1
Introduction

History of the problem

Consider the set $\mathcal{W}$ of all C^r regular closed curves in the plane $\mathbf{R}^2$ (i.e., C^r immersions $\mathbf{S}^1 \to \mathbf{R}^2$), furnished with the C^r topology ($r \geq 1$). The Whitney-Graustein theorem ([17], thm. 1) states that two such curves are homotopic through regular closed curves if and only if they have the same rotation number (where the latter is the number of full turns of the tangent vector to the curve).[1] Thus, the space $\mathcal{W}$ has an infinite number of connected components $\mathcal{W}_n$, one for each rotation number $n \in \mathbf{Z}$. A typical element of $\mathcal{W}_n$ ($n \neq 0$) is a circle traversed $|n|$ times, with the direction depending on the sign of n; $\mathcal{W}_0$ contains a figure eight curve.

For curves on the unit sphere $\mathbf{S}^2 \subset \mathbf{R}^3$, there is no natural notion of rotation number. Indeed, the corresponding space $\mathcal{I}$ of C^r immersions $\mathbf{S}^1 \to \mathbf{S}^2$ (i.e., regular closed curves on $\mathbf{S}^2$) has only two connected components $\mathcal{I}_+$ and $\mathcal{I}_-$; this is an immediate consequence of a much more general result of S. Smale ([16], thm. A). The component $\mathcal{I}_-$ contains all circles traversed an odd number of times, and the component $\mathcal{I}_+$ contains all circles traversed an even number of times. Actually, the Hirsch-Smale theorem implies that $\mathcal{I}_\pm \simeq \mathbf{SO}_3 \times \Omega\mathbf{S}^3$, where $\Omega\mathbf{S}^3$ denotes the set of all continuous closed curves on $\mathbf{S}^3$, with the compact-open topology; the properties of the latter space are well understood (see [1], §16).[2]

In 1970, J. A. Little formulated and solved the following problem: Let $\mathcal{L}$ denote the set of all C^2 closed curves on $\mathbf{S}^2$ which have nonvanishing geodesic curvature, with the C^2 topology; what are the connected components of $\mathcal{L}$? Although his motivation to investigate $\mathcal{L}$ appears to have been purely geometric, this space arises naturally in the study of a certain class of linear ordinary differential equations (see [12] for a discussion of this class and further references).

[1] Numbers enclosed in brackets refer to works listed in the bibliography at the end.
[2] The notation $X \simeq Y$ (resp. $X \approx Y$) means that X is homotopy equivalent (resp. homeomorphic) to Y.

Little was able to show (see [8], thm. 1) that $\mathcal{L}$ has six connected components, $\mathcal{L}_{\pm 1}$, $\mathcal{L}_{\pm 2}$ and $\mathcal{L}_{\pm 3}$, where the sign indicates the sign of the geodesic curvature of a curve in the corresponding component. A homeomorphism between $\mathcal{L}_i$ and $\mathcal{L}_{-i}$ is obtained by reversing the orientation of the curves in $\mathcal{L}_i$.

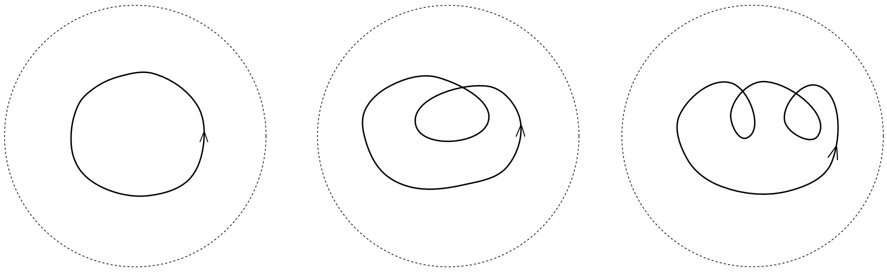

Figure 1: The curves depicted above provide representatives of the components $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$, respectively. All three are contained in the upper hemisphere of $\mathbf{S}^2$; the dashed line represents the equator seen from above.

The topology of the space $\mathcal{L}$ has been investigated by quite a few other people since Little. We mention here only B. Khesin, B. Shapiro and M. Shapiro, who studied $\mathcal{L}$ and similar spaces in the 1990's (cf. [6], [7], [14] and [15]). They showed that $\mathcal{L}_{\pm 1}$ are homotopy equivalent to $\mathbf{SO}_3$, and also determined the number of connected components of the spaces analogous to $\mathcal{L}$ in $\mathbf{R}^n$, $\mathbf{S}^n$ and $\mathbf{RP}^n$, for arbitrary n.

The first pieces of information about the homotopy and cohomology groups $\pi_k(\mathcal{L})$ and $H^k(\mathcal{L})$ for $k \geq 1$ were, however, only obtained a decade later by N. C. Saldanha in [10] and [11]. Finally, in the recent work [12], Saldanha gave a complete description of the homotopy type of $\mathcal{L}$ and other closely related spaces of curves on $\mathbf{S}^2$. He proved in particular that

$$\mathcal{L}_{\pm 2} \simeq \mathbf{SO}_3 \times \left(\Omega\mathbf{S}^3 \vee \mathbf{S}^2 \vee \mathbf{S}^6 \vee \mathbf{S}^{10} \vee \dots\right) \quad \text{and}$$
$$\mathcal{L}_{\pm 3} \simeq \mathbf{SO}_3 \times \left(\Omega\mathbf{S}^3 \vee \mathbf{S}^4 \vee \mathbf{S}^8 \vee \mathbf{S}^{12} \vee \dots\right).$$

The reason for the appearance of an $\mathbf{SO}_3$ factor in all of these results is that (unlike Saldanha, cf. [12]) we have not chosen a basepoint for the unit tangent bundle $UT\mathbf{S}^2 \approx \mathbf{SO}_3$; a careful discussion of this is given in §1.

Overview of this work

The main purpose of this thesis is to generalize Little's theorem to other spaces of closed curves on $\mathbf{S}^2$. Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ be given and let $\mathcal{L}_{\kappa_1}^{\kappa_2}$ be the set of all C^r closed curves on $\mathbf{S}^2$ whose geodesic curvatures are restricted to lie in the interval (κ_1, κ_2), furnished with the C^r topology (for some $r \geq 2$); in this notation, the spaces $\mathcal{L}$ and $\mathcal{I}$ discussed above become $\mathcal{L}_{-\infty}^{0} \sqcup \mathcal{L}_{0}^{+\infty}$ and $\mathcal{L}_{-\infty}^{+\infty}$, respectively. We present a direct characterization of the connected components of $\mathcal{L}_{\kappa_1}^{\kappa_2}$ in terms of the pair $\kappa_1 < \kappa_2$ and of the properties of curves in $\mathcal{L}_{\kappa_1}^{\kappa_2}$. It is shown in particular that the number of components is always finite, and a simple formula for it in terms of κ_1 and κ_2 is deduced.

More precisely, let $\rho_i = \operatorname{arccot}(\kappa_i)$, $i = 1, 2$, where we adopt the convention that arccot takes values in $[0, \pi]$, with $\operatorname{arccot}(+\infty) = 0$ and $\operatorname{arccot}(-\infty) = \pi$. Also, let $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x. Then $\mathcal{L}_{\kappa_1}^{\kappa_2}$ has n connected components $\mathcal{L}_1, \dots, \mathcal{L}_n$, where

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

and $\mathcal{L}_j$ contains circles traversed j times $(1 \leq j \leq n)$. The component $\mathcal{L}_{n-1}$ also contains circles traversed $(n-1) + 2k$ times, and $\mathcal{L}_n$ contains circles traversed $n + 2k$ times, for $k \in \mathbf{N}$. In addition, it will be seen that each of $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$ is homotopy equivalent to $\mathbf{SO}_3$ $(n \geq 3)$.

This result could be considered a first step towards the determination of the homotopy type of $\mathcal{L}_{\kappa_1}^{\kappa_2}$ in terms of κ_1 and κ_2. In this context, it is natural to ask whether the inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty} = \mathcal{I}$ is a homotopy equivalence; as we have already mentioned, the topology of the latter space is well understood. It will be shown that the answer is negative when $\rho_1 - \rho_2 \leq \frac{2\pi}{3}$. We expect this to be false except when $\kappa_1 = -\infty$ and $\kappa_2 = +\infty$. Actually, we conjecture that $\mathcal{L}_{\kappa_1}^{\kappa_2}$ and $\mathcal{L}_{\bar\kappa_1}^{\bar\kappa_2}$ have different homotopy types if and only if $\rho_1 - \rho_2 \neq \bar\rho_1 - \bar\rho_2$, but here it will only be proved that $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is homeomorphic to $\mathcal{L}_{\bar\kappa_1}^{\bar\kappa_2}$ if $\rho_1 - \rho_2 = \bar\rho_1 - \bar\rho_2$ $(\rho_i = \operatorname{arccot} \kappa_i$ and $\bar\rho_i = \operatorname{arccot} \bar\kappa_i)$.

Brief outline of the sections

It turns out that it is more convenient, but not essential, to work with curves which need not be C^2. The curves that we consider possess continuously varying unit tangent vectors at all points, but their geodesic curvatures are defined only almost everywhere. This class of curves is described in §1, where we also relate the resulting spaces of curves to the more familiar spaces of C^r curves. In this section we take the first steps toward the main theorem by proving that the topology of $\mathcal{L}_{\kappa_1}^{\kappa_2}$ depends only on $\rho_1 - \rho_2$. A corollary of this result is that any space $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is homeomorphic to a space of type $\mathcal{L}_{\kappa_0}^{+\infty}$; the latter class is usually more convenient to work with. Some variations of our definition are also investigated. In particular, in this section we consider spaces of non-closed curves.

In §2, we study curves which have image contained in a hemisphere. Almost all of this section is dedicated to proof that it is possible to assign to each such curve a distinguished hemisphere h_γ containing its image, in such a way that h_γ depends continuously on γ.

The main tools in the thesis are introduced in §3. Given a curve γ, we assign to γ certain maps B_γ and C_γ, called the regular and caustic bands spanned by γ, respectively. These are "fat" versions of the curve, and each of them carries in geometric form important information on the curve. We separate our curves into two main classes, called condensed and diffuse, depending on the properties of its caustic band. This distinction is essential throughout the work.

In §4, the grafting construction is explained. If the curve is diffuse, then we can use grafting to deform it into a circle traversed a certain number of times, which is the canonical curve in our spaces. We reach the same conclusion for condensed curves, using very different methods, in §5, where a notion of rotation numbers for curves of this type is also introduced. Although there exist curves which are neither condensed nor diffuse, any such curve is homotopic to a curve of one of these two types. The main results used to establish this are presented in §6.

In §7, we decide when it is possible to deform a circle traversed k times into a circle traversed $k+2$ times in $\mathcal{L}_{\kappa_0}^{+\infty}$. It is seen that this is possible if and only if $k \geq n-1 = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor$ (where $\rho_0 = \operatorname{arccot} \kappa_0$), and an explicit homotopy when this is the case is presented. It is also shown that the set of condensed curves in $\mathcal{L}_{\kappa_0}^{+\infty}$ with fixed rotation number $k < n-1$ is a connected component of this space.

The proofs of the main theorems are given in §8, after most of the work has been done. A direct characterization of the components of $\mathcal{L}_{\kappa_0}^{+\infty}$ ($\kappa_0 \in \mathbf{R}$) in terms of the properties of a curve is presented at the end of this section.

The last section is dedicated to the proof that the inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty} = \mathcal{I}$ is not a (weak) homotopy equivalence if $\rho_1 - \rho_2 \leq \frac{2\pi}{3}$

Finally, we present in an appendix some basic results on convexity in $\mathbf{S}^n$ that are used throughout the thesis. Although none of these results is new, complete proofs are given.