3
The Problem

3.1
Some basic geometry

We present an abstract setup for a global Lyapunov-Schmidt decomposition for the nonlinear operator to be defined in the next section.

Let X and Y be Banach spaces which are split as direct sums of *horizontal* and *vertical* subspaces, $X = W_X \oplus V_X$ and $Y = W_Y \oplus V_Y$. Here W_X and W_Y are closed subspaces and $k = \dim V_X = \dim V_Y < \infty$. There are unique projections $P_X : X \to W_X$ and $P_Y : Y \to W_Y$ with kernels V_X and V_Y respectively, and complementary projections $Q_X : X \to V_X$ and $Q_Y = Y \to V_Y$ given by $Q_X = I - P_Y$ and $Q_Y = I - P_X$. Sets of the form $x + W_X$ (resp. $y + W_Y$) or $x+V_X$ (resp. $y+V_Y$) will be denoted by horizontal and vertical *affine subspaces*. The *height* of a horizontal affine subspace $v + W_X$ (resp. $v + W_Y$) is $v \in V_X$ (resp. $v \in V_Y$). In the definitions below, F is a C^1 operator from X to Y, not necessarily linear.

Definition 1. *A fiber through a point $x \in X$ is the set $F^{-1}(y + V_Y)$, where $y = F(x)$.*

That is, a fiber is the inverse image of a vertical line. Fibers were used in [4] and [18] to provide very geometric proofs of results of Ambrosetti-Prodi type. They were also considered in the study of first order periodic differential equations in [14].

Definition 2. *Given an arbitrary $v \in V_X$, we define the projected restriction operator $F_v : W_X \to W_Y$ by $F_v(w) = PF(v + w)$.*

The working hypothesis on F is very stringent: we assume that F is a C^1 map for which $F_v : W_X \to W_Y$ is a diffeomorphism for any v. Thus, horizontal affine subspaces are sent injectively by F to their images, which are graphs of functions from W_Y to V_Y. For brevity, we then say that F is *flat*. Clearly, the definition depends on the decompositions of X and Y, but we will not mention them in order to simplify notation. There is a global form of operators for which F_v is as above.

Proposition 2. *Let $F : X \to Y$ be flat. Then the function*

$$\Phi : \tilde{X} = W_Y \oplus V_X \to W_X \oplus V_X, \quad \Phi(z,v) = ((F_v)^{-1}(z), v)$$

is a C^1 diffeomorphism such that $\tilde{F} = F \circ \Phi : \tilde{X} \to Y$ becomes $\tilde{F}(z,v) = (z, \phi(z,v))$ for a C^1 function $\phi : \tilde{Y} \to V_Y$.

Proof. We denote by $\partial_w F_v$, $\partial_v F_v$ the partial derivatives of the map $(w,v) \mapsto PF(w,v)$. Analyzing the diagram below,

$$\begin{array}{ccc} (w,v) & \overset{F}{\longmapsto} & (F_v(w), Q_Y F(w,v)) \\ \xi \searrow\nwarrow \Phi & & \nearrow F\xi^{-1} =: \tilde{F} \\ & (F_v(w), v) & \end{array}$$

we see that $\Phi = \xi^{-1}$, $\phi = Q_Y F \xi^{-1}$. The function ξ is one-to-one and onto and its derivative, in block-matrix notation, is

$$\xi'(w,v) = \begin{bmatrix} \partial_w F_v(w) & \partial_v F_v(w) \\ 0 & I \end{bmatrix} = \begin{bmatrix} F_v'(w) & \partial_v F_v(w) \\ 0 & I \end{bmatrix}.$$

Applying the inverse function theorem (F_v' is invertible and thus also ξ'), we see that ξ is a global diffeomorphism. □

Not only do fibers stretch out indefinitely, but they do so in a smooth way.

Proposition 3. *Let $F : X \to Y$ be flat. Then each fiber α is a C^1 surface of dimension $k = \dim V_X$, which intersects each horizontal affine subspace exactly once, always transversally. The height map $x \mapsto Q_X x$ is a diffeomorphism between α and V_X.*

The fact that α and a horizontal affine subspace $x + W_X$ meet transversally at a point x means that X is a direct sum of the tangent space of α at x and W_X. According to the proposition, the horizontal subspace parametrizes (bijectively) the set of fibers, and the vertical subspace is a parametrization of each fiber. Also, horizontal affine subspaces are sent injectively by F to their images, which are graphs of functions from W_Y to V_Y. On the other hand, fibers are not taken injectively (nor subjectively!) to vertical subspaces necessarily. In particular, the given hypothesis are not enough to imply the properness of the map $F : X \to Y$.

Proof. We use the change of variables $\Phi(z,v) = (F_v^{-1}(z), v)$ defined in the previous proposition. This map, from the domain of $\tilde{F}$ to the domain of F, clearly takes each vertical affine subspace in $\tilde{X}$ to a fiber of F diffeomorphically and so that that heights are preserved. Every statement about fibers now follows from its analogous counterpart for vertical affine subspaces in $\tilde{X}$. □

□

We now consider the effect of flatness on the linearizations.

Corollary 1. *Let $F : X \to Y$ be flat. The Jacobian $F'(x) : X \to Y$ is a Fredholm operator of index zero at $x \in X$. The restriction of $F'(x)$ to W_X is an isomorphism between W_X and its (closed) range, which is transversal to V_Y. If x_c is a critical point of F contained in the fiber α, then $\mathrm{Ker}(F'(x_c)) \subset T_{x_c}\alpha$.*

Again, transversality here means that $Y = F'(x)W_X \oplus V_Y$.

Proof. By flatness, the derivative $P_Y F'(x) : W_X \to W_Y$ is a linear isomorphism, hence a Fredholm operator of index 0. Thus the map $T : W_X \oplus V_X \to W_Y \oplus V_Y$ given by $T(w, v) = (P_Y F'(x), 0)$ is also Fredholm of index zero. The same is true for $F'(x) : X \to H^{-1}(\Omega)$, since $F'(x) - T$ is the finite range operator $w + v \mapsto QDF(x)w + F'(x)V$.

Transversality of $F'(x)W_X$ and V_Y follows from the fact that $F'(x) : W_X \to F'(x)W_X$ must be injective, with closed range.

At a critical point $x_c \in \alpha$, use the transversality of the intersection of α and $(x_c + W_X)$ proved in the previous proposition to split $X = W_X \oplus T_{x_c}\alpha$. Now combine $Y = F'(x)W_X \oplus V_Y$ with the fact that $F'(x) : W_X \to F'(x)W_X$ is an isomorphism and $F'(x)T_{x_c}\alpha \subset V_Y$ to conclude that $\mathrm{Ker}(F'(x_c)) \subset T_{x_c}\alpha$. □

3.2 The Nonlinear Operator

For this section we set $X = H_0^1(\Omega)$, $Y = H^{-1}(\Omega)$. The corresponding projections will be denoted P_1, Q_1, P_{-1}, Q_{-1}. The norm used in $H_0^1(\Omega)$ will be $\|u\| = |u|_1 = \langle u, u\rangle_1^{\frac{1}{2}}$. Notice that this is equivalent to the full H^1 norm, by Friedrich's inequality. We will use often this result. For the expansion of $u \in H_0^1(\Omega)$ we use the notation $u(x) = \sum u_i\varphi_i(x)$, with $u_i = \langle u, \varphi_i\rangle_1 / \langle \varphi_i, \varphi_i\rangle_1$. We have $H_0^1(\Omega) \simeq H^{-1}(\Omega)$ via $\langle \tilde{u}, \cdot\rangle = \langle u, \cdot\rangle_1$, where we denote with a tilde the functional induced by an element of $H_0^1(\Omega)$. From Hilbert space theory we also know that

$$\|\tilde{u}\|_{-1} = \|u\|_1 \quad \text{and} \quad \tilde{u}_n \overset{H^{-1}}{\to} \tilde{u} \Leftrightarrow u_n \overset{H_0^1}{\to} u.$$

For a C^1 function f of bounded derivative, we define $F : X \to Y$ by

$$F(u) = -\Delta u - f(u). \tag{3-1}$$

The Laplacian above is understood as the weak Laplacian, acting as $u \overset{-\Delta}{\mapsto} \langle u, \cdot \rangle_1$ and $f(u)$ is the functional associated to the $L^2(\Omega)$ function

$$f(u) : z \overset{f(u)}{\longmapsto} \langle f(u), z \rangle_0.$$

We wish now to split our spaces in direct sums of a certain finite-dimensional space and its orthogonal complement. Denote the eigenvalues of $-\Delta : H_0^1(\Omega) \to H^{-1}(\Omega)$ by $0 < \lambda_1 < \lambda_2 \leq \ldots$ with corresponding eigenvectors φ_i. The eigenvectors may be taken to be orthogonal functions (in the occasional situation of multiplicity) and orthogonality holds simultaneously for all considered Sobolev spaces.

Definition 3. *A set J of indices is said to be* complete *if $j \in J$ whenever $\lambda_i = \lambda_j$ and $i \in J$.*

That is, if a complete set includes an index of a multiple eigenvalue, then it contains *all* indices associated with it.

For J a given *finite* complete set of indices, define the spaces $V_1 = \text{Span}\{\varphi_j,\ j \in J\}$ and $V_{-1} = \text{Span}\{\tilde{\varphi}_j,\ j \in J\}$. Since each space is closed (they are finite-dimensional), we can split the whole spaces as

$$X = W_1 \oplus V_1, \quad Y = W_{-1} \oplus V_{-1}, \quad \text{where} \quad W_1 = V_1^{\perp}, \quad W_{-1} = V_{-1}^{\perp}.$$

Recall that the inner product in Y is $\langle \tilde{u}, \tilde{v} \rangle_{-1} = \langle u, v \rangle_1$.

Proposition 4. *The correspondence $\tilde{u} \leftrightarrow u$ is also a bijection between W_1 and W_{-1}. Moreover, $W_1 = \{w \in X\ :\ \forall \tilde{v} \in V_{-1},\ \langle \tilde{v}, w \rangle = 0\}$ and $\Delta W_1 = W_{-1}$.*

Proof. This follows directly from $\langle \tilde{v}, w \rangle = \langle v, w \rangle_1 = \langle \tilde{v}, \tilde{w} \rangle_{-1}$. and the fact that $\langle \Delta w, \tilde{v} \rangle_{-1} = \langle w, v \rangle_1$ □

Definition 4. *Given a complete set J, a C^1 function $f : \mathbb{R} \to \mathbb{R}$* interacts with J *if*

1. *the only eigenvalues λ_i in the image of f' are labeled by indices in J,*
2. *there are no eigenvalues in the boundary of the image of f'.*

As in the abstract case, we define the orthogonal projection $P_{-1} : Y \to W_{-1}$, defined by $\langle P_{-1}\tilde{u}, \tilde{w} \rangle_{-1} = \langle \tilde{u}, \tilde{w} \rangle_{-1} = \langle u, w \rangle_1$ for each $\tilde{w} \in W_{-1}$. For a given $v \in V_1$, we define the restricted projection $F_v : W_1 \to W_{-1}$ by

$$F_v(w) = P_{-1} F(v + w), \tag{3-2}$$

Our next goal is to prove that if f interacts with a complete set J, then the function F is flat with respect to the decompositions induced by J. The first step is the local version of this property.

Proposition 5. *Let* $J = \{l+1 \leq \ldots \leq r-1\}$ *be a complete set and* $f : \mathbb{R} \to \mathbb{R}$ *a* C^1 *function interacting with* J*, then the derivatives of the restricted projection* F_v *are uniformly bounded below.More precisely, there exists* $C > 0$ *such that*

$$\forall v \in V_1 \ \forall w \in W_1 \ \forall h \in W_1, \quad \|F_v'(w)h\|_{-1} \geq C\|h\|_1. \tag{3-3}$$

Also, all derivatives of F_v *are invertible.*

This estimate, for the case $J = \{1\}$, i.e., when the nonlinearity f interacts only with the first eigenvalue λ_1, has been extensively used ([1], [4]). It is also used in [9] in the case $J = \{1, \ldots, r\}$.

Proof. From Proposition 1, each restricted projection $F_v : W_1 \to W_{-1}$ is C^1 with derivative $F_v'(w) : W_1 \to W_{-1}$ given by $F_v'(w)h = -\Delta h - f'(w)h$. Let $\overline{\operatorname{Ran} f'} = [a, b]$, so that $\lambda_l < a < \lambda_{l+1}$ and $\lambda_{r-1} < b < \lambda_r$. Let $h \in W_1$ be of unit norm and let $\tilde{h}^0$ be the functional $\langle \tilde{h}^0, \cdot \rangle = \langle h, \cdot \rangle_0$ and $\gamma = (a+b)/2$. Adding and subtracting $\gamma \tilde{h}^0$ and setting $u = w + v$ we have

$$\begin{aligned} \|F_v'(w)h\|_{-1} &= \|P_{-1}(-\Delta h - \gamma \tilde{h}^0) - P_{-1}(f'(u)h - \gamma \tilde{h}^0)\|_{-1} \\ &\geq \|P_{-1}(-\Delta h - \gamma \tilde{h}^0)\|_{-1} - \|P_{-1}(f'(u)h - \gamma \tilde{h}^0)\|_{-1} \\ &\geq \|\tilde{A}\|_{-1} - \|\tilde{B}\|_{-1}. \end{aligned} \tag{3-4}$$

In what follows, we will write z for an arbitrary element of $H_0^1(\Omega)$, and w for one in W_1. Let us start with a bound for $\|\tilde{B}\|_{-1}$.

$$\begin{aligned} \|\tilde{B}\|_{-1} &= \sup_{\|z\|=1} \langle P_{-1}(f'(u)h - \gamma \tilde{h}^0), z \rangle = \sup_{\|w\|=1} \langle f'(u)h - \gamma \tilde{h}^0, w \rangle \\ &= \sup_{\|w\|=1} \langle (f'(u) - \gamma)h, w \rangle_0 \leq \|f' - \gamma\|_\infty \sup_{\|w\|=1} \langle |h|, |w| \rangle_0. \end{aligned}$$

By Cauchy-Schwartz, the supremum above is realized when $|w|$ is a scalar multiple of $|h|$, which is achieved by $w = \rho h$. Since h is assumed unitary, $\rho = 1$ and, defining $c = \|f' - \gamma\|_\infty$,

$$\|\tilde{B}\|_{-1} \leq c \langle |h|, |h| \rangle_0 = c\,\|h\|_0^2 = \sum_{i \notin J} c\, h_i^2 \|\varphi_i\|_0^2 = \sum_{i \notin J} (c/\lambda_i) h_i^2 |\varphi_i|_1^2. \tag{3-5}$$

We will use now the decomposition $W_1 = W_+ \oplus W_-$. The spaces are given by $W_- = \{u \,:\, u = \sum_{i \leq l} u_i \varphi_i\}$, $W_+ = \{u \,:\, u = \sum_{i \geq r} u_i \varphi_i\}$ and are orthogonal both in $\langle , \rangle_0$ and $\langle , \rangle_1$.

To estimate $\|\tilde{A}\|_{-1}$, start with

$$\begin{aligned}\|\tilde{A}\|_{-1} &= \sup_{\|z\|=1} \langle P_{-1}(-\Delta h - \gamma \tilde{h^0}), z\rangle = \sup_{\|w\|=1} \langle -\Delta h - \gamma \tilde{h^0}, w\rangle \\ &= \sup_{\|w\|=1} \left(\langle h, w\rangle_1 - \gamma \langle h, w\rangle_0\right).\end{aligned}$$

Now we choose $w = h_+ - h_-$ above, noting that it also has unit norm.

$$\begin{aligned}\|\tilde{A}\|_{-1} &\geq \langle h, h_+ - h_-\rangle_1 - \gamma\langle h, h_+ - h_-\rangle_0 \\ &= (|h_+|_1^2 - \gamma\|h_+\|_0^2) + (\gamma\|h_-\|_0^2 - |h_-|_1^2) \\ &= \sum_{i\geq r} h_i^2(|\varphi_i|_1^2 - \gamma\|\varphi_i\|_0^2) + \sum_{i\leq l} h_i^2(\gamma\|\varphi_i\|_0^2 - |\varphi_i|_1^2) \\ &= \sum_{i\geq r}(1-\gamma/\lambda_i)h_i^2|\varphi_i|_1^2 + \sum_{i\leq l}(\gamma/\lambda_i - 1)h_i^2|\varphi_i|_1^2.\end{aligned}$$

That the coefficients above are all positive follows from the completeness of the set J. We have then

$$\|\tilde{A}\|_{-1} \geq \sum_{i\notin J} |1-\gamma/\lambda_i|\, h_i^2|\varphi_i|_1^2 = \sum_{i\notin J}(C_i/\lambda_i)h_i^2|\varphi_i|_1^2. \tag{3-6}$$

Combining equations (3-4), (3-5) and (3-6) we get

$$\begin{aligned}\|F_v'(w)h\|_{-1} &\geq \sum_{i\notin J}(C_i - c)/\lambda_i\, h_i^2|\varphi_i|_1^2 \\ &\geq \left(\inf_{i\notin J}(C_i - c)/\lambda_i\right)\sum_{i\notin J} h_i^2|\varphi_i|_1^2 \\ &= \left(\inf_{i\notin J}(C_i - c)/\lambda_i\right)|h|_1^2 \\ &= C|h|_1^2 = C,\end{aligned}$$

which establishes (3-3), since $C \geq \min\{1 - b/\lambda_{r+1}, a/\lambda_{l-1} - 1\} > 0$. In particular, the derivative of $F_v(w)$ is always injective. To prove invertibility, we write

$$F_v'(w)h = P_{-1} \circ F'(v+w) \circ \iota\, h,$$

where ι denotes the inclusion from W_1 into $H_0^1(\Omega)$ and notice that the composition of these three Fredholm operators is also Fredholm, with index given by the sum of the individual indices, namely, zero.

□

The following result is a global inversion theorem, first obtained by Hadamard in the finite-dimensional case [3].

Lemma 1. *Let $\Phi : X \to Y$ be a C^1 map between Banach spaces X and Y such that $\Phi'(u)$ is invertible for each u. Suppose there exists $C > 0$ such that*

$$\forall u, h, \quad \|\Phi'(u)h\| \geq C\|h\|. \tag{3-7}$$

Then Φ is a global C^1-diffeomorphism.

Theorem 1. *Let J be a complete set and $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function interacting with J. Then each restricted projection $F_v : W_1 \to W_{-1}$ is a C^1 diffeomorphism. Thus $F : H_0^1(\Omega) \to H^{-1}(\Omega)$ is flat.*

Proof. Simply combine Proposition 5 and the lemma above. □

In the case where f does not interact with the spectrum, the full operator is a diffeomorphism. This result was fully obtained by [11], after an initial version of it by [12], and follows from a simple adaptation of the proof of the above theorem.

When f interacts with a set J containing a single element j, then only the j-th eigenvalue of the Jacobian F' may become zero.

Proposition 6. *If $f \in C^1$, $\overline{f'(\mathbb{R})} = [a, b]$, with $\lambda_{k-1} < a < \lambda_k < b < \lambda_{k+1}$ and u_c is a critical point of F, then the only zero eigenvalue of $F'(u_c)$ is the k-th one. In particular, it is simple.*

An analogous result holds for a general complete set J: the only zero eigenvalues of F' are labelled by indices in J.

Proof. By the Fredholm property, we must have a nonzero ξ with $F'(u_c)\xi = -\Delta\xi - f'(u_c)\xi = 0$. In other words, 1 is an eigenvalue of the generalized problem $-\Delta u = \mu f'(u_c)u$, which we write as $\mu_j(f'(u_c)) = 1$ for some j. An application of a comparison theorem yields then

$$\lambda_{k-1} < f'(u_c) < \lambda_{k+1} \Rightarrow \mu_j(\lambda_{k-1}) > \mu_j(f'(u_c)) > \mu_j(\lambda_{k+1}), \tag{3-8}$$

or, since $\mu_j(\lambda) = \lambda_j/\lambda$ for constant λ,

$$\lambda_j/\lambda_{k-1} > 1 > \lambda_j/\lambda_{k+1} \Rightarrow \lambda_{k-1} < \lambda_j < \lambda_{k+1} \Rightarrow j = k. \tag{3-9}$$

□

The bound in Proposition 5 allows to make precise the idea that fibers are uniformly steep and images under F of horizontal affine subspaces are uniformly flat.

Proposition 7. *Let J be a complete set of indices with $|J| = k$ and $f : \mathbb{R} \to \mathbb{R}$ a C^1 function interacting with J. Let $u(t) = w(t) + \sum_{j \in J} t_j \varphi_j$ be a* *parametrization of a fiber α, where $t = (t_1, \dots, t_k) \in \mathbb{R}^k$ and $w(t) \in W_1$. Then there exists a positive constant C, independent of t, such that*

$$\|\nabla_t w(t)\|_1 \le C \sum_{j \in J} \|\varphi_j\|_1.$$

In particular, there exist positive constants A, B, independent of t, such that

$$\|w(t)\|_1 \le A + B\|t\|.$$

Let $\mathcal{W}_u \subset H^{-1}(\Omega)$ be the image under F of an horizontal affine subspace $u + W_1$, passing by $u \in H_0^1(\Omega)$. Then the angle between a vector in the tangent space $T_{F(u)}\mathcal{W}_u$ at a point $F(u) \in \mathcal{W}_u$ and its orthogonal projection in W_{-1} is uniformly bounded above by a constant less then $\pi/2$ for all $u \in H_0^1(\Omega)$.

Proof. Fibers are inverses under F of vertical affine subspaces in $H^{-1}(\Omega)$. Thus $PF(u(t)) =$ const. and, taking derivatives,

$$(PF)'(u(t))\ \partial_{t_j} u(t) \ = \ PF'(u(t))\ \partial_{t_j} u(t) = 0. \tag{3-10}$$

Write $u(t) = w(t) + v(t)$ and expand $v(t) = \sum_{j \in J} t_j\, \varphi_j$, so that $\partial_{t_j} u(t) = \partial_{t_j} w(t) + \varphi_j$. For $h \in W_1$, we have $PF'(u(t))h = F'_{v(t)}(w(t))h$ and thus, setting $h = \partial_{t_j} w(t)$,

$$F'_v(w(t))\partial_{t_j} w(t) = PF'(u(t))\partial_{t_j} w(t) = -PF'(u(t))\varphi_j.$$

Using first the lower bound (3-3) and then the boundedness of F',

$$C_1\|\partial_{t_j} w(t)\|_1 \le \|F'_v(w(t))\partial_{t_j} w(t)\|_{-1} = \|PF'(u(t))\varphi_j\|_{-1} \le C_2\|\varphi_j\|_1,$$

for some positive constant C_2. Thus $\|\nabla_t w(t)\|_1 \le C \sum_{j \in J} \|\varphi_j\|_1$. A bound of the form $\|w(t)\|_1 \le A + B\|t\|$ is now immediate.

To see that the tangent space $T_{F(u)}\mathcal{W}_u$ is bounded away from the vertical subspace, consider the sequence of simple estimates

$$C_1\|h\|_1 \le \|PF'(u)h\|_{-1} \le \|F'(u)h\|_{-1} \le C_3\|h\|_1$$

The cosine between a vector $F'(u)h \in T_{F(u)}\mathcal{W}_u$ and the horizontal subspace W_{-1} is given by the quotient $\|PF'(u)h\|_{-1}/\|F'(u)h\|_{-1}$, which is bounded from below by C_1/C_3. □

The result may be interpreted as a source of stability for the numerics described in the next sections. We indicate a first application immediately. From Theorem 1, the function $F : V_1 \oplus W_1 \to V_{-1} \oplus W_{-1}$ admits a global

Lyapunov-Schmidt decomposition, where V_1 is generated by the eigenvectors $\varphi_j, j \in J$.

When performing numerics, however, we do not work with φ_j — indeed, a general domain Ω does not allow for a formula for the eigenvectors. Even when this happens, as for rectangles, we must still consider the fact that the computations are performed on a finite dimensional subspace. In our case (see Section 4.2), we are using finite elements of the standard type $\mathcal{P}_1$, generating an approximation X_h to the domain $H_0^1(\Omega)$ and counter-domain $H^{-1}(\Omega)$. Since $\varphi_j \notin X_h$, we have to consider approximations ${\varphi_j}^h \in X_h$.

An *ϵ-tilted Lyapunov-Schmidt decomposition* of F is a pair of splittings $F : \tilde{V}_X \oplus \tilde{W}_X \to \tilde{V}_Y \oplus \tilde{W}_Y$, for which F admits a global Lyapunov-Schmidt decomposition and the four subspaces $\tilde{V}_X$, $\tilde{W}_X$, $\tilde{V}_Y$ and $\tilde{W}_Y$ are ϵ-close to their untilted counterparts. Here, one may take the distance between two subspaces as the maximal angle between them.

Corollary 2. *For ϵ sufficiently small and subspaces $\tilde{V}_X$, $\tilde{W}_X$, $\tilde{V}_Y$ and $\tilde{W}_Y$ ϵ-close to their untilted counterparts, the splittings $F : \tilde{V}_X \oplus \tilde{W}_X \to \tilde{V}_Y \oplus \tilde{W}_Y$ induce a tilted Lyapunov-Schmidt decomposition of F.*

Proof. This is an immediate consequence of the above proposition. □

The results in this section considered the nonlinear operator F as acting between $H_0^1(\Omega)$ and $H^{-1}(\Omega)$. Analogous results (stable global Lyapunov-Schmidt decomposition, boundedness and coercivity estimates, uniform flatness and steepness) also hold for $F : H_0^2 \to L^2$.