5
Blenders

In this chapter, following [9, Section 6.2.1], we first introduce a simplified model of a blender with central direction $c > 1$. The main result in Section 5.1 is the intersection property in Proposition 5.1 implying that an invariant manifold of a saddle of s-index s topologically behaves as a saddle of bigger s-index. We see in Section 5.2 that these blenders are robust.

5.1
Geometric model of blender

Fix $c > 1$ and $n \geq 2$ and consider a diffeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ such that the maximal invariant set of f in the rectangle $R = [0,1]^n$ is an affine Smale horseshoe contained in the interior of R.

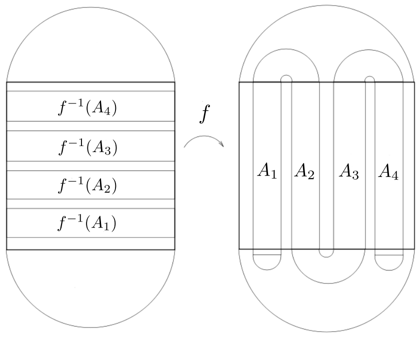

Figure 5.1: Rectangles

For notational simplicity, let us consider the case $n = 2$ and $c = 2$. Suppose that the affine horseshoe has four legs. Let $I_1, \ldots, I_4$ be pairwise disjoint closed intervals in $(0,1)$ such that $f(R) \cap R$ is the union of the vertical rectangles $A_i = I_i \times [0,1]$, $i = 1, \ldots, 4$. Analogously, let $J_1, \ldots, J_4$ be pairwise disjoint closed intervals in $(0,1)$ such that $f^{-1}(R) \cap R$ is the union of four horizontal rectangles $f^{-1}(A_i) = [0,1] \times J_i$, $i = 1, \ldots, 4$. Denote by (x_q, y_q) the fixed point of f in the rectangle A_1.

Let $\sigma < 1$ be the rate of contraction of f in R and take $1 < \lambda < \beta < \max\{2, \sigma^{-1}\}$. Consider $0 < \lambda_0, \beta_0 < 1$ with $\beta - \beta_0$, $\lambda - \lambda_0 \in (0,1)$.

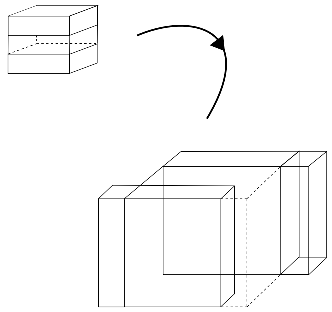

Figure 5.2: The map F

We define a diffeomorphism $F\colon \mathbb{R}^4 \to \mathbb{R}^4$ whose restriction to the cube $\mathbf{C} = [-1,1]^2 \times R$ is defined as follows:

$$F(t,s,x,y) = \begin{cases} \big(\lambda\, t, \beta\, s, f(x,y)\big) & \text{if } (x,y) \in f^{-1}(A_1), \\ \big(\lambda\, t, \beta\, s - \beta_0, f(x,y)\big) & \text{if } (x,y) \in f^{-1}(A_2), \\ \big(\lambda\, t - \lambda_0, \beta\, s, f(x,y)\big) & \text{if } (x,y) \in f^{-1}(A_3), \\ \big(\lambda\, t - \lambda_0, \beta\, s - \beta_0, f(x,y)\big) & \text{if } (x,y) \in f^{-1}(A_4), \end{cases} \tag{5.1}$$

where and $t, s \in [-1,1]$. Let Γ be the maximal invariant (hyperbolic) set of F in the cube $\mathbf{C}$. Under these assumptions the set $\mathbf{C} \cap F(\mathbf{C})$ is the union of the following four sets

$$\begin{aligned} \mathbb{A}_1 &= [-1,1] \times [-1,1] \times A_1, \\ \mathbb{A}_2 &= [-1,1] \times [-1, \beta - \beta_0] \times A_2, \\ \mathbb{A}_3 &= [-1, \lambda - \lambda_0] \times [-1,1] \times A_3, \\ \mathbb{A}_4 &= [-1, \lambda - \lambda_0] \times [-1, \beta - \beta_0] \times A_4. \end{aligned}$$

Consider the hyperbolic fixed point $Q = (0,0,x_q,y_q) \in \mathbb{A}_1$ of F of s-index 1 and its local stable manifold

$$W^s_{loc}(Q;F) \stackrel{\text{def}}{=} \{(0,0)\} \times [0,1] \times \{y_q\},$$

which is the connected component of $W^s(Q;F) \cap \mathbf{C}$ that contains Q.

We define *vertical segments, strips, and blocks* to the right of $W^s_{loc}(Q;F)$ as follows:

- A *vertical segment to the right of* $W^s_{loc}(Q;F)$ is a segment of the form $\varsigma = \{(t,s,x)\} \times [0,1]$, where $x \in [0,1]$ and $0 < t, s \leq 1$.
- A *vertical block to the right of* $W^s_{loc}(Q;F)$ is a set of the form $\mathrm{B} = [t_1,t_2] \times [s_1,s_2] \times \{x\} \times [0,1]$, where $x \in [0,1]$, $0 < t_1 < t_2 \leq 1$ and $0 < s_1 < s_2 \leq 1$. Consider $a(\mathrm{B}) = (t_2 - t_1) \cdot (s_2 - s_1)$ the area of the base of this vertical block.

- A *t-vertical strip to the right of* $W^s_{loc}(Q; F)$ is a set of the form $\Delta^t = [t_1, t_2] \times \{s\} \times \{x\} \times [0, 1]$, where $x \in [0, 1]$, $s \in (0, 1]$ and $0 < t_1 < t_2 \leq 1$. We denote by $w_t(\Delta^t) = (t_2 - t_1)$ the *t-width of* Δ^t.
- A *s-vertical strip to the right of* $W^s_{loc}(Q; F)$ is a set of the form $\Delta^s = \{t\} \times [s_1, s_2] \times \{x\} \times [0, 1]$, where $x \in [0, 1]$, $t \in (0, 1]$ and $0 < s_1 < s_2 \leq 1$. We denote by $w_s(\Delta^s) = (s_2 - s_1)$ the *s-width of* Δ^s.

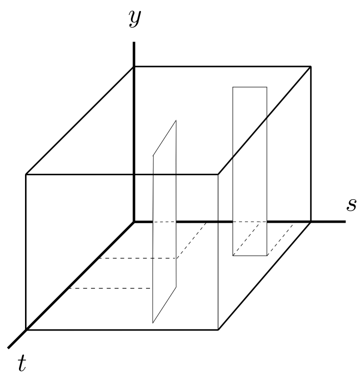

Figure 5.3: t-vertical strips and s-vertical strips

The following intersection result is the main step of this section.

Proposition 5.1. *Consider the map* F *and the cube* $\mathbf{C}$ *in* (5.1) *and the hyperbolic fixed point* Q *of* F. *Then the stable manifold of* Q *intersects every vertical block* B *to the right of* $W^s_{loc}(Q; F)$ *in* $\mathbf{C}$.

5.1.1 Proof of Proposition 5.1

To prove the proposition, we will rewrite it using iterated function systems (IFS). See Proposition 5.2 which is just a reformulation of Proposition 5.1 in terms of the IFS's obatined considering the quotient dynamics by the hyperbolic part.

More precisely, consider the maps

- $\alpha_1(t, s) = (\phi_1(t), \psi_1(s))$ for $t \in [-\frac{1}{\lambda}, \frac{1}{\lambda}]$ and $s \in [-\frac{1}{\beta}, \frac{1}{\beta}]$;
- $\alpha_2(t, s) = (\phi_1(t), \psi_2(s))$ for $t \in [-\frac{1}{\lambda}, \frac{1}{\lambda}]$ and $s \in [\frac{-1+\beta_0}{\beta}, 1]$;
- $\alpha_3(t, s) = (\phi_2(t), \psi_1(s))$ for $t \in [\frac{-1+\lambda_0}{\lambda}, 1]$ and $s \in [-\frac{1}{\beta}, \frac{1}{\beta}]$;
- $\alpha_4(t, s) = (\phi_2(t), \psi_2(s))$ for $t \in [\frac{-1+\lambda_0}{\lambda}, 1]$ and $s \in [\frac{-1+\beta_0}{\beta}, 1]$,

where $\phi_1, \phi_2, \psi_1, \psi_2 \colon \mathbb{R} \to \mathbb{R}$ are the expanding maps given by

- $\phi_1(t) = \lambda\, t$ and $\phi_2(t) = \lambda\, t - \lambda_0$,

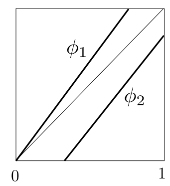

Figure 5.4: $\phi_1, \phi_2 \colon \mathbb{R} \to \mathbb{R}$

- $\psi_1(s) = \beta\, s$ and $\psi_2(s) = \beta\, s - \beta_0$.

Indeed, considering the quotient of the dynamics of F by the sum of the strong stable and strong unstable directions, one gets the bidimensional central dynamics given by the compositions of maps α_i, $i = 1, \dots, 4$, above. More precisely, observe that

- $F^{-1}(\mathbb{A}_1) = [-\frac{1}{\lambda}, \frac{1}{\lambda}] \times [-\frac{1}{\beta}, \frac{1}{\beta}] \times f^{-1}(A_1)$;
- $F^{-1}(\mathbb{A}_2) = [-\frac{1}{\lambda}, \frac{1}{\lambda}] \times [\frac{-1+\beta_0}{\beta}, 1] \times f^{-1}(A_2)$;
- $F^{-1}(\mathbb{A}_3) = [\frac{-1+\lambda_0}{\lambda}, 1] \times [-\frac{1}{\beta}, \frac{1}{\beta}] \times f^{-1}(A_3)$;
- $F^{-1}(\mathbb{A}_4) = [\frac{-1+\lambda_0}{\lambda}, 1] \times [\frac{-1+\beta_0}{\beta}, 1] \times f^{-1}(A_4)$

and that the map α_i is the "central part"of the restriction of F to $F^{-1}(\mathbb{A}_i)$.

Proposition 5.2. *Let $\mathcal{G} = \mathcal{G}(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be the iterated function system generated by the compositions of the maps α_i, $i = 1, \dots, 4$. Then given any open set $O \subset [0,1]^2$ there is a map $\alpha \in \mathcal{G}$ such that $(0,0) \in \alpha(O)$.*

We need the following one-dimensional version of the proposition above.

Lemma 5.3 (Lemma 6.7 in [9])**.** *Denote by $\mathcal{G}(\phi_1, \phi_2)$ the iterated function system generated by the maps*

$$\phi_1(t) = \lambda\, t \quad \textit{and} \quad \phi_2(t) = \lambda\, t - \lambda_0,$$

where $0 < \lambda_0 < 1 < \lambda < 2$. Then given any open interval $I \subset [0,1]$ there is $\phi \in \mathcal{G}(\phi_1, \phi_2)$ such that $0 \in \phi(I)$.

Proof. Note that $\phi_2(\frac{\lambda_0}{\lambda}) = 0$, then it is enough to prove that there is a map $\varphi \in \mathcal{G}(\phi_1, \phi_2)$ such that $\frac{\lambda_0}{\lambda} \in \varphi(I)$. Then $\phi = \phi_2 \circ \varphi$ satisfies the conclusion in the lemma. Let $I = I_0$ be an open interval in $[0,1]$ that does not contain $\frac{\lambda_0}{\lambda}$. We will define inductively intervals $I_n \subset [0,1]$ as follows: if I_n does not contain $\frac{\lambda_0}{\lambda}$ we let

$$I_{n+1} = \begin{cases} \phi_1(I_n) & \text{if} \quad I_n \subset (0, \frac{1}{\lambda}], \\ \phi_2(I_n) & \text{if} \quad I_n \subset (\frac{\lambda_0}{\lambda}, 1]. \end{cases}$$

Note that the length of $|I_{n+1}|$ of I_{n+1} is $\lambda\,|I_n|$. Thus if the intervals I_n are inductively defined and do not contain $\frac{\lambda_0}{\lambda}$ then $|I_{n+1}| = \lambda^n\,|I_0|$. Since $\lambda > 1$, this implies that there is a first $n_0 > 0$ such that I_{n_0} contains $\frac{\lambda_0}{\lambda}$. Taking φ the composition of the corresponding ϕ_i we prove the claim and thus the lemma. □

Proof of Proposition 5.2. Given an open set O in $[0,1]^2$, there are intervals $I_t, I_s \subset [0,1]$ such that $O \supset I_t \times I_s$. Applying Lemma 5.3 to the intervals I_t and I_s, and to the maps (ϕ_1, ϕ_2) and (ψ_1, ψ_2), respectively, we get maps

$$\phi = \phi_{i_n} \circ \cdots \circ \phi_{i_1} \in \mathcal{G}(\phi_1, \phi_2) \quad \text{and} \quad \psi = \psi_{j_m} \circ \cdots \circ \psi_{j_1} \in \mathcal{G}(\psi_1, \psi_2)$$

such that $0 \in \phi(I_t)$ and $0 \in \psi(I_s)$, respectively. Suppose that $n \leq m$. Then, since $\phi_1(0) = 0$, to prove the proposition it is enough to consider the maps

$$\alpha_k = \begin{cases} (\phi_{i_k}, \psi_{j_k}) & \text{for} \quad k = 1, \ldots, n \\ (\phi_1, \psi_{j_k}) & \text{for} \quad k = n+1, \ldots, m, \end{cases}$$

and their composition. □

Proof of Proposition 5.1. Let $\mathrm{B} = \mathrm{B}_0 = [t_1, t_2] \times [s_1, s_2] \times \{x\} \times [0,1]$ be a vertical block to the right of $W^s_{loc}(Q; F)$. It is enough to proof that

$$F^N(\mathrm{B}_0) \cap W^s_{loc}(Q; F) \neq \emptyset \quad \text{for some } N \geq 0. \tag{5.2}$$

Recall that $a(\mathrm{B}_0) = (t_2 - t_1) \cdot (s_2 - s_1) > 0$ is the area of the base of the block B_0. Write $I_t = [t_1, t_2]$ and $I_s = [s_1, s_2]$ and consider $w_t(\mathrm{B}_0) = t_2 - t_1$ and $w_s(\mathrm{B}_0) = s_2 - s_1$ the t and s-widths of the block.

First recall that $W^s_{loc}(Q; F) = \{(0,0)\} \times [0,1] \times \{y_q\}$. In the two last coordinates the action of the map F is given by the map f, more precisely, we have a contraction and an expansion in each coordinate, respectively. To get the intersection in (5.2), we need to prove that there is $N > 0$ such that $F^N(\mathrm{B}_0) \supset \{(0, 0, x(N))\} \times [0,1]$ for some $x(N) \in [0,1]$. Indeed, this inclusion is given by Proposition 5.2. Let us go to the details. We call *t-interval* the first interval of a block and *s-interval* the second one.

The intervals $I_t = I_t(0)$ and $I_s = I_s(0)$, that form the base of the block B_0, might satisfy one of the following cases:

i) $I_t \subseteq (0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$ and $I_s \subseteq (0, \frac{\beta_0}{\beta}) \cup (\frac{\beta_0}{\beta}, \frac{1}{\beta}]$;

ii) $I_t \subseteq (0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$ and $I_s \subseteq (\frac{\beta_0}{\beta}, 1]$;

iii) $I_t \subseteq (\frac{\lambda_0}{\lambda}, 1]$ and $I_s \subseteq (0, \frac{\beta_0}{\beta}) \cup (\frac{\beta_0}{\beta}, \frac{1}{\beta}]$;

iv) $I_t \subseteq (\frac{\lambda_0}{\lambda}, 1]$ and $I_s \subseteq (\frac{\beta_0}{\beta}, 1]$;

v) $I_t \subseteq (0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$ and $I_s \ni \frac{\beta_0}{\beta}$;

vi) $I_t \subseteq (\frac{\lambda_0}{\lambda}, 1]$ and $I_s \ni \frac{\beta_0}{\beta}$;

vii) $I_t \ni \frac{\lambda_0}{\lambda}$ and $I_s \subseteq (0, \frac{\beta_0}{\beta}) \cup (\frac{\beta_0}{\beta}, \frac{1}{\beta}]$;

viii) $I_t \ni \frac{\lambda_0}{\lambda}$ and $I_s \subseteq (\frac{\beta_0}{\beta}, 1]$;

ix) $I_t \ni \frac{\lambda_0}{\lambda}$ and $I_s \ni \frac{\beta_0}{\beta}$.

We claim that there are essentially three cases to consider (cases (ix), (v), and (i)) and that the other cases follow analogously.

Case (ix)**:** If the intervals I_t and I_s satisfy case (ix), then $F(\mathrm{B}_0) \cap \mathbb{A}_4$ intersects $W^s_{loc}(Q; F)$, thus we are done and $N = 1$.

Cases (v)**,** (vi)**,** (vii)**, and** (viii)**:** First, if the intervals I_t and I_s satisfy case (v), then the s-interval of $F(\mathrm{B}_0) \cap \mathbb{A}_2$ contains 0, and $F(\mathrm{B}_0) \cap \mathbb{A}_2$ contains a t-vertical strip

$$\Delta^t(1) = I_t(1) \times \{s(1)\} \times \{x(1)\} \times [0, 1]$$

to the right of $W^s_{loc}(Q; F)$ with $w_t(\Delta^t(1)) \geq \lambda\, w_t(\mathrm{B}_0)$. There are three possibilities for $\Delta^t(1)$:

- $I_t(1) \subset [0, \frac{\lambda_0}{\lambda}) \cup (\frac{\lambda_0}{\lambda}, \frac{1}{\lambda}]$. Then $F(F(\mathrm{B}_0) \cap \mathbb{A}_2) \cap \mathbb{A}_2$ contains a t-vertical strip $\Delta^t(2) = I_t(2) \times \{s(2)\} \times \{x(2)\} \times [0, 1]$ to the right of $W^s_{loc}(Q; F)$ with $w_t(\Delta^t(2)) \geq \lambda^2\, w_t(\mathrm{B}_0)$.
- $I_t(1) \subset (\frac{\lambda_0}{\lambda}, 1]$. Then $F(F(\mathrm{B}_0) \cap \mathbb{A}_2) \cap \mathbb{A}_4$ contains a t-vertical strip $\Delta^t(2) = I_t(2) \times \{s(2)\} \times \{x(2)\} \times [0, 1]$ to the right of $W^s_{loc}(Q; F)$ with $w_t(\Delta^t(2)) \geq \lambda^2\, w_t(\mathrm{B}_0)$.
- $I_t(1) \ni \frac{\lambda_0}{\lambda}$. Then $F(F(\mathrm{B}_0) \cap \mathbb{A}_2) \cap \mathbb{A}_4$ intersects $W^s_{loc}(Q; F)$.

Arguing as in Lemma 5.3 and considering the t-vertical strip $\Delta^t(2)$ there are three possibilities as above and we repeat the process until getting a first N such that the interval $I_t(N)$ contains $\frac{\lambda_0}{\lambda}$, (this natural number N exists because $\lambda > 1$, then the width of the strips are increasing).

The three cases below follow similarly:

- for case (vi), $F(\mathrm{B}_0) \cap \mathbb{A}_4$ contains a t-vertical strip $\Delta^t(1)$ to the right of $W^s_{loc}(Q; F)$ with $w_t(\Delta^t(1)) \geq \lambda\, w_t(\mathrm{B}_0)$.
- for case (vii), $F(\mathrm{B}_0) \cap \mathbb{A}_3$ contains a s-vertical strip $\Delta^s(1)$ to the right of $W^s_{loc}(Q; F)$ with $w_s(\Delta^s(1)) \geq \beta\, w_s(\mathrm{B}_0)$.

- for case (viii), $F(\mathrm{B}_0) \cap \mathbb{A}_4$ contains a s-vertical strip $\Delta^s(1)$ to the right of $W^s_{loc}(Q;F)$ with $w_s(\Delta^s(1)) \geq \beta\, w_s(\mathrm{B}_0)$.

Cases (i)**,** (ii)**,** (iii)**, and** (iv)**:** First, suppose that the intervals I_t and I_s of the block B_0 satisfy case (i). Then $F(\mathrm{B}_0) \cap \mathbb{A}_1$ contains a vertical block $\mathrm{B}_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0,1]$ to the right of $W^s_{loc}(Q;F)$ with $a(\mathrm{B}_1) \geq \lambda\,\beta\, a(\mathrm{B}_0)$. There are nine possibilities analogous to (i) $\sim$ (ix) for the intervals $I_t(1)$ and $I_s(1)$. Arguing again as in Proposition 5.2 (this is possible since $\lambda > 1$ and $\beta > 1$) we repeat the process until getting a first N such that intervals $I_t(N)$ and $I_s(N)$ satisfy $I_t(N) \ni \frac{\lambda_0}{\lambda}$ and $I_s(N) \ni \frac{\beta_0}{\beta}$. The number N exists because $\lambda > 1$ and $\beta > 1$ and thus then the area/width of the base/strips are increasing.

The three cases bellow are similar to case (i):

- for case (ii), $F(\mathrm{B}_0) \cap \mathbb{A}_2$ contains a vertical block $\mathrm{B}_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0,1]$ to the right of $W^s_{loc}(Q;F)$ with $a(\mathrm{B}_1) \geq \lambda\,\beta\, a(\mathrm{B}_0)$.
- for case (iii), $F(\mathrm{B}_0) \cap \mathbb{A}_3$ contains a vertical block $\mathrm{B}_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0,1]$ to the right of $W^s_{loc}(Q;F)$ with $a(\mathrm{B}_1) \geq \lambda\,\beta\, a(\mathrm{B}_0)$.
- for case (iv), $F(\mathrm{B}_0) \cap \mathbb{A}_4$ contains a vertical block $\mathrm{B}_1 = I_t(1) \times I_s(1) \times \{x(1)\} \times [0,1]$ to the right of $W^s_{loc}(Q;F)$ with $a(\mathrm{B}_1) \geq \lambda\,\beta\, a(\mathrm{B}_0)$.

The proof of the proposition is now complete. □

5.2 Cone fields and robustness

5.2.1 Cone fields

Consider the map F defined in Equation 5.1, $F\colon \mathbf{C} \to \mathbb{R}^{n+2}$, where $\mathbf{C} = [-1,1]^2 \times [0,1]^n$, and the maximal invariant set of the map F in the cube $\mathbf{C}$ is Γ. We call the set Γ a *model of blender*. Note that Γ has a dominated splitting (recall Section 2.1) of the form $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$, where $\mathbb{X}$ and $\mathbb{Y}$ are the strong stable and strong unstable bundles (corresponding to the stable and unstable bundles of the quotient horseshoe), respectively, and $\mathbb{T}$ and $\mathbb{S}$ the weak unstable bundles (corresponding to the weak expansions λ and β). This dominated splitting can be extended (and we do) to the whole cube $\mathbf{C}$. To emphasize this domination, from now on, we will write the strong stable direction in the first coordinate and the strong unstable in the last one. That is if $[0,1]^n = [0,1]^s + [0,1]^u$, where s is the dimension of the stable direction of the horseshoe and u is the dimension of the unstable one, we write

$$\mathbf{C} = [0,1]^s \times [-1,1]^2 \times [0,1]^u.$$

Consider cone fields $\mathfrak{C}^s, \mathfrak{C}^{uu}, \mathfrak{C}^u, \mathfrak{C}^1$ and $\mathfrak{C}^2$ defined as follows. Given $x \in \mathbf{C}$ and the decomposition of the tangent bundle $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$ over x, we consider (for simplicity the dependence on x is omitted)

$$\begin{aligned}
\mathfrak{C}^s &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^1, v^2, v^u)\| < \|v^s\|\},\\
\mathfrak{C}^{uu} &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^s, v^1, v^2)\| < \|v^u\|\},\\
\mathfrak{C}^u &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|v^s\| < \|(v^1, v^2, v^u)\|\},\\
\mathfrak{C}^1 &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^2, v^u)\| < \|(v^s, v^1)\|\},\\
\mathfrak{C}^2 &= \{(v^s, v^1, v^2, v^u) \in \mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}, \|(v^s, v^1)\| < \|(v^2, v^u)\|\}.
\end{aligned}$$

Due to the domination[1] of $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$ we have that the cones $\mathfrak{C}^s, \mathfrak{C}^{uu}, \mathfrak{C}^u, \mathfrak{C}^1$ and $\mathfrak{C}^2$ are invariant under the derivative $DF^{\pm 1}$. More precisely, the following inclusions hold for all $x \in \mathbf{C}$ with $F^{\pm 1}(x) \in \mathbf{C}$:

$$\begin{aligned}
&DF^{-1}_{F(x)}\, \mathfrak{C}^s\big(F(x)\big) \subset \mathfrak{C}^s(x),\\
&DF_x\, \mathfrak{C}^u(x) \subset \mathfrak{C}^u\big(F(x)\big),\\
&DF_x\, \mathfrak{C}^{uu}(x) \subset \mathfrak{C}^{uu}\big(F(x)\big),\\
&DF^{-1}_{F(x)}\, \mathfrak{C}^1\big(F(x)\big) \subset \mathfrak{C}^1(x),\\
&DF_x\, \mathfrak{C}^2(x) \subset \mathfrak{C}^2\big(F(x)\big).
\end{aligned}$$

Similarly as in the previous section, using these cones we now define *almost vertical disks, strips, and blocks* to the right of $W^s(Q;F)$. For simplicity we use the same notation as in the previous section for them. First, we define *top* and the *base* of the cube by:

$$\mathbf{C}_{\mathrm{top}} \stackrel{\mathrm{def}}{=} [0,1]^s \times [-1,1]^2 \times \{1^u\} \quad \text{and} \quad \mathbf{C}_{\mathrm{base}} \stackrel{\mathrm{def}}{=} [0,1]^s \times [-1,1]^2 \times \{0^u\}.$$

Analogously, the t and s-*top*, and the t and s-*base* of the cube are defined as follows:

$$\begin{aligned}
\mathbf{C}^t_{\mathrm{top}} &\stackrel{\mathrm{def}}{=} [0,1]^s \times \{1\} \times [-1,1] \times [0,1]^u,\\
\mathbf{C}^t_{\mathrm{base}} &\stackrel{\mathrm{def}}{=} [0,1]^s \times \{-1\} \times [-1,1] \times [0,1]^u,\\
\mathbf{C}^s_{\mathrm{top}} &\stackrel{\mathrm{def}}{=} [0,1]^s \times [-1,1] \times \{1\} \times [0,1]^u,\\
\mathbf{C}^s_{\mathrm{base}} &\stackrel{\mathrm{def}}{=} [0,1]^s \times [-1,1] \times \{-1\} \times [0,1]^u.
\end{aligned}$$

[1] As in Section 2.1, the splitting with four bundles $\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}$ is dominated if the bundles $\mathbb{X} \oplus (\mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y})$, $(\mathbb{X} \oplus \mathbb{T}) \oplus (\mathbb{S} \oplus \mathbb{Y})$ and $(\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S}) \oplus \mathbb{Y}$ are all dominated.

In what follows we consider disks and strips contained in $\mathbf{C}$.

- An *almost vertical disk* ς *to the right of* $W^s_{loc}(Q;F)$ is a disk (of dimension u) tangent to $\mathfrak{C}^{uu}$, that is, $T_x\varsigma \subset \mathfrak{C}^{uu}(x)$ for all $x \in \varsigma$, and such that ς intersects the top and the base of $\mathbf{C}$ (note that by "transversality" these intersections consist of just a point).

- An *almost t-vertical strip to the right of* $W^s_{loc}(Q;F)$ is a $(u+1)$-dimensional disk tangent to $\mathfrak{C}^1$ and foliated by almost vertical segments.
- An *almost s-vertical strip to the right of* $W^s_{loc}(Q;F)$ is a $(u+1)$-dimensional disk tangent to $\mathfrak{C}^2$ and foliated by almost vertical segments.
- An *almost vertical block* B *to the right of* $W^s_{loc}(Q;F)$ is a $(u+2)$-dimensional disk tangent to unstable cone field $\mathfrak{C}^u$, foliated by almost t-vertical strips and by almost s-vertical strips.

Given an almost t-vertical (resp. s-vertical) strip Δ^t (resp. Δ^s), we define the *t-width* of Δ^t (resp. *s-width* of Δ^s) as the infimum of the lengths of the curves α contained in Δ^t (resp. Δ^s) joining $\mathbf{C}^t_{\text{top}}$ and $\mathbf{C}^t_{\text{base}}$ (resp. $\mathbf{C}^s_{\text{top}}$ and $\mathbf{C}^s_{\text{base}}$) and tangent to $\mathfrak{C}^1$ (resp. $\mathfrak{C}^2$).

Given an almost vertical block B (which is foliated by t-vertical and s-vertical strips), we define the *area of the base of a block* as the product of the infimum of the t-width of t-vertical strips in B and the infimum of the s-width of s-vertical strips in B.

Next result is the version of Proposition 5.1 for almost vertical blocks:

Lemma 5.4. *The stable manifold of Q intersects every almost vertical block* B *to the right of* $W^s_{loc}(Q;F)$.

Proof. To show this result we argue as in the proof of Proposition 5.1. Let B be an almost vertical block to the right of $W^s_{loc}(Q;F)$. Due to the domination and the forward invariance of the unstable cones $\mathfrak{C}^u$, we have that the area of the base of B exponentially increases under forward iterations of F (in fact, this area is increasing in the rate $\lambda\,\beta > 1$). Then arguing as in Proposition 5.1, we have that $W^s(Q;F)$ intersects B. This ends the sketch of proof of the lemma. □

5.2.2 Robustness

In previous section we got invariant cone fields under the derivative $DF^{\pm 1}$ associated to the sum of the bundles of the dominated splitting

$$\mathbb{X} \oplus \mathbb{T} \oplus \mathbb{S} \oplus \mathbb{Y}.$$

We note that this invariance is an open property, thus the cones are also $DG^{\pm 1}$ invariant for every map G which is C^1-close to F.

Observe also that the maximal invariant set Γ of F is contained in the interior of the cube $\mathbf{C}$. Thus the same holds for the maximal invariant set of G in $\mathbf{C}$ that we denote by Γ_G and call it the continuation of Γ. Note also that the continuation Q_G of the point Q for G is well defined. Thus we have analogous definitions for almost vertical disks, t-strips, s-strips and blocks to the right of $W^s_{loc}(Q_G; G)$. We also have that every map G close to F are expansions in the central coordinate, then the expansion properties (for G) are satisfied for these sets. Therefore Lemma 5.4 holds for every map G sufficiently close to F. Thus we have proved the following:

Proposition 5.5. *For every map G C^1-close enough to F there are defined the continuation Q_G of Q and almost vertical disks, t-strips, s-strips and blocks to the right of $W^s_{loc}(Q_G; G)$. Then the stable manifold of Q_G intersects every almost vertical block* B *to the right of $W^s_{loc}(Q_G; G)$.*