2
Basic Theory

2.1
Notation

Throughout this text, Ω denotes an *open*, *simply connected*, *bounded* set of $\mathbb{R}^n$ (mostly $n = 1$ or $n = 2$). The boundary of our domain is denoted by $\partial\Omega$ and is assumed to be smooth, except at finitely many points, where it is at least *Lipschitz* . For technical results for this class of boundaries we refer to [16]. We define the bilinear forms

$$\langle u, v\rangle_0 = \int_\Omega uv, \quad \langle u, v\rangle_1 = \int_\Omega \nabla u \cdot \nabla v,$$

which give rise to the norms resp. seminorms

$$\|u\|_0 = \langle u, u\rangle_0^{1/2}, \quad |u|_1 = \langle u, u\rangle_1^{1/2}.$$

For more general exponents $1 \le p < \infty$,

$$|u|_{j,p}^p = \sum_{|\alpha|=j} \int_\Omega \|D^\alpha u\|^p, \quad \|u\|_{j,p}^p = \sum_{i\le j} |u|_{i,p}^p,$$

and

$$|u|_{j,\infty} = \sum_{|\alpha|=j} \text{ess sup}_\Omega \|D^\alpha u\|, \quad \|u\|_{j,\infty} = \sum_{i\le j} |u|_{i,\infty},$$

where for $\alpha = (\alpha_1, \ldots, \alpha_n)$ a multiindex, $D^\alpha u = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} u$. Whenever an operator F is said to be differentiable, it is understood to be in the Fréchet sense. The partial derivative of F with respect to, say, u is denoted as $\partial_u F$. A generic constant (not always the same one) will be denoted simply by C and, unless otherwise stated, is assumed to depend only upon the dimension of the space and the domain Ω.

2.2 Differentiability of Nemytskii Operators

Recall that a function $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a *Carathéodory* function if $s \mapsto f(x,s)$ is continuous for almost every x and $x \mapsto f(x,s)$ is measurable for every s. An operator of the form $u(x) \mapsto f(x, u(x))$, where f is Carathéodory, is referred to as a *Nemytskii* operator. For a discussion about their differentiability as maps from L^p to $L^{p'}$ ($\frac{1}{p} + \frac{1}{p'} = 1$) see, e.g. [2]. We are interested in *autonomous* operators induced by f, i.e. $x \mapsto f(u(x))$, where u lies in Sobolev spaces of higher order, and such that $f \in C^1$, with bounded derivative. Note that this assumption clearly implies the Carathéodory property and the inequality $|f(s)| \leq a + b|s|$ for some $a, b > 0$. Given the stronger assumptions, the Nemytskii operators we consider have nicer properties.

Proposition 1. *If $f : \mathbb{R} \to \mathbb{R}$ is a C^1-function with bounded derivative, then the Nemytskii operator $N_f : H^1(\Omega) \to L^2(\Omega)$ defined by $u \mapsto f(u)$ is continuously differentiable with derivative given by $N_f'(u)z = f'(u)z$. The operators $N_f : H^1(\Omega) \to H^{-1}(\Omega)$ and $N_f : H^2(\Omega) \to L^2(\Omega)$ together with their restrictions to $H_0^1(\Omega)$ and $H_0^2(\Omega)$ are continuously differentiable, with compact derivatives.*

Proof. The operator $N_f : H^1(\Omega) \to L^2(\Omega)$ is well defined: there exist $a, b > 0$ so that

$$\|f(u)\|_0 \leq \|a\|_0 + b\|u\|_0 \leq a|\Omega| + b\|u\|_1.$$

Write the superlinear remainder $e(h) := f(u+h) - f(u) - f'(u)h$ as $e(h) = \delta\, h$, where

$$\delta(x, h(x)) := \int_0^1 f'(u(x) + s\, h(x)) - f'(u(x))\, ds.$$

We have then $\|e(h)\|_0 \leq \|\delta\|_\infty \|h\|_0 \leq \|\delta\|_\infty \|h\|_1$. To establish differentiability it suffices to show that $\|\delta\|_\infty \to 0$ as $\|h\|_1 \to 0$, and that multiplication by $f'(u)$ is a bounded operator. Since $\|h\|_0 \leq \|h\|_1$, $h \to 0$ also in L^2. We assume then, switching to a subsequence if necessary, that $h \to 0$ pointwise a.e., so that the integrand in δ converges to zero pointwise a.e. ($f \in C^1$ and $|s| \leq 1$). From the bounded convergence theorem, $\|\delta\|_\infty \to 0$. Thus N_f is Fréchet differentiable. The boundedness of $z \mapsto f'(u)z$ follows from $\|f'(u)z\|_0 \leq \|f'\|_\infty \|z\|_0 \leq \|f'\|_\infty \|z\|_1$.

We now show the continuity of the derivative. For an arbitrary $u \in H^1(\Omega)$, we have to show that $\|N_f'(u+h) - N_f'(u)\| \to 0$ whenever $\|h\|_1 \to 0$. Suppose $v \in H^1(\Omega)$. Defining $g(h) = f'(u+h) - f'(u)$ and using Hölder's inequality with exponents $\frac{1}{p} + \frac{1}{p'} = 1$ we obtain

$$\|(f'(u+h) - f'(u))\, v\|_{0,2} = \|g(h)v\|_{0,2} \leq \|g(h)\|_{0,2p'} \|v\|_{0,2p}. \tag{2-1}$$

To estimate $\|v\|_{0,2p}$ we divide in cases.

If $n = 1$, we can choose $p = \infty$ ($p' = 1$), since by Morrey's inequality we have

$$\|v\|_{0,2p} = \|v\|_{\bar{\Omega},0,\infty} \leq C\|v\|_{1,2}.$$

If $n \geq 3$, we set $2p = q^* = \frac{2n}{n-2} > 2$ ($p' = n/2$), the Sobolev conjugate of $q = 2$. Gagliardo-Nirenberg-Sobolev inequality yields then

$$\|v\|_{0,2p} = \|v\|_{0,q^*} \leq C\|v\|_{1,q} = C\|v\|_{1,2}.$$

Finally, for $n = 2$, we apply again Gagliardo-Nirenberg-Sobolev, this time with $2p = q^* = 4$ ($p' = 2$), the Sobolev conjugate of $q = 4/3 < 2$. Since Ω is bounded, we can then write

$$\|v\|_{0,2p} = \|v\|_{0,q^*} \leq C\|v\|_{1,q} \leq C\left(1 + 2|\Omega|\right)^{\frac{1}{q}} \|v\|_{1,2},$$

where we split $\Omega = \{|v| < 1\} \cup \{|v| \geq 1\}$ in the integrals and used a scaling argument. In all cases, after taking the supremum over all unitary $v \in H^1(\Omega)$, equation (2-1) becomes

$$\|N_f(u+h) - N_f(u)\| \leq C\|g(h)\|_{0,2p'}.$$

The argument used previously allows us to assume $h \to 0$ (and also $g(h) \to 0$) pointwise a.e. Since in each case we have $1 \leq p' < \infty$, the bounded convergence theorem guarantees that $\|g(h)\|_{0,2p'} \to 0$.

The analogous statements for the other Nemytskii operators in the statement of the proposition follow from the natural compact inclusions among Sobolev spaces. □