8
Symbolic blenders in the one-step setting

In this section, we prove the existence of symbolic blender-horseshoes in the one-step setting, Definition 1.11. We begin studying the relation between one-step skew product maps and their associated iterated function systems (IFS). To construct symbolic blender-horseshoes we use the covering property and the Hutchinson attractor of the associated IFS.

8.1
One-step skew products and IFS's

Given a one-step map $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ we denote by $\mathrm{IFS}(\phi_1, \dots, \phi_k)$, or shortly $\mathrm{IFS}(\Phi)$, the set of all compositions of the maps $\phi_1, \dots, \phi_k$ and we will refer to this as the associated *iterated function system*, or shortly IFS, of Φ.

The orbit of a point $x \in G$ for $\mathrm{IFS}(\phi_1, \dots, \phi_k)$, shortly the $\mathcal{G}_\Phi$-orbit of x, is the set

$$\mathrm{Orb}_\Phi(x) \stackrel{\text{def}}{=} \{\phi(x) : \phi \in \mathrm{IFS}(\phi_1, \dots, \phi_k)\}.$$

Next proposition shows that if (ϑ, p) is a fixed point of Φ then $\mathrm{Orb}_\Phi(p)$ is the projection into the fiber space of the strong unstable set of (ϑ, p). This result was proved in [19], since the proof is short, for completeness we include it here. A consequence of this proposition is that the density property (1.9) of the strong unstable set in Definition 1.11 of one-step symbolic blender-horseshoes is reduced to the density of the orbit of the "fixed point" p for the associated iteration function system.

Proposition 8.1. *[19, Proposition 2.16] Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ an one-step map and let (ϑ, p) be a fixed point of Φ. Then*

$$\mathscr{P}(W^{uu}(\vartheta, p); \Phi)) = \mathrm{Orb}_\Phi(p).$$

Proof. Since (ϑ, p) is a fixed point of Φ then

$$W^{uu}(\vartheta, p); \Phi) = \bigcup_{n=0}^{\infty} \Phi^n\big(W^{uu}_{loc}((\vartheta, p); \Phi)\big).$$

On the other hand, we have that for each $n \geq 1$

$$\Phi^n(W^{uu}_{loc}\big((\vartheta,p);\Phi\big) = \{(\tau^n(\zeta), \phi_{\tau^{n-1}(\zeta)} \circ \ldots \circ \phi_\zeta(p)) : \zeta \in W^u_{loc}(\vartheta;\tau)\}.$$

Since Φ is one-step, we have that $\phi_{\tau^i(\zeta)} = \phi_{\zeta_i}$ for all $i \geq 0$. Note that since $\zeta \in W^u_{loc}(\vartheta;\tau)$ we have that $\phi_\zeta(p) = \phi_\vartheta(p) = p$, then

$$\begin{aligned}\mathscr{P}\big(\Phi^n(W^{uu}_{loc}((\vartheta,p);\Phi)\big) &= \{\phi_{\tau^{n-1}(\zeta)} \circ \ldots \circ \phi_{\tau(\zeta)}(p) : \zeta \in W^u_{loc}(\vartheta;\tau)\} \\ &= \{\phi_{i_{n-1}} \circ \cdots \circ \phi_{i_1}(p) : i_j \in \{1,\ldots,k\},\ 1 \leq j < n\}.\end{aligned}$$

Hence the projection on the fiber space of the strong unstable set is $\mathrm{Orb}_\Phi(p)$, concluding the proof of the proposition. □

Recall that $\mathcal{Q}^0_{k,\lambda,\beta}(D)$ is the subset of $\mathcal{S} := \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}$ (Definition 1.7) consisting of one-step skew product maps. In this section, for simplicity, we denote $\mathcal{Q}$ in the place of $\mathcal{Q}^0_{k,\lambda,\beta}(D)$, with $\beta < 1$. A neighborhood $\mathcal{V}$ of Φ in $\mathcal{Q}$ is a neighborhood in the topology of $\mathcal{S}$ intersected with $\mathcal{Q}$. As the topology of $\mathcal{S}$ is induced by the distance in (1.6), noting that for every $\Psi \in \mathcal{Q}$ its Hölder constant is $C_\Psi = 0$, we have that if $\Psi = \tau \ltimes (\psi_1,\ldots,\psi_k)$ and $\Phi = \tau \ltimes (\phi_1,\ldots,\phi_k)$ are δ-close then

$$d_{\mathcal{Q}}(\Psi,\Phi) = \max_{i=1,\ldots,k} d_{C^0}(\psi_i|_D, \phi_i|_D) < \delta.$$

A periodic point (ϑ,p) of a skew product map $\Phi = \tau \ltimes \phi_\xi$ is *fiber-hyperbolic* for Φ if p is a hyperbolic point of ϕ^n_ϑ, where n is the period of (ϑ,p). We analogously define *fiber-attractors* and *fiber-repellors.*

Proposition 8.2. *Consider $\Phi \in \mathcal{Q}$, a non-empty open set $B \subset D$, and a fiber-hyperbolic fixed point $(\vartheta,p) \in \Sigma_k \times D$ of Φ. The following properties are equivalent:*

i) There is a neighborhood $\mathcal{V}$ of Φ in $\mathcal{Q}$ such that for every $\Psi \in \mathcal{V}$, one has that

$$W^{uu}\big((\vartheta,p_\Psi);\Psi\big) \cap \big(W^s_{loc}(\xi;\tau) \times U\big) \neq \emptyset,$$

for every $\xi \in \Sigma_k$ and every non-empty open subset U in B, where p_Ψ is the continuation of p.

ii) $B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}$ for every $\Psi \in \mathcal{Q}$ close to Φ.

Proof. From Proposition 8.1, for a fixed point (ϑ,p_Ψ) of Ψ, we have that $\mathscr{P}\big(W^{uu}((\vartheta,p_\Psi);\Psi)\big) = \mathrm{Orb}_\Psi(p_\Psi)$. Therefore, item (i) implies that $B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}$ for every $\Psi \in \mathcal{Q}$ close to Φ.

For the converse take the fixed point (ϑ, p_Ψ) of $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ close to $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ and fix $U \subset B$ and $\xi \in \Sigma_k$. By item (ii), there is $\psi_{i_n} \circ \dots \circ \psi_{i_1} \in \mathrm{IFS}(\psi_1, \dots \psi_k)$ such that the point $x = \psi_{i_n} \circ \dots \circ \psi_{i_1}(p_\Psi) \in U$. Take

$$\zeta = (\dots \vartheta_{-1}\vartheta_0, i_1, \dots, i_n; \xi_0, \xi_1, \dots),$$

and note that $(\zeta, x) \in W^s_{loc}(\xi; \tau) \times U$. It is enough to see that $(\zeta, x) \in W^{uu}\big((\vartheta, p_\Psi); \Phi\big)$. Since (ϑ, p_Ψ) is a fixed point of Ψ, by the choice of x we have that

$$\Psi^{-n-1}(\zeta, x) = \big((\dots, \vartheta_{-1}; \vartheta_0, i_1, \dots, i_n, \xi_0, \xi_1, \dots), p_\Psi\big) \in W^u_{loc}(\vartheta; \tau) \times \{p_\Psi\}.$$

Therefore

$$(\zeta, x) \in \Psi^{n+1}(W^u_{loc}(\vartheta; \tau) \times \{p_\Psi\}) = \Psi^{n+1}(W^{uu}_{loc}((\vartheta, p_\Psi); \Psi)) \subset W^{uu}((\vartheta, p_\Psi); \Psi).$$

Hence

$$(\zeta, x) \in W^{uu}\big((\vartheta, p_\Psi); \Psi\big) \cap \big(W^s_{loc}(\xi; \tau) \times U\big),$$

completing the proof of the proposition. □

Remark 8.3. *If (ϑ, p) in Proposition 8.2 is a fiber-attracting fixed point of $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ with B contained in the attracting region of p for* $\mathrm{IFS}(\phi_1, \dots, \phi_k)$*, then item* (ii) *is equivalent to*

$$B \subset \overline{\mathrm{Orb}_\Psi(x)}, \quad \textit{for every } x \in B \textit{ and every } \Psi \in \mathcal{Q} \textit{ close to } \Phi. \tag{8.1}$$

To see why this remark is so note first that Equation (8.1) implies item (ii) immediately (just take $x = p$). To see the converse take a perturbation $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ of Φ in $\mathcal{Q}$, a non-empty open set U in B, and $x \in B$. By hypotheses, there is $\psi \in \mathrm{IFS}(\psi_1, \dots, \psi_k)$ such that $\psi(p_\Psi) \in U$. As U is open there is a neighborhood V of p_Ψ such that $\psi(V) \subset U$. If Ψ is close enough to Φ then B is also in the attracting region of p_Ψ for $\psi_\vartheta = \psi_i$ where $i = \vartheta_0$. Thus there is $n \in \mathbb{N}$ such that $\psi_i^n(x) \in V$ and hence $\psi \circ \psi_i^n(x) \in U$, proving (8.1).

Motivated by (8.1), we give the following definition:

Definition 8.4 (Blending regions)**.** *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}$. A non-empty open set $B \subset M$ is called a* blending region *for Φ (or for the* $\mathrm{IFS}(\phi_1, \dots, \phi_k)$*) if for every $\Psi = \tau \ltimes (\psi_1, \dots, \psi_k)$ close to Φ it holds*

$$B \subset \overline{\mathrm{Orb}_\Psi(x)} \quad \textit{for all } x \in B.$$

Proposition 8.5. *Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}$ and consider a blending region $B \subset D$ of Φ. Suppose that there are a hyperbolic fixed point $p \in D$ of some*

ϕ_i and a map $\phi \in \mathrm{IFS}(\phi_1,\dots,\phi_k)$ with $\phi(p) \in B$. Then the maximal invariant set of Φ in $\Sigma_k \times \overline{D}$ is a one-step symbolic blender-horseshoe.

Proof. By Proposition 8.2, it is enough to see that $B \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}$, for every $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k)$ close to Φ, where p_Ψ the continuation of p for Ψ. By hypothesis, there are $i_n,\dots,i_1$ such that $\phi_{i_n} \circ \cdots \circ \phi_{i_1}(p) \in B$. Since B is an open set, if $\Psi = \tau \ltimes (\psi_1,\dots,\psi_k)$ is close enough to Φ then $\psi_{i_n} \circ \ldots \circ \psi_{i_1}(p_\Psi) \in B$. Since B is a blending region for $\mathrm{IFS}(\phi_1,\dots,\phi_k)$ it follows that

$$B \subset \overline{\mathrm{Orb}_\Psi(\psi_{i_n} \circ \cdots \circ \psi_{i_1}(p_\Psi))} \subset \overline{\mathrm{Orb}_\Psi(p_\Psi)}.$$

This concludes the proof of the proposition. □

8.2
Blending regions for contracting IFS: The Hutchinson attractor

In this section, we will prove that the covering property implies the existence of one-step symbolic blender-horseshoes. Associated to a one-step map $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{Q} \subset \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}$, with $\beta < 1$, or to the contracting $\mathrm{IFS}(\phi_1,\dots,\phi_k)$, the *Hutchinson's operator* is defined by

$$\mathcal{G}_\Phi \colon \mathcal{K}(\overline{D}) \to \mathcal{K}(\overline{D}), \qquad \mathcal{G}_\Phi(A) \stackrel{\text{def}}{=} \phi_1(A) \cup \ldots \cup \phi_k(A), \tag{8.2}$$

where $\mathcal{K}(\overline{D})$ denotes the set of compact subsets of $\overline{D}$ and $A \in \mathcal{K}(\overline{D})$.

Given a one-step map $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k)$, we also define $\mathrm{Per}(\mathrm{IFS}(\Phi))$ as the projection of $\mathscr{P}(\mathrm{Per}(\Phi))$ in the fiber space, that is, the set of fixed points of the maps in $\mathrm{IFS}(\phi_1,\dots,\phi_k)$.

Since the maps ϕ_i are contractions, the map $\mathcal{G}_\Phi$ is also a contraction. This fact leads to the following result:

Proposition 8.6 ([25, 16])**.** *Let $\Phi = \tau \ltimes (\phi_1,\dots,\phi_k) \in \mathcal{Q}$ Then there exists a unique compact set $K_{\mathcal{G}_\Phi} \in \mathcal{K}(\overline{D})$ such that*

$$K_{\mathcal{G}_\Phi} = \mathcal{G}_\Phi(K_{\mathcal{G}_\Phi}) = \overline{\mathrm{Per}(\mathrm{IFS}(\Phi)) \cap D} = K_\Phi.$$

Moreover, the set $K_{\mathcal{G}_\Phi}$ depends continuously (in the set $\mathcal{Q}$) on the map Φ and is the global attractor of $\mathcal{G}_\Phi$, that is, for every $A \in \mathcal{K}(\overline{D})$ it holds $\lim_{m\to\infty} d_H\big(\mathcal{G}^m_\Phi(A), K_{\mathcal{G}_\Phi}\big) = 0$.

We call the compact set $K_{\mathcal{G}_\Phi}$ (in the sequel denoted by K_Φ) the *Hutchinson's attractor* of the contracting one-step map Φ or of its associated $\mathrm{IFS}(\Phi)$.

Let us recall that given $x \in D$ its orbit is defined by

$$\mathrm{Orb}_\Phi(x) = \{\phi(x) \colon \phi \in \mathrm{IFS}(\Phi)\} = \{\phi_{i_n} \circ \cdots \circ \phi_{i_1}(x) \colon n \le 1,\, i_j \in \{1,\dots,k\}\}.$$

By Proposition 8.6, we have that $\mathcal{G}_\Phi^m(x) \overset{m\to\infty}{\longrightarrow} K_\Phi$ for all $x \in D$ and thus

$$K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}. \tag{8.3}$$

We now have the following consequences of Proposition 8.6:

Corollary 8.7. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}$ and let K_Φ be its Hutchinson's attractor.*

i) For every $A \in \mathcal{K}(\overline{D})$ with $A \subset \mathcal{G}_\Phi(A)$ one has that $A \subset K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in \overline{D}$.

ii) For every $p \in K_\Phi$ there is a sequence $(\sigma_n)_{n\in\mathbb{N}} \in \{1, \dots, k\}^\mathbb{N}$ such that

$$\phi_{\sigma_n}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1}(p) \in K_\Phi \quad \text{for all } n \in \mathbb{N}.$$

iii) For each open set V such that $V \cap K_\Phi \neq \emptyset$ there exist $n \in \mathbb{N}$ and $(i_1, \dots, i_n) \in \{1, \dots, k\}^n$ such that $\phi_{i_n} \circ \cdots \circ \phi_{i_1}(K_\Phi) \subset V$.

Proof. To prove the first item note that, by hypothesis,

$$A \subset \mathcal{G}_\Phi(A) \subset \ldots \subset \mathcal{G}_\Phi^m(A),$$

for all $m \geq 1$. Since $\mathcal{G}_\Phi^m(A) \to K_\Phi$ this implies that $A \subset K_\Phi$. Since, by (8.3), one has that $K_\Phi \subset \overline{\mathrm{Orb}_\Phi(x)}$ for all $x \in \overline{D}$, this proves (i).

To prove item (ii) recall that by Proposition 8.6 one has that $K_\Phi = \phi_1(K_\Phi) \cup \ldots \cup \phi_k(K_\Phi)$. Thus given any $p \in K_\Phi$ there exits $\sigma_1 \in \{1, \dots, k\}$ such that $\phi_{\sigma_1}^{-1}(p) \in K_\Phi$. Arguing inductively, we get a sequence $(\sigma_n)_{n\in\mathbb{N}}$ such that $\phi_{\sigma_n}^{-1} \circ \cdots \circ \phi_{\sigma_1}^{-1}(p) \in K_\Phi$ for all $n \in \mathbb{N}$.

To prove item (iii), fix an open set V with $V \cap K_\Phi \neq \emptyset$. Consider $i \in \{1, \dots, k\}$ and the fixed point s of ϕ_i. By the first item we have $K_\Phi \subset \overline{\mathrm{Orb}_\Phi(s)}$. Hence, there are $m \in \mathbb{N}$ and $(\sigma_1, \dots, \sigma_m) \in \{1, \dots, k\}^m$ such that $\phi_{\sigma_m} \circ \cdots \circ \phi_{\sigma_1}(s) \in V$ and thus $\phi_{\sigma_1}^{-1} \circ \cdots \circ \phi_{\sigma_m}^{-1}(V)$ is a neighborhood of s. Since ϕ_i^{-1} is an expansion, the set $\phi_{\sigma_1}^{-1} \circ \cdots \circ \phi_{\sigma_m}^{-1}(V)$ contains the repelling point s of ϕ_i^{-1}, and K_Φ is bounded, there exists $\ell \in \mathbb{N}$ such that

$$K_\Phi \subset \phi_i^{-\ell} \circ \phi_{\sigma_1}^{-1} \circ \cdots \circ \phi_{\sigma_m}^{-1}(V).$$

Now it is enough to take $n = \ell + m$ and the sequence $(i, \overset{\ell}{\ldots}, i, \sigma_1, \dots, \sigma_m)$. This completes the proof of the corollary. □

For next result, recall that an open set B has the covering property for IFS$(\phi_1, \dots, \phi_k)$ or for $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k)$ if

$$\overline{B} \subset \phi_1(B) \cup \cdots \cup \phi_k(B).$$

Next result shows that an open set satisfying the covering property for a contractive IFS is a blending region.

Corollary 8.8. *Consider $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q}$. Let $B \subset \overline{D}$ be a non-empty bounded open set satisfying the covering property for Φ. Then for every $\Psi \in \mathcal{Q}$ close enough to Φ one has that $\overline{B} \subset K_\Psi \subset \overline{\mathrm{Orb}_\Psi(x)}$, for all $x \in \overline{D}$, where K_Ψ is the Hutchinson attractor of Ψ.*

The corollary above and Proposition 8.5 imply that covering property generates one-step symbolic blender-horseshoes (Definition 1.11).

Corollary 8.9. *Let $\Phi = \tau \ltimes (\phi_1, \dots, \phi_k) \in \mathcal{Q} \subset \mathcal{S}^{0,\alpha}_{k,\lambda,\beta}$ with $\beta < 1$ and let $B \subset \overline{D}$ be a non-empty open set satisfying the covering property for Φ. Then the maximal invariant set of Φ is a one-step symbolic blender-horseshoe.*

Proof of Corollary 8.8. Recalling (8.2), if the skew product map $\Psi \in \mathcal{Q}$ is close to Φ then $d_H(\mathcal{G}_\Psi(\overline{B}), \mathcal{G}_\Phi(\overline{B}))$ is small. From this proximity and since $\mathcal{G}_\Phi(B) = \phi_1(B) \cup \cdots \cup \phi_k(B)$ is open, one has that $\overline{B} \subset \mathcal{G}_\Psi(B) \subset \mathcal{G}_\Psi(\overline{B})$. Inductively, we get

$$\overline{B} \subset \mathcal{G}^m_\Psi(\overline{B}), \quad \text{for all } m \geq 0.$$

Since the Hutchinson attractor K_Ψ of Ψ is closed and $d_H(\mathcal{G}^m_\Psi(\overline{B}), K_\Psi) \to 0$ we get $\overline{B} \subset K_\Psi \subset \overline{\mathrm{Orb}_\Psi(x)}$ for all $x \in \overline{D}$. □