6
Robust cycles from blenders

In this section, using blenders, we prove Theorem 1.4 (strong homoclinic intersections of partially hyperbolic periodic points yield robust cycles) and the stabilization of $(\mathbb{C}, \mathbb{C})$-cycles.

6.1
Strong homoclinic intersections yield robust cycles

Let f be diffeomorphism with a (non-hyperbolic) periodic point Z of period $\pi(Z)$ which has bidimensional central direction and a strong homoclinic intersection point R, that is, $R \in W^{ss}(\mathcal{O}(Z); f) \cap W^{uu}(\mathcal{O}(Z); f)$ and $R \notin \mathcal{O}(Z)$.

The first part of the proof is make some small perturbations on the diffeomorphism f to have a linear dynamics in the neighborhood of the periodic point Z and its strong homoclinic intersection R. Let us go into the details.

After a small perturbation of f, we can assume that there are a large (even) number $n > 0$ and a small neighborhood $\mathcal{U}_Z$ of Z such that in local coordinates $[-1,1]^s \times [-1,1]^2 \times [-1,1]^u$ around Z, the restriction of $(f^{\pi(Z)})^n \stackrel{\text{def}}{=} f^{n_0}$ to the neighborhood $\mathcal{U}_Z$ is a linear map preserving the splitting $E^{ss} \oplus E^c \oplus E^{uu}$ (this part is similar to Claim 2.3), say

$$f^{n_0} = (f^s, \mathrm{Id}, f^u) \colon \mathbb{R}^s \times \mathbb{R}^2 \times \mathbb{R}^u \to \mathbb{R}^s \times \mathbb{R}^2 \times \mathbb{R}^u,$$

where f^s and f^u are contracting and expanding linear maps and Id is the identity map.

Define $W^{ss}_{loc}(Z; f)$ and $W^{uu}_{loc}(Z; f)$ (the local strong stable and local strong unstable manifolds of Z) as the connected components of $W^{ss}(Z; f) \cap \mathcal{U}_Z$ and $W^{uu}(Z; f) \cap \mathcal{U}_Z$ containing P, respectively. In local coordinates we have

$$W^{ss}_{loc}(Z; f) = [-1,1]^s \times \{(0^2, 0^u)\} \quad \text{and} \quad W^{uu}_{loc}(Z; f) = \{(0^s, 0^2)\} \times [-1,1]^u.$$

Consider the first $m_1, m_2 > 0$ such that

$$B \stackrel{\text{def}}{=} f^{-m_1}(R) \in W^{uu}_{loc}(Z; f), \qquad A \stackrel{\text{def}}{=} f^{m_2}(R) \in W^{ss}_{loc}(Z; f).$$

Note that for $m = m_1+m_2$ we have $f^m(B) = A$. By shrinking the neighborhood $\mathcal{U}_Z$, we can assume that m is even and arbitrarily large. Using this fact we can perform a small perturbation of f along the segment of the orbit joining B and A such that the restriction of f^m to a small neighborhood $\mathcal{U}_B$ of B is a linear contraction in the s-coordinate, a linear expansion in u-coordinate, and the identity in the central coordinate (this is identical to Equation (6.1)).

Note that the hyperplane $\Pi^{su} = \mathbb{R}^s \times \{0^2\} \times \mathbb{R}^u$ is f^{n_0}-invariant in $\mathcal{U}_Z$ and f^m-invariant in $\mathcal{U}_B$. Then there is $\bar{n} = m\, n_0$ such that $f^{\bar{n}}$ restricted to the hyperplane Π^{su} has a Smale linear horseshoe containing the points Z, A and B, see Figure 6.1.

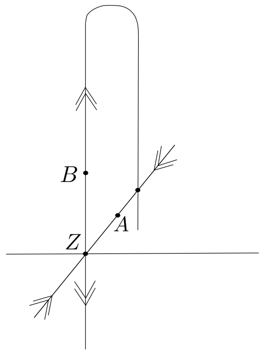

Figure 6.1: Points Z, A and B

Since this horseshoe has many strong homoclinic intersections there are points[1]

$$A_1, \ldots, A_4 \in W^{ss}_{loc}(Z; f) \quad \text{and} \quad B_1, \ldots, B_4 \in W^{uu}_{loc}(Z; f)$$

such that $f^{n_i}(B_i) = A_i$, for some $n_i > \bar{n}$, $i = 1, \ldots, 4$.

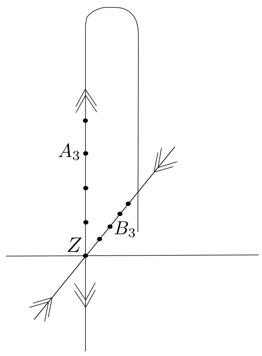

Figure 6.2: Points $A_1, \ldots, A_4$ and $B_1, \ldots, B_4$

[1]Note that the points $A_1, \ldots, A_4$ are in different orbits.

Consider the cube $\mathbf{C} = [-1,1]^s \times [-1,1]^2 \times [-1,1]^u$ and pairwise disjoint u-disks C_i^u in $[0,1]^u$ as follows. Take five "horizontal" sub-cubes of the form

$$C_i = [0,1]^s \times [-1,1]^2 \times C_i^u, \quad i = 0, 1, \ldots, 4,$$

such that

- C_0 contains the point Z and the four points A_i's;
- C_i contains B_i, for each $i = 1, \ldots, 4$;
- $f^{n_0}(C_0)$ and $f^{n_i}(C_i)$ are "vertical" sub-cubes of the form $C_i^s \times [-1,1]^2 \times [0,1]^u$, where C_i^s are s-disks pairwise disjoint contained in $[0,1]^s$.

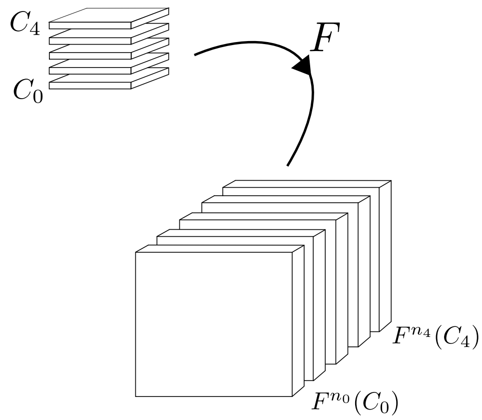

Figure 6.3: Sub-cubes

We now consider the local diffeomorphism $F\colon \cup_{i=0}^4 C_i \to \mathbf{C}$ given by

$$F\mid_{C_0} = f^{n_0} \qquad \text{and} \qquad F\mid_{C_i} = f^{n_i}, \qquad i = 1, \ldots, 4.$$

Note that using the coordinates above we have that $F = (F^s, \mathrm{Id}, F^u)$, where F^s is an affine contraction and F^u is an affine expansion.

Consider now a small perturbation F_ϕ of F of the form $F_\phi = (F^s, \phi, F^u)$, where $\phi\colon [-1,1]^2 \to [-1,1]^2$ is a map close to the identity having a repellor $(0,0)$ and an attractor (δ_1, δ_2), (small $\delta_1, \delta_2 > 0$) such that

$$W^u\big((0,0);\phi\big) \cap W^s\big((\delta_1,\delta_2);\phi\big) \neq \emptyset. \tag{6.1}$$

Note that the map F_ϕ has "two strong homoclinic intersections" (in this case associated to hyperbolic points). More precisely, we have the following:

- F_ϕ has two saddles Z and $Z_\delta = (0^s, \delta_1, \delta_2, 0^u)$ of s-indices s and $s+2$, respectively.

- $A_i = (a_i^s, 0^2, 0^u) \in W^{ss}(Z; F_\phi)$ and $B_i \in W^{uu}(Z; F_\phi)$ for $i = 1, \dots, 4$. Moreover, $F_\phi(B_i) = A_i$.
- There are points

$$\tilde{A}_i = (a_i^s, \delta_1, \delta_2, 0^u) \in W^{ss}(Z_\delta; F_\phi) \quad \text{and} \quad \tilde{B}_i \in W^{uu}(Z_\delta; F_\phi),$$

where $F_\phi(\tilde{B}_i) = \tilde{A}_i$, for $i = 1, \dots, 4$.

Note that by construction of the map F, to the perturbation F_ϕ of F is associated a perturbation f_ϕ of f such that $f_\phi^{n_i}(B_i) = A_i$ and $f_\phi^{n_i}(\tilde{B}_i) = \tilde{A}_i$.

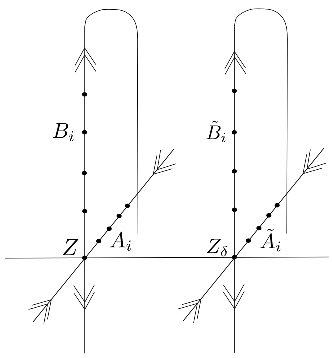

Figure 6.4: A_i's and $\tilde{A}_i$'s

We now consider a three-parameter perturbation $f_{t,s,r}$ of f_ϕ (small $t, s, r > 0$) defined as follows. Outside small neighborhoods of the union of the sets $f^{n_i - 1}(C_i)$ we have $f_{t,s,r} = f_\phi$. We modify f_ϕ in neighborhoods of $f^{n_i-1}(C_i)$ such that the restriction of $f_{t,s,r}^{n_i}$ to C_i is of the form:

- $f_{t,s,r}^{n_0}(x) = f_\phi^{n_0}(x)$, for $x \in C_0$;
- $f_{t,s,r}^{n_1}(x) = f_\phi^{n_1}(x) - (0^s, t, 0, 0^u)$, for $x \in C_1$;
- $f_{t,s,r}^{n_2}(x) = f_\phi^{n_2}(x) - (0^s, 0, s, 0^u)$, for $x \in C_2$;
- $f_{t,s,r}^{n_3}(x) = f_\phi^{n_3}(x) - (0^s, t, s, 0^u)$, for $x \in C_3$;
- $f_{t,s,r}^{n_4}(x) = f_\phi^{n_4}(x) - (0^s, r, r, 0^u)$, for $x \in C_4$.

In this way, the map $F_{t,s,r}$ associated to $f_{t,s,r}$ and defined in $\cup_{i=0}^4 C_i$ satisfies the following:

$$F_{t,s,r}(x) = \begin{cases} F_\phi(x), & \text{if} \quad x \in C_0, \\ F_\phi(x) - (0^s, t, 0, 0^u), & \text{if} \quad x \in C_1, \\ F_\phi(x) - (0^s, s, 0, 0^u), & \text{if} \quad x \in C_2, \\ F_\phi(x) - (0^s, t, s, 0^u), & \text{if} \quad x \in C_3, \\ F_\phi(x) - (0^s, r, r, 0^u), & \text{if} \quad x \in C_4. \end{cases} \tag{6.2}$$

Fix small $\delta > 0$ ($\delta < \delta_i$, $i = 1, 2$), consider the subset $\mathbf{C}_\delta$ of $\mathbf{C}$ of the form

$$\mathbf{C}_\delta = [0,1]^s \times [-\delta, \delta] \times [-\delta, \delta] \times [0,1]^u,$$

and let Γ be the maximal invariant set of $F_{t,s,r}$ in $(\cup_{i=0}^3 C_i) \cap \mathbf{C}_\delta$,

$$\Gamma \stackrel{\text{def}}{=} \bigcap_{k \in \mathbb{Z}} F_{t,s,r}^k \Big(\cup_{i=0}^3 C_i \cap \mathbf{C}_\delta \Big).$$

Observe that Γ is a hyperbolic set of s-index s containing the saddle Z.

Note that the local stable manifold of Z for $F_{t,s,r}$ in local coordinates is:

$$W^s_{\text{loc}}(Z; F_{t,s,r}) = [0,1]^s \times \{(0^2, 0^u)\}.$$

As in previous section, we define a *vertical disk* to the right of $W^s_{\text{loc}}(Z; F_{t,s,r})$ as u-disk (of dimension u) such that in local coordinates is of the form

$$\{(x^s, t, s)\} \times [0,1]^u, \quad x^s \in [0,1]^s,\ 0 < t \leq \delta,\ 0 < s \leq \delta.$$

Remark 6.1. *For* $\max\{\delta_1 - \delta, \delta_2 - \delta\} < r < \min\{\delta_1, \delta_2\}$, *we have that the*

$$\{(a_4^s, \delta_1 - r, \delta_2 - r)\} \times [0,1]^u$$

is a vertical disk to the right of $W^s_{\text{loc}}(Z; F_{t,s,r})$.

Similarly, we define a *vertical* $(u+2)$*-block* to the right of $W^s_{\text{loc}}(Z; F_{t,s,r})$ as a set of dimension $u+2$ such that in local coordinates is of the form

$$\{x^s\} \times [t_1, t_2] \times [s_1, s_2] \times [0,1]^u, \quad x^s \in [0,1]^s,\ 0 < t_1 < t_2 \leq \delta,\ 0 < s_1 < s_2 \leq \delta.$$

Next result is Proposition 5.1 in this context. Note that the point Z corresponds to $Q = (x_q, 0, 0, y_q)$ of Section 5.1, the map ϕ is a small perturbation of the identity, then the rate of expansion (in the central coordinate of $F_{t,s,r}$) is smaller than the rate of expansion of $(f^s)^{-1}$ and f^u, and the map $F_{t,s,r}$ corresponds to the map F in Equation (5.1).

Corollary 6.2. *The stable manifold of* Z *intersects every vertical* $(u+2)$*-block to the right of* $W^s_{loc}(Z; F_{t,s,r})$.

Fix small $r > 0$ such that $\max\{\delta_1 - \delta, \delta_2 - \delta\} < r < \min\{\delta_1, \delta_2\}$ as in Remark 6.1. Now we are ready to prove that $F_{t,s,r}$ has a robust heterodimensional cycle associated to Z_δ and Γ.

Claim 6.3. $W^s(Z_\delta; F_{t,s,r}) \cap W^u_{loc}(\Gamma; F_{t,s,r}) \neq \emptyset$.

Proof. From the intersection in Equation (6.1) one immediately gets that $W^s(Z_\delta; F_{t,s,r}) \pitchfork W^u(Z; F_{t,s,r}) \neq \emptyset$. The claim follows noting that $Z \in \Gamma$. □

Claim 6.4. $W^u(Z_\delta; F_{t,s,r}) \cap W^s_{loc}(\Gamma; F_{t,s,r}) \neq \emptyset$.

Proof. First, recall that in local coordinates we have that

$$\tilde{B}_4 \in W^u_{\rm loc}(Z_\delta; F_\phi) = \{(0^s, \delta_1, \delta_2)\} \times [0,1]^u.$$

Recall also that $\tilde{B}_4 \in C_4$ and $f^{n_4}_\phi(\tilde{B}_4) = \tilde{A}_4$. Also by construction of the map $F_{t,s,r}$ we have that

$$\sigma \stackrel{\rm def}{=} \{(a^s_4, \delta_1 - r, \delta_2 - r)\} \times [0,1]^u \subset W^u_{\rm loc}(Z_\delta; F_{t,s,r}).$$

By Remark 6.1, σ is a vertical disk to the right of $W^s_{\rm loc}(Z; F_{t,s,r})$. Since vertical blocks are foliated by vertical disks, applying Corollary 6.2 for "smaller and smaller" (nested) vertical blocks containing σ, we have that

$$\overline{W^s(Z; F_{t,s,r})} \cap \sigma \neq \emptyset \quad \text{then} \quad \overline{W^s(Z; F_{t,s,r})} \cap W^u_{\rm loc}(Z_\delta; F_{t,s,r}) \neq \emptyset.$$

Since $Z \in \Gamma$ we have that $W^s_{loc}(\Gamma; F_{t,s,r}) \cap W^u_{\rm loc}(Z_\delta; F_{t,s,r}) \neq \emptyset$, ending the proof of the claim. □

Note that in Claim 6.3, the dimensions of $W^s(Z_\delta; F_{t,s,r})$ and $W^u(\Gamma; F_{t,s,r})$ are $s+2$ and $u+2$, respectively, where $(s+2)+(u+2)$ is bigger than the dimension of the ambient, then the intersection is transverse and moreover, robust. By robustness of blenders, Claim 6.4 holds for every map C^1-close to $F_{t,s,r}$, proving that the map $F_{t,s,r}$ has a robust cycle. Since $F_{t,s,r}$ is associated to $f_{t,s,r}$, we have that $f_{t,s,r}$ also must have a robust heterodimensional cycle. The proof of the theorem is now complete. □

6.2 Stabilization of $(\mathbb{C}, \mathbb{C})$-cycles

In this section we conclude the proof of Theorem A checking that a $(\mathbb{C}, \mathbb{C})$-cycles can be stabilized after arbitrarily small perturbations. This is a consequence of Proposition 3.9 and results in previous sections.

Let f be a diffeomorphism having a $(\mathbb{C}, \mathbb{C})$-cycle associated with the saddles P and Q. By Proposition 3.9 there is a small perturbation g_1 of f having a cycle (associated with P and Q) with three heteroclinic points of type $\vec{PQ}$ (say X_1, X_2, X_3) with different orbits. We select one of these points, say X_1, and perform series of perturbations for constructing a diffeomorphism g_2 having a blender Γ_{g_2} involved in a robust cycle (see the proof of Proposition 3.1 and

Section 6.1). Note that the perturbation g_2 of g_1 does not involve the points X_2 and X_3 and therefore we can assume that besides the blender having a robust cycle this diffeomorphism also has a cycle associated to P and Q such that X_2 and X_3 are $\vec{PQ}$ heteroclinic points of it.

Recall that the blender persists after perturbations and it contains a saddle Z of s-index s. We now perform local perturbations of g_2 at the heteroclinic points X_2 and X_3 to get the following two lemmas whose prove we postpone.

Lemma 6.5. *There is a local perturbation of g_3 of g_2 at the point X_2 such that the saddles Q and Z^2 are homoclinically related[3] for g_3.*

Note that, since the perturbation is local, the diffeomorphism g_3 has a cycle associated to P and Q (and X_3 is a heteroclinic point of it).

Observe also that this implies that there is a transitive hyperbolic set $\tilde{\Gamma}_{g_3}$ containing the blender Γ_{g_3} (the continuation of Γ_{g_2} for g_3) and the saddle Q.

Let $\delta, \delta_1, \delta_2 > 0$ and $\mathbf{C}_\delta$ be a small cube such that the blender Γ_{g_3} is the maximal invariant set of g_3 in $\mathbf{C}_\delta$. In the local coordinates around the saddle $Z = (0^s, 1, 0, 0^u)$ one has

$$\mathbf{C}_\delta = [0,\delta]^s \times [1-\delta_1, 1+\delta_1] \times [-\delta_2, \delta_2] \times [0,\delta]^u. \tag{6.3}$$

In these local coodinates we also have $W^s_{loc}(Z; g) = [0,\delta]^s \times \{(1, 0, 0^u)\}$.

As in Section 5.2, we consider (small) invariant cone fields and define *almost vertical disks, t-strips, s-strips, and blocks* to the right of $W^s(Z; g_3)$.

Lemma 6.6. *There is a local perturbation g of g_3 at the point X_3 such that $W^u(P; g)$ contains an almost vertical disk σ to the right of $W^s(Z; g)$.*

By Lemma 6.6 and by the intersection property of the blender in Proposition 5.5 we have that $W^s(Z; g)$ intersects every almost vertical block containing σ. Since these blocks can taken arbitrarily small (i.e., their intersection is just the disk σ) it follows that $\overline{W^s(Z; g)} \cap W^u(P; g) \neq \emptyset$. Therefore

$$W^s_{loc}(\Gamma_g; g) \cap W^u(P; g) \neq \emptyset.$$

As the blender $\Gamma_g \subset \tilde{\Gamma}_g$,

$$W^s_{loc}(\tilde{\Gamma}_g; g) \cap W^u(P; g) \neq \emptyset.$$

[2]Note that Z is the continuation of the saddle of g_2. For simplicity, we omit the dependence on g_2 and just write Z.

[3]Recall Equation (3.5).

As $W^s(P,g) \cap W^u(Q,g) \neq \emptyset$ and $Q \in \tilde{\Gamma}_g$ (recall that Q and Z are homoclinic related) we have that g has a (robust) heterodimensional cycle associated with the saddle P and the transitive hyperbolic set $\tilde{\Gamma}_g \ni Q$.

To complete the proof of the stabilization of the cycle it remains to prove the two lemmas above.

Proof of the Lemma 6.5. We need to prove that

$$W^u(Q; g_3) \pitchfork W^s(Z; g_3) \neq \emptyset. \quad \text{and} \quad W^s(Q; g_3) \pitchfork W^u(Z; g_3) \neq \emptyset \tag{6.4}$$

Recall that by construction (Proposition 3.1),

$$W^u(Q; g_2) \pitchfork W^s(Z; g_2) \neq \emptyset \quad \text{and} \quad W^s(P; g_2) \pitchfork W^u(Z; g_2) \neq \emptyset. \tag{6.5}$$

Thus it remains to see that the second intersection holds (recall also that the first intersection in (6.4) persists after perturbations).

To get this intersection note that since $W^u(Z; g_2) \pitchfork W^s(P; g_2) \neq \emptyset$ by the λ-lemma we have that $W^u(Z; g_2)$ accumulates to $X_2 \in W^s(Q; g_2)$. Thus there is a perturbation g_3 of g_2 such that $W^u(Z; g_2) \pitchfork W^s(Q; g_2) \neq \emptyset$.

Let us observe that in our setting this intersection can be obtained doing a small perturbation at X_2 preserving the heteroclinic point X_3. This ends the proof of the lemma. □

Proof of Lemma 6.6. To prove the lemma we consider a small perturbation unfolding the cycle in the heteroclinic point X_3.

Recall the Definition 3.4 of the quotient family $(\mathcal{Q}^{\alpha,\beta}_{m,\ell,t=(t_1,t_2)})_{m,\ell,t}$ (we are omitting the superscripts $i, j = +, -, \pm$) in Section 3. Using this family to prove the assertion in the lemma it is enough to see that there are arbitrarily large $\ell \in \mathbb{N}$ and small $|t_0|$, and $\hat{\alpha}$ and $\hat{\beta}$ close to α and β such that

$$\mathcal{Q}^{\hat{\alpha},\hat{\beta}}_{0,\ell,t_0}(0^2) \in (1, 1+\delta_1) \times (0, \delta_2),$$

where δ_1, δ_2 are the (fixed) central sizes in the definition of the cube in (6.3) associated to the blender above. This is done as in Proposition 3.5. □