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Introduction

In this work we consider autonomous semi-linear elliptic equations

$$F(u) = -\Delta u - f(u) = g, \quad u|_{\partial\Omega} = 0,$$

for different classes of nonlinearities f and domains $\Omega \in \mathbb{R}^n$, taken to be bounded, open, connected subsets of \mathbb{R}^n with Lipschitz boundary $\partial\Omega$.

There are natural identifications between the theoretical tools and the numerical methods used in the study of these equations. After appropriate function spaces are chosen, local behavior at regular points concerns both the inverse function theorem and Newton's algorithm to invert a point given a good initial approximation. For nonlocal issues, homotopy methods like degree theory go along well with continuation methods. The celebrated mountain pass lemma ([19], [10]) is the starting point of an algorithm presented in [7]. More recently, ideas used in computer assisted proofs were combined with the topological toolbox with striking effect by Breuer, McKenna and Plum [6].

Surely there are difficulties: as one proceeds to obtain preimages of a path γ by a continuation method, it may happen that γ trespasses the image of the critical set of F , and one is faced with the possible creation or annihilation of preimages. This leads to consider singularity/bifurcation theory and their numerical counterparts. These are intrinsically nonlinear difficulties and hard problems abound: how to count solutions or at least how to spot a few, how to find reasonable starting points. As stated in [17], computer assisted arguments require good approximations for the eventual validation of a solutions.

On the one hand, numerics develop our intuition of elliptic equations by collecting expressive examples. On the other, the subject is sufficiently mature that algorithms should stand side by side with theoretical information. The situation may be compared to the simpler study of functions of one variable in a basic calculus course. Some functions, like parabolas, may be handled without substantial computational effort. But the range of our understanding increases once we start drawing graphs, which are obtained by following standard procedures.

The scope of the algorithm presented in this work is defined in terms of

the allowed nonlinearities. For simplicity, we only treat the autonomous case. Index the eigenvalues λ_i of the Dirichlet Laplacian in nondecreasing order,

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots$$

We make two hypothesis on the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$.

1. $\overline{f'(\mathbb{R})} = \overline{(a, b)}$, where the extreme points a, b are not eigenvalues (the *non-resonant case*; here a may be $-\infty$);
2. $\overline{f'(\mathbb{R})}$ contains a finite set of ordered eigenvalues λ_i of the Dirichlet Laplacian, indexed by a set J which is *complete*, in the sense that it labels all the eigenvalues (including multiplicities) contained in $\overline{f'(\mathbb{R})}$.

1.0.1

Global Lyapunov-Schmidt decompositions and fibers

The history of the theoretical (and some computational) aspects of semi-linear elliptic theory is very well described in [6]. A good introduction to the application of computer assisted proofs to nonlinear elliptic equations is [17]. Here we emphasize the techniques we intend to convert into algorithms.

In more geometric terms, we want to compute the preimages of a point g for the nonlinear operator $F : H_0^1(\Omega) \rightarrow H^{-1}(\Omega)$ between Sobolev spaces given by $F(u) = -\Delta u - f(u)$: in other words, we want to solve $F(u) = g$.

Hammerstein [12] showed that if $\overline{f'(\mathbb{R})}$ lies below the smallest eigenvalue λ_1 and is nonnegative, then F is a diffeomorphism. Dolph [11] extended the result for the case when $\overline{f'(\mathbb{R})}$ does not contain any eigenvalue λ_i . This is clearly a sound starting point for numerics: to solve $F(u) = g$, take any point u_0 with image $F(u_0) = g_0$ and proceed to invert points along the segment joining g_0 and g by, say, Newton's method. The process will only require (standard) algorithms for inverting $v \mapsto -\Delta v - f'(u)v$ with Dirichlet boundary conditions.

The first example for which F admits a multiplicity of preimages was identified by Ambrosetti and Prodi [1]. Their beautiful result, immediately amplified by Manes and Micheletti [15], essentially states that if f is convex and $\overline{f'(\mathbb{R})}$ only contains the eigenvalue λ_1 , then g can only have 0, 1 or 2 preimages. Their arguments made a subtle use of local theory (the fact that all critical points of F are fold points) and global properties (mostly, the injectivity of F restricted to its critical set), yielding the counting of preimages.

Later, Berger and Podolak [4] came up with a geometric description of the Ambrosetti-Prodi operator F . Split domain and counter-domain as an orthogonal sum of the *vertical* subspace V spanned by the first eigenfunction

φ_1 and its orthogonal complement V^\perp . Rather surprisingly, this gives rise to a *global* Lyapunov-Schmidt decomposition. More precisely, the projection of F on V^\perp may be interpreted as a parameterized set of diffeomorphisms from affine subspaces in $H_0^1(\Omega)$ parallel to V^\perp to $V^\perp \in H^{-1}(\Omega)$ and the projection on V leads to the bifurcation equation of the problem, $P_V F(u) = P_V g$.

Berger and Podolak went one step further: they considered the *fibers*, which are the preimages of vertical lines. For the Ambrosetti-Prodi scenario, they essentially showed that each fiber is a differentiable curve on which the restriction of F becomes $x \mapsto -x^2$, after global changes of variable in domain and target space. Moreover, fibers foliate the domain. Said differently, F is a global fold: (global) changes of variables in domain and counter-domain convert F into $(x, y) \mapsto (-x^2, y)$, where $y \in V^\perp$ parameterizes the set of fibers (or, equivalently, the set of vertical lines in the counter-domain) and $x \in V$ parameterizes each fiber. In a subsequent paper [18], Podolak considered fibers in the case f is convex and $\overline{f'(\mathbb{R})}$ only contains the eigenvalue λ_2 .

Fibers were used to show that the map $G(u(t)) = u'(t) + u^3(t) - u(t)$ is a global cusp between $C^1([0, 1])$ functions with periodic boundary conditions to $C([0, 1])$ (see [14]). In other words, after global changes of variables, G becomes $(x, y) \mapsto (x^3 - xy, y)$. In this case, the appropriate decomposition splits domain and counter-domain as direct sums of constant functions and functions of zero average. Restricted to each fibers, G looks like $x \mapsto x^3 - ax$ for some real number a which depends on the fiber. One is then left with showing that the fibers stack together in the right fashion.

1.0.2

A sketch of the algorithm

Such decompositions are opportunities for numerics. Again, in the Ambrosetti-Prodi case, let α_g be the fiber given by the inverse under F of the vertical line through g . Clearly, all the solutions of the equation $F(u) = g$ belong to α_g . Thus, one might solve $F(u) = g$ by first identifying α_g , and then searching for solutions along this well behaved curve. As we shall see (and is clear already in [4]), the space decompositions do not require the convexity of f . So, in principle, one might perform numerics for more general nonlinearities already when interacting only with λ_1 .

The general procedure goes as follows. The hypotheses on the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ allow for the required decompositions in domain and counter-domain: the vertical subspace V is spanned by the eigenvectors of the Dirichlet Laplacian labeled by indices in the complete set J , which describes which eigenvalues interact with the nonlinearity. Fibers, which are the inverse

images under F of the affine subspaces parallel to V , will have dimension k , the cardinality of J . In order to solve $F(u) = g$, first move along fibers so as to identify the fiber α_g which contains all the solutions of the equation. As we shall see, the equation obtained by projecting on V^\perp essentially states that there is a diffeomorphism between the set of fibers and the set of vertical affine subspaces. One task of this work is to show how this converts the search for α_g into a (stable) Newton's method which requires the inversion of an appropriate integro-differential operator in each iteration. Once $\alpha_g \simeq \mathbb{R}^k$ is found, the problem reduces to the search for roots of a (computable, robust) function from \mathbb{R}^k to \mathbb{R}^k , as we shall see.

The first step of the method is where the infinite dimensional computations are performed, or better, emulated by finite elements methods. This part is robust and globally stable: errors self-correct in the spirit of Newton-type iterations, and the linear operators which require inversion are both uniformly bounded and uniformly coercive. The second step is subtler. When J consists of one or two elements, we provide graphs representing the restriction of F to the appropriate fiber, and the visual hints may help the numerics. We do not consider here an algorithm for the general (finite dimensional) inversion problem. For the general case, one might use any software for the general n-dimensional root finder. At this point, we are considering extending the software available for the two dimensional case [13]¹, but the subject is barely beginning.

Insight on the numerics may be obtained also from the understanding of the discrete counterparts of such operators F . This is considered in [20], but we give no details here.

Section 2.2 is dedicated to the basic estimates ensuring that F is indeed a C^1 map. Then in Section 3.1, a few basic geometric properties are obtained from a general function $F : X \rightarrow Y$, where the Banach spaces X and Y are supposed to admit the decompositions which will be used in the global Lyapunov-Schmidt decomposition of Section 3.2. The algorithm is described first theoretically in Section 4.1, then in more technical terms (Section 4.2). We conclude the work with a few numerical examples, in which we consider nonlinearities interacting first with λ_1 , next with λ_2 and finally with both eigenvalues.

¹For this program, see <http://www.mat.puc-rio.br/hjbortol/2x2/>