Bibliography

[1] BARRIENTOS, P. Blenders and non-hyperbolic dynamics arising in generic unfoldings of nilpotent singularities (ph.d. thesis), 2012.

[2] BONATTI, C. Survey: Towards a global view of dynamical systems, for the C^1-topology. **Ergodic Theory Dynam. Systems**, v.31, n.4, p. 959–993, 2011.

[3] BONATTI, C.; CROVISIER, S. Récurrence et généricité. **Invent. Math.**, v.158, n.1, p. 33–104, 2004.

[4] BONATTI, C.; DÍAZ, L. J. Persistent nonhyperbolic transitive diffeomorphisms. **Ann. of Math. (2)**, v.143, n.2, p. 357–396, 1996.

[5] BONATTI, C.; DÍAZ, L. J. Robust heterodimensional cycles and C1-generic dynamics. **J. Inst. Math. Jussieu**, v.7, n.3, p. 469–525, 2008.

[6] BONATTI, C.; DÍAZ, L. J. Abundance of C^1-robust homoclinic tangencies. **Trans. Amer. Math. Soc.**, v.364, n.10, p. 5111–5148, 2012.

[7] BONATTI, C.; DÍAZ, L. J. Fragile cycles. **J. Differential Equations**, v.252, n.7, p. 4176–4199, 2012.

[8] BONATTI, C.; DÍAZ, L. J. ; KIRIKI, S. Stabilization of heterodimensional cycles. **Nonlinearity**, v.25, n.4, p. 931–960, 2012.

[9] BONATTI, C.; DÍAZ, L. J. ; VIANA, M. Discontinuity of the Hausdorff dimension of hyperbolic sets. **C. R. Acad. Sci. Paris Sér. I Math.**, v.320, n.6, p. 713–718, 1995.

[10] BROOMHEAD, D.; HADJILOUCAS, D. ; NICOL, M. Random and deterministic perturbation of a class of skew-product systems. **Dynam. Stability Systems**, v.14, n.2, p. 115–128, 1999.

[11] CROVISIER, S.; PUJALS, E. R. Essential hyperbolicity and homoclinic bifurcations: a dichotomy phenomenon/mechanism for diffeomorphisms. **ArXiv e-prints**, Nov. 2010.

[12] DÍAZ, L. J.; URES, R. Persistent homoclinic tangencies and the unfolding of cycles. **Ann. Inst. H. Poincaré Anal. Non Linéaire**, v.11, n.6, p. 643–659, 1994.

[13] GORODETSKI, A. S. Regularity of central leaves of partially hyperbolic sets and applications. **Izv. Ross. Akad. Nauk Ser. Mat.**, v.70, n.6, p. 19–44, 2006.

[14] GORODETSKI, A. S.; ILYASHENKO, Y. S. Some properties of skew products over a horseshoe and a solenoid. **Tr. Mat. Inst. Steklova**, v.231, n.Din. Sist., Avtom. i Beskon. Gruppy, p. 96–118, 2000.

[15] HIRSCH, M. W.; PUGH, C. C. ; SHUB, M. **Invariant manifolds**. Lecture Notes in Mathematics, Vol. 583. Berlin: Springer-Verlag, 1977. ii+149p.

[16] HUTCHINSON, J. E. Fractals and self-similarity. **Indiana Univ. Math. J.**, v.30, n.5, p. 713–747, 1981.

[17] ILYASHENKO, Y.; NEGUT, A. Hölder properties of perturbed skew products and fubini regained. **Nonlinearity**, v.25, n.8, p. 2377–2399, 2012.

[18] MOREIRA, C. G. There are no C^1-stable intersections of regular Cantor sets. **Acta Math.**, v.206, n.2, p. 311–323, 2011.

[19] NASSIRI, M.; PUJALS, E. R. Robust transitivity in hamiltonian dynamics. **Ann. Sci. Éc. Norm. Supér.**, v.45, n.2, 2012.

[20] NEWHOUSE, S. E. **Nondensity of axiom A(a) on S2**. In: global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968), p. 191–202. Providence, R.I.: Amer. Math. Soc., 1970.

[21] PALIS, J. A global view of dynamics and a conjecture on the denseness of finitude of attractors. **Astérisque**, n.261, p. xiii–xiv, 335–347, 2000. Géométrie complexe et systèmes dynamiques (Orsay, 1995).

[22] PUJALS, E. R.; SAMBARINO, M. Homoclinic tangencies and hyperbolicity for surface diffeomorphisms. **Ann. of Math. (2)**, v.151, n.3, p. 961–1023, 2000.

[23] RODRIGUEZ-HERTZ, F.; RODRIGUEZ-HERTZ, M. A.; TAHZIBI, A. ; URES, R. A criterion for ergodicity of non-uniformly hyperbolic diffeomorphisms. **Electron. Res. Announc. Math. Sci.**, v.14, p. 74–81, 2007.

[24] STARK, J. Regularity of invariant graphs for forced systems. **Ergodic Theory Dynam. Systems**, v.19, n.1, p. 155–199, 1999.

[25] WILLIAMS, R. F. Composition of contractions. **Bol. Soc. Brasil. Mat.**, v.2, n.2, p. 55–59, 1971.

A

In this appendix, recalling notation in Section 3.2, our goal is prove equation (3.6):

$$\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{0,\tilde{k},\tilde{t}}(0^2) = x^c_Q.$$

More precisely, we want to prove that given a quotient family $(\mathcal{Q}^{\alpha,\beta,i,j}_{m,\ell,t})$ there are natural numbers $k, m, \ell, \tilde{m}, \tilde{\ell}$, with $(m,\ell) \neq (\tilde{m},\tilde{\ell})$, and numbers $\hat{\alpha}$ and $\hat{\beta}$ close to α and β, and small t such that x^c_Q is a common fixed point of $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{m,\ell,t}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{\tilde{m},\tilde{\ell},t}$, and $\mathcal{Q}^{\hat{\alpha},\hat{\beta},i,j}_{0,\tilde{k},\tilde{t}}(0^2) = x^c_Q$.

Here we consider the case $(i,j) = (+,+)$, the other cases follows similarly. Without lost of generality, we can assume that in the local coordinates $x^c_P = (1,0)$ and $x^c_Q = (1,0)$. Recall that $\alpha = \rho\, e^{2\pi i \phi}$ and $\beta = \varrho\, e^{2\pi i \varphi}$, where $0 < \rho < 1 < \varrho$ and $\phi, \varphi \in [0,1)$.

Recall that for $t = (t_1, t_2)$ the bidimensional quotient map $\mathcal{Q}^{\alpha,\beta,+,+}_{m,\ell,t}(x,y)$ is of the form:

$$\varrho^\ell \begin{pmatrix} \cos \ell\, 2\pi\varphi & -\sin \ell\, 2\pi\varphi \\ \sin \ell\, 2\pi\varphi & \cos \ell\, 2\pi\varphi \end{pmatrix} \left[\rho^m \begin{pmatrix} \cos m\, 2\pi\phi & -\sin m\, 2\pi\phi \\ \sin m\, 2\pi\phi & \cos m\, 2\pi\phi \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right].$$

We choose $k > 0$ (we will explain this choice later) and consider

$$\tilde{t} = (\tilde{t}_1, \tilde{t}_2) = (\rho^{-k}\, \cos k\, 2\pi\phi, -\rho^{-k}\, \sin k\, 2\pi\phi).$$

We want to prove that

$$\mathcal{Q}^{\alpha,\beta,+,+}_{m,\ell,\tilde{t}} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Using trigonometric formulae and considering $x^c_Q = (1,0)$ and $\tilde{t}$ above this equality can be read as follows

$$\varrho^\ell \begin{pmatrix} \cos \ell 2\pi\varphi & -\sin \ell 2\pi\varphi \\ \sin \ell 2\pi\varphi & \cos \ell 2\pi\varphi \end{pmatrix} \left[\rho^m \begin{pmatrix} \cos m 2\pi\phi & -\sin m 2\pi\phi \\ \sin m 2\pi\phi & \cos m 2\pi\phi \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} \tilde{t}_1 \\ \tilde{t}_2 \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\varrho^{\ell}\begin{pmatrix}\cos\ell 2\pi\varphi & -\sin\ell 2\pi\varphi\\ \sin\ell 2\pi\varphi & \cos\ell 2\pi\varphi\end{pmatrix}\begin{pmatrix}\rho^m\cos m2\pi\phi+\varrho^{-k}\cos k2\pi\varphi\\ \rho^m\sin m2\pi\phi-\varrho^{-k}\sin k2\pi\varphi\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

$$\varrho^{\ell}\begin{pmatrix}\rho^m\cos(\ell 2\pi\varphi+m2\pi\phi)+\varrho^{-k}\cos(\ell 2\pi\varphi-k2\pi\varphi\\ \rho^m\sin(\ell 2\pi\varphi+m2\pi\phi)+\varrho^{-k}\sin(\ell 2\pi\varphi-k2\pi\varphi\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}$$

Which is equivalent to the following system:

$$\begin{cases}\varrho^{\ell}\rho^m\cos(\ell 2\pi\varphi+m2\pi\phi)+\varrho^{\ell-k}\cos(\ell 2\pi\varphi-k2\pi\varphi)=1\\ \varrho^{\ell}\rho^m\sin(\ell 2\pi\varphi+m2\pi\phi)+\varrho^{\ell-k}\sin(\ell 2\pi\varphi-k2\pi\varphi)=0.\end{cases}$$

Multiplying second equation by i and adding first equation we get

$$\begin{aligned}&\rho^m\varrho^{\ell}\,e^{i\,(\ell\,2\pi\varphi+m\,2\pi\phi)}+\varrho^{\ell-k}\,e^{i\,(\ell\,2\pi\varphi-k\,2\pi\varphi)}=1\\ &\varrho^{\ell}\,e^{i\,\ell\,2\pi\varphi}\left(\rho^m\,e^{i\,m\,2\pi\phi}+\varrho^{-k}\,e^{-i\,k\,2\pi\varphi}\right)=1\\ &\left(\rho\,e^{i\,2\pi\phi}\right)^m+\left(\varrho\,e^{i\,2\pi\varphi}\right)^{-k}=\left(\varrho\,e^{i\,2\pi\varphi}\right)^{-\ell}\\ &\big(\underbrace{\rho\,e^{i\,2\pi\phi}}_{\alpha}\big)^m=\big(\underbrace{\varrho\,e^{i\,2\pi\varphi}}_{\beta}\big)^{-\ell}-\big(\underbrace{\varrho\,e^{i\,2\pi\varphi}}_{\beta}\big)^{-k}.\end{aligned}$$

Therefore we need to find a pair of natural numbers m and ℓ satisfying the previous equality. The rest of the proof follows exactly as in the proof of Proposition 3.5.

B

In the appendix we see that Lema 3.10 also holds for $(\mathbb{C},\mathbb{R})$-cycle, see Remark 3.13.

We want to prove that given a quotient family $(\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},i,j}_{m,\ell,t})$ as Definition 4.5 for the case $(\mathbb{C},\mathbb{R})$-cycle, there are natural numbers $k, m, \ell, \tilde{m}, \tilde{\ell}$, with $(m,\ell) \neq (\tilde{m},\tilde{\ell})$, and numbers $\hat{\alpha}$ and $\hat{\beta}_{s+1}$ close to α and β_{s+1}, repectively, and small t such that x^c_Q is a common fixed point of $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{m,\ell,t}$ and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{\tilde{m},\tilde{\ell},t}$, and $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},i,j}_{0,\tilde{k},\tilde{t}}(0^2) = x^c_Q$.

The proof follows as Proposition 4.6. Let us remark the main differences.

Without lost of generality, we can assume that in local coordinates $x^c_P = (1,0)$ and $x^c_Q = (1,0)$. After an arbitrarily small perturbation we can assume that α has a rational argument ϕ. Fix $n > 0$ such that the map $C^n_\alpha = \rho^n\, R^n_\phi = \rho^n\,\mathrm{Id}$, where R_ϕ denotes the rotation of angle ϕ.

We consider the case $(i,j) = (+,+)$. Recalling Equations (4.5) and (4.4), for the case $(+,+)$ we have

$$\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},+,+}_{n,\ell,(t_1,t_2)}(x,y) = \big((\beta_{s+1})^\ell[\rho^n\, x + t_1],\, (\beta_{s+2})^\ell(\rho^n\, M_1\, M_2\, y + t_2]\big).$$

Let $t = (t_1,t_2) = (t_1,0)$ and consider a point $(x,0)$. Then

$$\mathcal{Q}^{\alpha,\beta_{s+1},\beta_{s+2},+,+}_{n,\ell,(t_1,0)}(x,0) = \big((\beta_{s+1})^\ell[\rho^n\, x + t_1],\, 0\big). \tag{B.1}$$

We will choose pairs $(m,\ell\, j)$ and $(\tilde{m},\ell\,(j+1))$ and a parameter t_1 such that (after a small perturbation) the point $x^c_Q = (1,0)$ is a fixed point for these compositions.

After an arbitrarily small perturbation of β_{s+1} we can assume that there are arbitrarily large (even) ℓ, j and (multiple of n) m such that

$$\rho^m = (\beta_{s+1})^{-\ell\, j} - (\beta_{s+1})^{-\ell\,(j+1)}.$$

Consider a $\tilde{m} >> m$ such that $\tilde{\rho}^{\tilde{m}}$ is close to zero for all $\tilde{\rho}$ close to ρ. Take

$k > 0$ (close to $\ell\,(j+1)$), $\hat{\beta}_{s+1}$ close to β_{s+1} and $\hat{\rho}$ close to ρ such that

$$(\hat{\rho})^m - (\hat{\rho})^{\tilde{m}} = (\hat{\beta}_{s+1})^{-\ell\,j} - (\hat{\beta}_{s+1})^{-\ell\,(j+1)}, \tag{B.2}$$

$$(\hat{\beta}_{s+1})^{-k} = -(\hat{\rho})^m + (\hat{\beta}_{s+1})^{-\ell\,j}. \tag{B.3}$$

Let

$$t_1 = (\hat{\beta}_{s+1})^{-k} \quad \text{and} \quad t_2 = 0. \tag{B.4}$$

With these choices we have the following claims that prove our assertion:

Claim B.1. *The point* $(1,0)$ *is fixed for* $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{m,\ell\,j,(t_1,0)}$ *and* $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{\tilde{m},\ell\,(j+1),(t_1,0)}$.

Proof. Note that by (B.1), we have that

$$\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{m,\ell\,j,(t_1,0)}(1,0) = \big((\hat{\beta}_{s+1})^{\ell\,j}[(\hat{\rho})^m + t_1], 0\big)$$

and the choice of t_1 in (B.4) proves the first assertion of the claim.

Similarly, for the second assertion it is enough to see that

$$(\hat{\beta}_{s+1})^{\ell\,(j+1)}\,[(\hat{\rho})^{\tilde{m}} + t_1] = 1.$$

Indeed, by the definition of t_1 in (B.4), (B.3) and (B.2), respectively, we have

$$\begin{aligned}
(\hat{\beta}_{s+1})^{\ell\,(j+1)}\,\big[(\hat{\rho})^{\tilde{m}} + t_1\big] &= (\hat{\beta}_{s+1})^{\ell\,(j+1)}\,\big[(\hat{\rho})^{\tilde{m}} + (\hat{\beta}_{s+1})^{-k}\big] \\
&= (\hat{\beta}_{s+1})^{\ell\,(j+1)}\,\big[(\hat{\rho})^{\tilde{m}} - (\hat{\rho})^m + (\hat{\beta}_{s+1})^{-\ell\,j}\,\big] \\
&= (\hat{\beta}_{s+1})^{\ell\,(j+1)}\,\big[(\hat{\rho})^{\tilde{m}} - (\hat{\rho})^{\tilde{m}} + (\hat{\beta}_{s+1})^{-\ell\,(j+1)}\,\big] = 1,
\end{aligned}$$

proving the claim. □

Claim B.2. $\mathcal{Q}^{\hat{\alpha},\hat{\beta}_{s+1},\beta_{s+2},+,+}_{0,k,(t_1,0)}(0,0) = (1,0)$.

Proof. By the choice of t_1 and Equation (B.2) we have that

$$t_1 = -(\hat{\rho})^m + (\hat{\beta}_{s+1})^{-\ell\,j} = -(\hat{\rho})^{\tilde{m}} + (\hat{\beta}_{s+1})^{-\ell\,(j+1)}.$$

Using Equation (B.1), note that by (B.3) and the choice of t_1 above we get

$$(\hat{\beta}_{s+1})^k[\hat{\rho}^0 + t_1] = (\hat{\beta}_{s+1})^k[(\hat{\beta}_{s+1})^{-k}] = 1$$

ending the proof of the claim. □