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A
A Quick Survey of the Finite Element Method

In this appendix we present the very basics of Finite Element Theory. The literature in the field is vast and we suggest [8] and [5] for those seeking more details on the subject. Let us start by motivating the method in its simplest application.

A.1
Variational Formulation

The *classical* formulation of the Poisson Equation with Dirichlet boundary conditions is the problem of finding a function $u \in C^2\left(\overline{\Omega}\right)$ satisfying

$$-\Delta u(x) = g(x), \quad x \in \Omega, \quad u(x) = 0,\ x \in \partial\Omega. \tag{A-1}$$

Multiplying equation (A-1) above by a function $v \in C_0^\infty(\Omega)$ and integrating over the domain we obtain

$$\int_\Omega -v(x)\Delta u(x) = \int_\Omega g(x)v(x). \tag{A-2}$$

Integrating by parts (i.e., using Green's first identity), the compact support of v annihilates the boundary terms and we obtain

$$\int_\Omega \nabla v(x) \cdot \nabla u(x) = \int_\Omega g(x)v(x), \quad v \in C_0^\infty(\Omega)\,. \tag{A-3}$$

The equation above makes sense not only for functions $v \in C_0^\infty(\Omega)$, but for the broader class of functions in $V = H_0^1(\Omega)$. This is the *variational formulation* of Poisson's problem:

$$\int_\Omega \nabla v(x) \cdot \nabla u(x) = \int_\Omega g(x)v(x), \quad \forall v \in V. \tag{A-4}$$

In fact, (A-4) is *equivalent* to the original equation (A-1). We no longer need additional information on the boundary. This is already built-in in the choice of the space $V = H_0^1(\Omega)$.

That (A-4) has a unique solution $u \in H_0^1(\Omega)$ is a straightforward consequence of Riesz Representation Theorem applied to the Hilbert space V. The *Finite Element Method* begins by replacing the infinite-dimensional space V above by a *finite dimensional* subspace V_h. The functions of V_h are called finite elements. In order to still have functions vanishing in the boundary, we require that each element also have this property. Once in X_h (A-4) reduces to a finite-dimensional linear system. Indeed, if $\{\psi_1, \ldots, \psi_N\}$ is a basis of X_h, it is sufficient that (A-4) is satisfied for each of the ψ_i. Also expanding $u(x) = \sum_j \underline{\mathrm{u}}_j \psi_j(x)$ we obtain the equivalent set of $N = \dim V$ equations:

$$\left(\sum_j \int_\Omega \nabla\psi_i(x) \cdot \nabla\psi_j(x) \right) u_j = \int_\Omega g(x)\psi_i(x), \quad i = 1, \ldots, N. \tag{A-5}$$

It is easy to see that this is an $N \times N$ system

$$\mathrm{K}\,\underline{\mathrm{u}} = \hat{g} \tag{A-6}$$

where the *stiffness matrix* K is given by $\mathrm{K}_{ij} = \int_\Omega \nabla\psi_i(x) \cdot \nabla\psi_j(x)$ and $\hat{g}_i = \int_\Omega g(x)\psi_i(x)$.

A.2 Triangulation and $\mathcal{P}_1$ Elements

Regardless of the element space or even the choice of basis, the stiffness matrix is always a positive definite matrix. Indeed, for $u \neq 0$,

$$\begin{aligned}
\langle \mathrm{K}\underline{\mathrm{u}}, \underline{\mathrm{u}} \rangle &= \sum_{i,j} \int_\Omega \nabla\psi_i(x) \cdot \nabla\psi_j(x) \underline{\mathrm{u}}_i \underline{\mathrm{u}}_j = \sum_{i,j} \int_\Omega \nabla\left(\underline{\mathrm{u}}_i \psi_i(x)\right) \cdot \nabla\left(\underline{\mathrm{u}}_j \psi_j(x)\right) \\
&= \int_\Omega \sum_i \nabla\left(\underline{\mathrm{u}}_i \psi_i(x)\right) \cdot \sum_j \nabla\left(\underline{\mathrm{u}}_j \psi_j(x)\right) = \int_\Omega \nabla u(x) \cdot \nabla u(x) > 0.
\end{aligned}$$

On the other hand, the sparsity pattern of K depends on the choice of basis for X_h. It is desirable to have the support of the ψ_i's overlapping as little as possible (this is not the only possibility, but it is the one we pursue here; an alternative would be spectral elements).

Let us describe briefly a way of designing the finite element used in this work. We want to take V_h consisting of continuous, piecewise linear functions. To allow for interpolation, we split the (rectangular) domain into triangles, as on the left of Figure A.1. Here, instead, we consider the more regular triangulation given by the figure on the right.

A function $f \in X_h$ can be described by its values on each vertex ν_i of the triangulation. This space is called in the literature $P_1\mathcal{P}_1$. Keeping in mind sparsity, we choose as a basis of X_h the *nodal functions* ψ_i defined by

$$\psi_i(\nu_j) = \delta_{ij}, \quad i, j = 1, \ldots, \dim X_h. \tag{A-7}$$

It is clear that the support of ψ_i, naturally associated with vertex ν_i, will overlap at most that of the nodal functions corresponding to neighboring

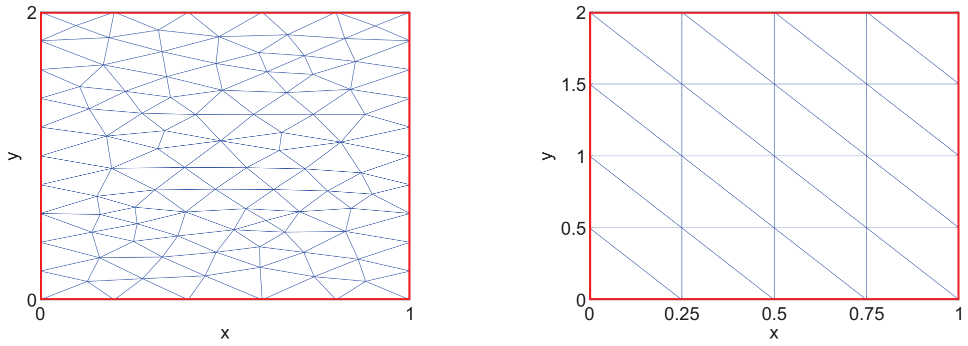

Figure A.1: Two Triangulations of $[0, 1] \times [0, 2]$.

vertices. Figure A.2 shows the graph of a typical ψ_i in a regular mesh. Since the

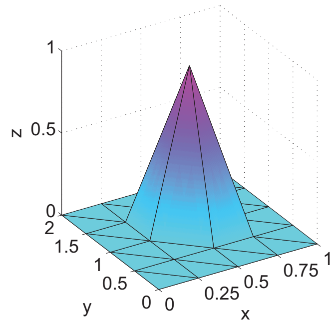

Figure A.2: A Nodal Basis Function

functions are piecewise linear, the integral needed to compute K, at least on each triangle, is the integral of a constant. That is a good motivation to range over triangles as we assemble K, instead of doing it scanning all possible indices (i, j). That is, denoting by T a generic triangle in the underlying triangulation of Ω, we have

$$\mathrm{K}_{ij} = \int_\Omega \nabla\psi_i(x) \cdot \nabla\psi_j(x) = \sum_T \int_T \nabla\psi_i(x) \cdot \nabla\psi_j(x).$$

What we do then is start with a zero matrix, range over all triangles computing a *local* 3×3 stiffness matrix and add this contribution to the actual K.

A.3 Error Bounds, Convergence

We are interested in having control over the error incurred when we replace our original problem in V by a finite-dimensional version X_h. The subscript h in X_h refers to a typical triangle size in a triangulation. A first remark is that the solution u_h of the linear system (A-6) is the closest we can get to the true solution u, measured in the H^1-seminorm — this is *Céa's Lemma.*

Lemma 2 (Céa's Lemma)**.** *For any finite element v_h function we have*

$$|u - u_h|_1 \leq |u - v_h|_1. \tag{A-8}$$

We then study the case where v_h above is the interpolation $\Pi_h u$ of the solution u in the space X_h. The idea is similar to a Taylor expansion. If our solution u is regular enough, it is possible to estimate the difference $u - \Pi_h u$ in terms of its higher derivatives and obtain a bound of the form

$$|u - \Pi_h u|_1 \leq Ch. \tag{A-9}$$

Notice that we do not need to know the solution u a priori, only some estimate on its higher order derivatives. Its regularity is also a consequence of domain regularity, which we do have in the rectangular case. The constant in (A-9) is also uniform provided for instance if we keep refining a given triangulation in a way that the triangles do not get too deformed. This provides a check for the computations we performed — halving triangles should halve the error.