F_{t_0})$ is an infinite set (non-trivial).

Result above is a similar result of Proposition 3.3 in this context. Since the proof follows exactly from it, we omit it. The estimates on the central eigenvalues and the constant C follow immediately from (ii) in Proposition 4.6.

Next result is an immediate consequence of proposition above.

Corollary 4.10. *Consider the model family $(F_t)_{t\in[-\epsilon,\epsilon]^2}$ in Proposition 4.9 associated to a family of diffeomorphisms $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ unfolding a $(\mathbb{R},\mathbb{C})$-simple cycle (associated to saddles P and Q) in a simple way. Let t_0, (m,ℓ), $(\tilde{m},\tilde{\ell})$, $k\in\mathbb{N}$ and $C>0$ as in Proposition 4.6 and 4.9. Then there is a periodic point $A\in\mathcal{U}_{X_Q}$ of f_{t_0} of period $m+2+\ell$ such that*

i) If $\lambda_{s+1}(A)$ is one of the central eigenvalue of $D(f_{t_0})_A^{m+2+\ell}$ then

$$\lambda_{s+1}(A)\in[C^{-1},C];$$

ii) the intersection $W^{uu}(A;f_{t_0})\cap W^{ss}(A;f_{t_0})$ is non-trivial;

iii) $W^{uu}(A;f_{t_0})\cap W^{s}(Q;f_{t_0})\neq\emptyset$ and $W^{ss}(A;f_{t_0})\cap W^{u}(Q;f_{t_0})\neq\emptyset$;

iv) $W^{uu}(A;f_{t_0})\cap W^{s}(P;f_{t_0})\neq\emptyset$.

4.3 Strong homoclinic intersections

In this section we prove proposition below.

Proposition 4.11. *Let f be a diffeomorphism having a co-index two $(\mathbb{R},\mathbb{C})$-cycle associated with a pair of saddles P and Q. Then there are diffeomorphisms g arbitrarily C^1-close to f having strong homoclinic intersection with one-dimensional central direction. Moreover,*

$$\begin{aligned}&W^{ss}(A;g)\cap W^{u}(Q_g;g)\neq\emptyset,\\&W^{uu}(A;g)\cap W^{s}(Q_g;g)\neq\emptyset,\quad \textit{and}\\&W^{uu}(A;g)\cap W^{s}(P_g;g)\neq\emptyset.\end{aligned}$$

where Q_g and P_g are the continuations of Q and P, respectively, for g.

The difference of Proposition 4.11 and Proposition 3.1 is that the periodic point of proposition above has just one eigenvalue equal to one, that is, it has a partially hyperbolic splitting $E^{ss}\oplus E^{c}\oplus E^{uu}$, where E^{ss} is a contracting bundle, E^{uu} is a expanding bundle and E^{c} is a one-dimensional bundle.

Therefore Propositions 4.2 and 4.11, and Theorem 1.5 give the following result:

Corollary 4.12. *Let f be a diffeomorphism having a co-index two $(\mathbb{R},\mathbb{C})$-cycle. Then every C^1-neighborhood of f contains diffeomorphisms with C^1-robust heterodimensional cycles of co-index one.*

Moreover, join Propositions 4.2 and 4.11, and Theorem 3.5 in [8] we have the following result related to the semi-stabilization of the cycle.

Corollary 4.13. *Let f be a diffeomorphism having a co-index two $(\mathbb{R},\mathbb{C})$-cycle associated with saddles P and Q. Then every C^1-neighborhood of f contains a diffeomorphism h with C^1-robust heterodimensional cycles of co-index one associated to the continuation Q_h of Q and a transitive hyperbolic set Γ_h.*

By completeness, we include Theorem 3.5 of [8]:

Theorem 4.14 (Theorem 3.5 [8]). *Let g be a diffeomorphism, Q a saddle of g, and A a partially hyperbolic saddle-node (or flip) of g such that:*

i) u-index of Q is equal to $\dim(W^{uu}(A;g))+1=u+1$;

ii) A has a strong homoclinic intersection;

iii) $W^{ss}(A;g)\cap W^{u}(Q;g)\neq\emptyset$; and

iv) $W^{uu}(A;g)\cap W^{s}(Q;g)\neq\emptyset$.

Then there is a diffeomorphism h arbitrarily C^1-close to g with C^1-robust heterodimensional cycles of co-index one associated to the continuation Q_h of Q and a transitive hyperbolic set Γ_h containing a hyperbolic continuation A_h of A of u-index u.

4.3.1 End of the proof of Proposition 4.11

To prove this proposition note first that, after an arbitrarily small perturbation we can assume that the cycle is $(\mathbb{R},\mathbb{C})$-simple (Proposition 4.2). We can consider a family $(f_t)_{t\in[-\epsilon,\epsilon]^2}$ unfolding this simple cycle in a simple way, Proposition 4.2. We can assume (after an arbitrarily small perturbation if necessary) that the central family $\mathcal{Q}_{m,\ell,t}$ of this model family satisfies Proposition 4.6 for some arbitrarily small t_0 and large (m,ℓ) and $(\tilde m,\tilde\ell)$. Corollary 4.10 implies that f_{t_0} has a strong homoclinic intersection associated to a saddle A with period $m+\ell+2$ and one central eigenvalue $\lambda_{s+1}(A)\in[C^{-1},C]$. Since this bound is independent of m and ℓ (note that m,ℓ can be chosen arbitrarily large), after an arbitrarily small perturbation (preserving the strong homoclinic intersection) we can assume that the central eigenvalue $\lambda_{s+1}(A)$ has modulus one. This completes the proof of Proposition 4.11. $\square$