10
The Inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}$

The objective of this section is to prove that the inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2} \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}$ is not a homotopy equivalence when

$$\rho_1 - \rho_2 \leq \frac{2\pi}{3},$$

where $\rho_i = \operatorname{arccot}(\kappa_i)$, $i = 1, 2$. This section is to a large extent independent of the rest of the work. In particular, we do not use the caustic band, only the regular band. Since we will be working mostly with spaces of type $\mathcal{L}_{-\kappa_1}^{+\kappa_1}$ (as we are allowed to do, by (2.24)), we start by modifying its definition to suit our needs.

The band spanned by a curve

Throughout this subsection, let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ be fixed and let $\rho_1 = \operatorname{arccot} \kappa_1$, $\rho_2 = \operatorname{arccot} \kappa_2$. In order to get rid of the distinguished position of the endpoints of $[0, 1]$, we shall extend the domain of definition of all (closed) curves $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ to $\mathbf{R}$ by declaring them to be 1-periodic.

(10.1) Definitions. Let $\gamma \colon \mathbf{R} \to \mathbf{S}^2$, $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$.

(a) The (*regular*) *band* B_γ spanned by γ is the map:

$$B_\gamma \colon \mathbf{R} \times [\rho_1 - \pi, \rho_2] \to \mathbf{S}^2, \quad B_\gamma(t, \theta) = \cos\theta\, \gamma(t) + \sin\theta\, \mathbf{n}(t). \tag{1}$$

(b) B_γ is *simple* if it is injective when restricted to $[0, 1) \times [\rho_1 - \pi, \rho_2]$.

(c) B_γ is *quasi-simple* if it is injective when restricted to $[0, 1) \times (\rho_1 - \pi, \rho_2)$.

(d) The *boundary curves* of B_γ are the curves $\beta_+, \beta_- \colon \mathbf{R} \to \mathbf{S}^2$ given by:

$$\beta_+ \colon t \mapsto B_\gamma(t, \rho_2) \quad \text{and} \quad \beta_- \colon t \mapsto B_\gamma(t, \rho_1 - \pi). \tag{2}$$

Clearly, B_γ is also 1-periodic in t. Aside from the periodicity, the definition of regular band in (4.6) is subsumed in (10.1). Here are some further basic properties of B_γ.

(10.2) Lemma. *Let $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ and let $B_\gamma \colon \mathbf{R} \times [\rho_1 - \pi, \rho_2] \to \mathbf{S}^2$ be the band spanned by γ. Then:*

(a) *The derivative of B_γ is an isomorphism at every point.*

(b) *$\frac{\partial B_\gamma}{\partial \theta}(t,\theta)$ has norm 1 and is orthogonal to $\frac{\partial B_\gamma}{\partial t}(t,\theta)$. Moreover,*

$$\frac{\partial B_\gamma}{\partial t}(t,\theta) \times \frac{\partial B_\gamma}{\partial \theta}(t,\theta) = \lambda B_\gamma(t,\theta), \text{ with } \lambda > 0.$$

(c) *If B_γ is quasi-simple, then the restriction of B_γ to $(\rho_1 - \pi, \rho_2)$ is a covering map onto its image.*

Proof. The proofs of (a) and (b) are practically identical to those of the corresponding items in (4.7), so they will be ommitted. For part (c), consider the unique map $\bar{B}_\gamma \colon \mathbf{S}^1 \times (\rho_1 - \pi, \rho_2) \to \mathbf{S}^2$ making the following diagram commute:

$$\begin{tikzcd} \mathbf{R} \times (\rho_1 - \pi, \rho_2) \arrow[r, "B_\gamma"] \arrow[d, "\mathrm{pr} \times \mathrm{id}"'] & \mathbf{S}^2 \\ \mathbf{S}^1 \times (\rho_1 - \pi, \rho_2) \arrow[ur, dotted, "\bar{B}_\gamma"'] & \end{tikzcd} \tag{3}$$

where $\mathrm{pr}(t) = \exp(2\pi i t)$. Since $\bar{B}_\gamma$ is a diffeomorphism and $\mathrm{pr} \times \mathrm{id}$ is a covering map, B_γ is also a covering map (onto its image). $\square$

(10.3) Lemma. *Let $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ and suppose that B_γ is quasi-simple. For fixed φ satisfying $\rho_1 - \pi < \varphi < \rho_2$, the curve $\gamma_\varphi \colon t \mapsto B_\gamma(t,\varphi)$ separates $\mathbf{S}^2$ into two connected components, one containing $B_\gamma\big(\mathbf{R} \times [\rho_1 - \pi, \varphi)\big)$ and the other containing $B_\gamma\big(\mathbf{R} \times (\varphi, \rho_2]\big)$.*

Proof. By (10.2 (b)), B_γ is an immersion. Consequently,

$$U = B_\gamma\big(\mathbf{R} \times (\rho_1 - \pi, \rho_2)\big) \quad \text{and} \quad U_\varepsilon = B_\gamma\big(\mathbf{R} \times (\varphi - \varepsilon, \varphi + \varepsilon)\big)$$

are open sets, for any $\varepsilon > 0$ satisfying $\varepsilon < \min\{\rho_2 - \varphi, \varphi + \pi - \rho_1\}$. Let S denote the image of γ_φ. If $\beta_+(t') \in S$ for some t', then $B_\gamma(t',\theta) \in U_\varepsilon$ for all θ close to ρ_2. This contradicts the fact that B_γ is quasi-simple. Hence γ_φ does not intersect β_+, and for the same reason it does not intersect β_- either. Now let

$$A_- = B_\gamma\big(\mathbf{R} \times [\rho_1 - \pi, \varphi)\big), \quad A_+ = B_\gamma\big(\mathbf{R} \times (\varphi, \rho_2]\big).$$

By the Jordan curve theorem, the simple closed curve γ_φ separates $\mathbf{S}^2$ into two connected components V_+, V_-, and S is the boundary of each. Let V_+ be the component which contains A_+. If $A_- \subset V_+$ then all of $U \smallsetminus S$ would be contained in V_+. Since U is a neighborhood of S, this would give $\partial V_- \cap S = \emptyset$, a contradiction. Hence $A_- \subset V_-$. $\square$

(10.4) Lemma. *Let $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$, with B_γ quasi-simple. If $B_\gamma(t_1, \theta_1) = B_\gamma(t_2, \theta_2)$ then:*

(a) $\theta_1 = \theta_2 \in \{\rho_1 - \pi, \rho_2\}$.

(b) $\frac{\partial B_\gamma}{\partial \theta}(t_2, \theta_2) = -\frac{\partial B_\gamma}{\partial \theta}(t_1, \theta_1)$, $\frac{\partial B_\gamma}{\partial t}(t_2, \theta_2) = -\mu \frac{\partial B_\gamma}{\partial t}(t_1, \theta_1)$, $\mu > 0$, *and* $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$, *unless* $t_1 - t_2 \in \mathbf{Z}$.

In other words, if B_γ is quasi-simple then all of its self-intersections are either self-intersections of β_+ or of β_-, and they are actually points of self-tangency.

Proof. Part (a) is an immediate corollary of (10.3). Assume that $t_1 - t_2 \notin \mathbf{Z}$ and, for the sake of concreteness, that $\theta_1 = \theta_2 = \rho_2$. Choose $\varepsilon > 0$ such that $(t_1 - \varepsilon, t_1 + \varepsilon) + \mathbf{Z}$ does not intersect $(t_2 - \varepsilon, t_2 + \varepsilon)$ and let U_1 be the open set

$$U_1 = B_\gamma\big((t_1 - \varepsilon, t_1 + \varepsilon) \times (\rho_1 - \pi, \rho_2)\big).$$

If $\frac{\partial B_\gamma}{\partial t}(t_2, \rho_2)$ is not a positive or negative multiple of $\frac{\partial B_\gamma}{\partial t}(t_1, \rho_2)$, then either $B_\gamma(t_2 + u, \rho_2) \in U_1$ or $B_\gamma(t_2 - u, \rho_2) \in U_1$ for all sufficiently small $u > 0$. This contradicts the fact that B_γ is quasi-simple. Hence

$$\frac{\partial B_\gamma}{\partial t}(t_2, \rho_2) = \pm\mu \frac{\partial B_\gamma}{\partial t}(t_1, \rho_2), \ \mu > 0, \quad \text{and} \quad \frac{\partial B_\gamma}{\partial \theta}(t_2, \rho_2) = \pm \frac{\partial B_\gamma}{\partial \theta}(t_1, \rho_2),$$

the latter being a consequence of the former, by (10.2(b)). If we had $\frac{\partial B_\gamma}{\partial \theta}(t_1, \rho_2) = \frac{\partial B_\gamma}{\partial \theta}(t_2, \rho_2)$, then $B_\gamma(t_2, \rho_2 - u) \in U_1$ for all sufficiently small $u > 0$, again contradicting the fact that B_γ is quasi-simple. Hence

$$\frac{\partial B_\gamma}{\partial \theta}(t_2, \rho_2) = -\frac{\partial B_\gamma}{\partial \theta}(t_1, \rho_2)$$

and (10.2(b)) then yields $\frac{\partial B_\gamma}{\partial t}(t_2, \rho_2) = -\mu \frac{\partial B_\gamma}{\partial t}(t_1, \rho_2)$, $\mu > 0$. Together with eq. (6) of §3, this implies that $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$. □

(10.5) Lemma. *Let $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$, with B_γ quasi-simple. Let $\alpha \colon [0, 1] \to \mathbf{S}^2$ be a C^1 curve of length L such that $\alpha(0)$ lies in the image of β_- and $\alpha(1)$ in the image of β_+. Then $L \geq \pi - (\rho_1 - \rho_2)$, and equality holds if and only if α is an orientation-preserving reparametrization of a curve $\theta \mapsto B_\gamma(t_0, \theta)$, $\theta \in [\rho_1 - \pi, \rho_2]$.*

More concisely: If a curve crosses the band, it must have length $\geq \pi - (\rho_1 - \rho_2)$.

Proof. Let

$$t_0 = \sup\big\{t \in [0,1] : \alpha(t) \in \beta_-(\mathbf{R})\big\} \quad \text{and} \quad t_1 = \inf\big\{t > t_0 : \alpha(t) \in \beta_+(\mathbf{R})\big\}$$

By (10.4), the images of β_- and β_+ do not intersect each other, whence $t_0 < t_1$. We lose no generality in assuming that $t_0 = 0$, $t_1 = 1$.

Let $\tau_0 \in [0,1)$ and $\theta_0 \in (\rho_1 - \pi, \rho_2)$ be the unique numbers satisfying $B_\gamma(\tau_0, \theta_0) = \alpha(\frac{1}{2})$. The image of $(0,1)$ by α is completely contained in $B_\gamma\big(\mathbf{R}\times(\rho_1-\pi,\rho_2)\big)$, because of the way t_0 and t_1 were chosen. Consequently, by (10.2 (c)), there exist unique C^1 functions $\tau\colon (0,1) \to \mathbf{R}$, $\theta\colon (0,1) \to (\rho_1-\pi,\rho_2)$ making the following diagram of pointed maps commute:

$$\begin{tikzcd}
\big((0,1)\,,\,\tfrac{1}{2}\big) \arrow[rr, "\alpha"] \arrow[d, dotted, "\tau\times\theta"'] & & \big(\mathbf{S}^2\,,\,\alpha(\tfrac{1}{2})\big) \\
\big(\mathbf{R}\times(\rho_1-\pi,\rho_2)\,,\,(\tau_0,\theta_0)\big) \arrow[urr, "B_\gamma"'] & &
\end{tikzcd}$$

The length L of α is therefore given by:

$$\begin{aligned}
L &= \int_0^1 |\dot{\alpha}(u)|\, du = \lim_{\delta\to 0+} \int_\delta^{1-\delta} |\dot{\alpha}(u)|\, du \\
&= \lim_{\delta\to 0^+} \int_\delta^{1-\delta} \Big|\dot{\tau}(u)\frac{\partial B_\gamma}{\partial t}(\tau(u),\theta(u)) + \dot{\theta}(u)\frac{\partial B_\gamma}{\partial \theta}(\tau(u),\theta(u))\Big|\, du \\
&\geq \lim_{\delta\to 0^+} \int_\delta^{1-\delta} |\dot{\theta}(u)|\, du \geq \lim_{\delta\to 0^+} \Big| \int_\delta^{1-\delta} \dot{\theta}(u)\, du\Big| = |\theta(1-) - \theta(0+)|\,,
\end{aligned}$$

where in the first inequality we have used the facts that $\frac{\partial B_\gamma}{\partial t} \perp \frac{\partial B_\gamma}{\partial \theta}$ and that the latter has norm 1, as proved in (10.2 (b)).

We claim that the limits $\theta(0+)$ and $\theta(1-)$ exist and are equal to $\rho_1 - \pi$ and ρ_2, respectively. Let $\varphi \in (\rho_1 - \pi, \rho_2)$ be given and let

$$A_- = B_\gamma\big(\mathbf{R}\times[\rho_1-\pi,\varphi)\big) \quad \text{and} \quad A_+ = B_\gamma\big(\mathbf{R}\times(\varphi,\rho_2]\big).$$

As we saw in (10.3), A_- and A_+ are contained in different connected components of $\mathbf{S}^2 \smallsetminus \gamma_\varphi(\mathbf{R})$. These components are open sets and $\alpha(0) \in A_-$ by hypothesis, hence, by continuity, there exists $\delta > 0$ such that $\alpha([0,\delta)) \subset A_-$. This implies that $\theta(u) < \varphi$ for all $u \in (0,\delta)$. Because we can choose φ arbitrarily close to $\rho_1 - \pi$, this shows that $\theta(0+) = \rho_1 - \pi$. Similarly, $\theta(1-) = \rho_2$. Therefore $L \geq \pi - (\rho_1 - \rho_2)$.

Furthermore, $L = \pi - (\rho_1 - \rho_2)$ if and only if $\dot{\theta}$ does not change sign in $(0,1)$ and $\dot{\tau}(u) = 0$ for all $u \in (0,1)$ (recall that, by (10.2 (a)), $\frac{\partial B_\gamma}{\partial t}$ never vanishes). In other words, α is an orientation-preserving reparametrization of the curve $\theta \mapsto B_\gamma\big(\tau\big(\frac{1}{2}\big), \theta\big)$, $\theta \in [\rho_1 - \pi, \rho_2]$. $\square$

The Topology of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ for $0 < \kappa_1 \leq \sqrt{3}$

Our next goal is to prove some basic facts about the topology of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ for any (fixed) κ_1 satisfying $0 < \kappa_1 \leq \sqrt{3}$. We shall extend the domain of curves in this space to $\mathbf{R}$ by declaring them to be 1-periodic.

(10.6) Definition. Let $\mathcal{A}$ denote the subspace of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ ($0 < \kappa_1 \leq \sqrt{3}$) consisting of all curves γ such that

$$\gamma\big(t + \tfrac{1}{2}\big) = -\gamma(t) \quad \text{for all } t \in \mathbf{R}. \tag{4}$$

By (10.1), the band of a curve in $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I) \supset \mathcal{A}$ ($0 < \kappa_1 \leq \sqrt{3}$) is defined on

$$\mathbf{R} \times [-\rho_1, \rho_1] \supset \mathbf{R} \times \big[-\tfrac{\pi}{6}, \tfrac{\pi}{6}\big].$$

For our purposes it will suffice to consider the restriction of B_γ to the latter set.

(10.7) *Remark.* If $\gamma \in \mathcal{A}$, then $\Phi_\gamma(\frac{1}{2}) = Q_{\boldsymbol{k}}$, where

$$Q_{\boldsymbol{k}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the image of the quaternion $\boldsymbol{k}$ (and of $-\boldsymbol{k}$) under the projection $\mathbf{S}^3 \to \mathbf{SO}_3$. In fact, $\mathcal{A} \approx \mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\boldsymbol{k}})$, because (4) implies that a curve in $\mathcal{A}$ is uniquely determined by its restriction to $[0, \frac{1}{2}]$.

Remark. A curve $\gamma\colon \mathbf{R} \to \mathbf{S}^2$ in $\mathcal{A}$ corresponds to a closed curve $\bar{\gamma}\colon \mathbf{S}^1 \to \mathbf{RP}^2$ induced by γ as follows:

$$\begin{tikzcd} \mathbf{R} \arrow[r, "\gamma"] \arrow[d, "p"'] & \mathbf{S}^2 \arrow[d, "q"] \\ \mathbf{S}^1 \arrow[r, dotted, "\bar{\gamma}"'] & \mathbf{RP}^2 \end{tikzcd}$$

Here $p(t) = \exp(4\pi i t)$ and q is the usual covering map. Thus, we could also view $\mathcal{A}$ as a space of closed curves in $\mathbf{RP}^2$ having curvature bounded by κ_1. (Even though $\mathbf{RP}^2$ is not orientable, we can still speak of the unsigned curvature of a curve in $\mathbf{RP}^2$.) We shall not make any use of this interpretation in the sequel, however.

Examples of curves contained in $\mathcal{A}$ are the geodesics $\sigma_m\colon \mathbf{R} \to \mathbf{S}^2$ given by:

$$\sigma_m(t) = \big(\cos\big((2m+1)\,2\pi t\big), \sin\big((2m+1)\,2\pi t\big), 0\big) \quad (m = 0, 1, 2). \tag{5}$$

One checks directly that

$$\tilde{\Phi}_{\sigma_m}(t) = \cos\big((2m+1)\pi t\big)\,\mathbf{1} + \sin\big((2m+1)\pi t\big)\,\boldsymbol{k} \quad (m = 0, 1, 2),$$

so that $\tilde{\Phi}_{\sigma_m}(\frac{1}{2}) = (-1)^m \boldsymbol{k}$. Therefore, by (a slightly different version of) lemma (2.13), σ_1 does not lie in the same connected component of $\mathcal{A}$ as σ_0, σ_2. We shall see later that σ_0 and σ_2 do not lie in the same connected component either. In fact, we have the following result.

(10.8) Proposition. *Let $0 < \kappa_1 \leq \sqrt{3}$. The subspace*

$$\mathcal{A}_0 = \big\{\gamma \in \mathcal{A} : B_\gamma \text{ is simple}\big\} \subset \mathcal{A},$$

which contains σ_0 but not σ_2, is both open and closed in $\mathcal{A}$.

To prove this result we will need several lemmas.

(10.9) Definition. Let $\gamma \in \mathcal{A}$, let $B_\gamma \colon \mathbf{R} \times [-\frac{\pi}{6}, \frac{\pi}{6}] \to \mathbf{S}^2$ be its band and let $C \subset \mathbf{S}^2$ be a great circle. We shall say that $[\tau_1, \tau_2] \subset \mathbf{R}$ is a *crossing interval* of B_γ with respect to C if:

(i) $B_\gamma\big(\{\tau_1\} \times \big[-\frac{\pi}{6}, \frac{\pi}{6}\big]\big)$ is contained in a closed disk bounded by C;

(ii) $B_\gamma\big(\{\tau_2\} \times \big[-\frac{\pi}{6}, \frac{\pi}{6}\big]\big)$ is contained in the other closed disk bounded by C;

(iii) $B_\gamma\big(\{t\} \times \big[-\frac{\pi}{6}, \frac{\pi}{6}\big]\big)$ is not contained in either of the closed disks bounded by C for $t \in (\tau_1, \tau_2)$.

Thus, $[\tau_1, \tau_2]$ is a crossing interval if it is a minimal interval during which the band passes from one side of C to the other. In view of the 1-periodicity of B_γ in t, we shall identify two crossing intervals which differ by a translation by an integer.

(10.10) Lemma. *Let $\gamma \in \mathcal{A}$, let $C \subset \mathbf{S}^2$ be a great circle and $[\tau_1, \tau_2]$ a crossing interval of B_γ. Then:*

(a) *$B_\gamma(t + \frac{1}{2}, \theta) = -B_\gamma(t, -\theta)$ for all $t \in \mathbf{R}$, $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$. In particular, the images of β_+ and β_- are antipodal sets.*

(b) *$[\tau_1 + \frac{1}{2}, \tau_2 + \frac{1}{2}]$ is also a crossing interval.*

(c) *Two crossing intervals are either equal or have disjoint interiors.*

(d) $\tau_2 - \tau_1 \leq \frac{1}{2}$.

Proof. Part (a) follows from definition (10.1) and the relation $\gamma(t+\frac{1}{2}) = -\gamma(t)$, which is valid for any $\gamma \in \mathcal{A}$.

Since C is a great circle, the two disks bounded by C are antipodal sets. Together with (a), this implies that $[\tau_1+\frac{1}{2}, \tau_2+\frac{1}{2}]$ is a crossing interval if $[\tau_1, \tau_2]$ is, and proves (b).

Part (c) is an immediate consequence of definition (10.9).

Part (d) follows from (b) and (c): If $\tau_2 - \tau_1 > \frac{1}{2}$, then $(\tau_1, \tau_2) \cap (\tau_1 + \frac{1}{2}, \tau_2 + \frac{1}{2}) \neq \emptyset$. □

(10.11) Lemma. *Let $\gamma \in \mathcal{A}$, let $C \subset \mathbf{S}^2$ be a great circle and let $[\tau_1, \tau_2]$ be a crossing interval of B_γ. Then the following conditions are equivalent:*

(i) $C \cap B_\gamma\big(\{t\}\times[-\frac{\pi}{6}, \frac{\pi}{6}]\big)$ *consists of more than one point for some* $t \in [\tau_1, \tau_2]$.

(ii) $B_\gamma\big(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}]\big)$ *is completely contained in C for some* $t \in [\tau_1, \tau_2]$.

(iii) $\gamma(t) \in C$ *and* $\dot\gamma(t)$ *is orthogonal to C for some* $t \in [\tau_1, \tau_2]$.

(iv) $B_\gamma(t, \theta) \in C$ *and* $\frac{\partial B_\gamma}{\partial t}(t, \theta)$ *is orthogonal to C for some $t \in [\tau_1, \tau_2]$ and all* $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$.

(v) $\tau_1 = \tau_2$.

Proof. Suppose that (i) holds, and let Γ_t be parametrized by:

$$u \mapsto \cos u\,\gamma(t) + \sin u\,\mathbf{n}(t) \quad (u \in [-\pi, \pi)). \tag{6}$$

By hypothesis, the great circles C and Γ_t have at least two non-antipodal points in common. Hence, they must coincide, and (ii) holds.

If (ii) holds then $\frac{\partial B_\gamma}{\partial \theta}(t, 0)$ is tangent to C. Hence, by (10.2(b)), $\dot\gamma(t) = \frac{\partial B_\gamma}{\partial t}(t, 0)$ is orthogonal to C, and (iii) holds.

Suppose that (iii) holds. Then $\frac{\partial B_\gamma}{\partial \theta}(t, 0)$ is tangent to C, which means that C and the circle Γ_t defined in (6) are two great circles which are tangent at $\gamma(t)$. Therefore $C = \Gamma_t$, and $B_\gamma(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}]) \subset C$. Since $\frac{\partial B_\gamma}{\partial t}(t, \theta)$ is a positive multiple of $\dot\gamma(t)$, it, too, is orthogonal to C, for every $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$.

Suppose now that (iv) holds. Then there exists $\delta > 0$ such that $B_\gamma(u, \theta) \notin C$ for $0 < |u - t| < \delta$ and all $\theta \in [-\frac{\pi}{6}, \frac{\pi}{6}]$. This implies that $\tau_1 = t = \tau_2$.

Finally, suppose (v) holds and let $t = \tau_1 = \tau_2$. Then, according to definition (10.9), $B_\gamma\big(\{t\} \times [-\frac{\pi}{6}, \frac{\pi}{6}]\big)$ must be contained in both of the closed disks bounded by C, that is, it must be contained in C, whence (i) holds. □

(10.12) Lemma. *Let $\gamma \in \mathcal{A}$, with B_γ quasi-simple. Let $C \subset \mathbf{S}^2$ be a great circle and $[\tau_1, \tau_2]$ a crossing interval of B_γ (with respect to C). Then $C \cap B_\gamma\big([\tau_1, \tau_2] \times [-\frac{\pi}{6}, \frac{\pi}{6}]\big)$ has total length $L \geq \frac{\pi}{3}$. Moreover, equality holds if and only if $\tau_1 = \tau_2$.*

Proof. If $\tau_1 = \tau_2$ then the equivalence (ii)$\leftrightarrow$(v) in (10.11) shows that $L = \frac{\pi}{3}$. Assume now that $\tau_1 < \tau_2$. Then, from the equivalence (i)$\leftrightarrow$(v) in (10.11), we deduce that for each $t \in [\tau_1, \tau_2]$ there exists exactly one $\theta(t) \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ such that $B_\gamma(t, \theta(t)) \in C$. Again by (10.11), $\frac{\partial B_\gamma}{\partial t}(t, \theta(t))$ is not orthogonal to C for any $t \in [\tau_1, \tau_2]$. Hence, the implicit function theorem guarantees that $t \mapsto \theta(t)$ is a C^1 map, and $\alpha(t) = B_\gamma(t, \theta(t))$ defines a regular curve $\alpha \colon [\tau_1, \tau_2] \to \mathbf{S}^2$.

Let $\theta_i = \theta(\tau_i)$, $i = 1, 2$. We claim first that $\theta_1, \theta_2 \in \left\{\pm\frac{\pi}{6}\right\}$. Otherwise, $B_\gamma(\tau_i \times [-\frac{\pi}{6}, \frac{\pi}{6}])$ would contain points on both sides of C. Further, we claim that $\theta_2 = -\theta_1$. Otherwise, say, $\theta_1 = \theta_2 = -\frac{\pi}{6}$. If $\theta(t) \neq \frac{\pi}{6}$ for all $t \in [\tau_1, \tau_2]$, then the curve $t \mapsto B_\gamma(t, \frac{\pi}{6})$ would not cross C in $[\tau_1, \tau_2]$, a contradiction. Let $\bar{\tau}_2 = \inf\left\{t \in [\tau_1, \tau_2] : \theta(t) = \frac{\pi}{6}\right\}$. Then $[\tau_1, \bar{\tau}_2] \subset [\tau_1, \tau_2]$ is a crossing interval, hence we must have $\bar{\tau}_2 = \tau_2$ and $\theta_2 = \frac{\pi}{6}$, again a contradiction. Therefore $\alpha \colon [\tau_1, \tau_2] \to \mathbf{S}^2$ is a curve satisfying the hypotheses of (10.5), so it has length $\geq \frac{\pi}{3}$, and so does $C \cap B_\gamma\big([\tau_1, \tau_2] \times [-\frac{\pi}{6}, \frac{\pi}{6}]\big)$. The remaining assertion follows from the case of equality in (10.5). $\square$

(10.13) Lemma. *Let $\gamma_0, \gamma_1 \in \mathcal{A}$ lie in the same connected component, and suppose that B_{γ_0} is simple. Then B_{γ_1} is also simple.*

This result implies that σ_0 and σ_2 (see eq. (5)) are not in the same connected component. In particular, the number of components of $\mathcal{A}$ is at least 3. More importantly for us, this lemma implies (10.8): $\mathcal{A}_0$ is a union of connected components, hence $\mathcal{A}_0$ is open, and its complement is also a union of connected components, hence $\mathcal{A}_0$ is closed. (Here we are using the fact that $\mathcal{A}$ is locally path-connected: As explained in (10.7), it is homeomorphic to the Hilbert manifold $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\boldsymbol{k}})$.)

Proof. Let γ_s, $s \in [0, 1]$, be a path joining γ_0 to γ_1 in $\mathcal{A}$, and let us denote B_{γ_s} simply by B_s.

We claim first that if B_{s_0} is simple, then so is B_s for all s sufficiently close to s_0. Indeed, to say that B_s is simple is the same as to say that the unique map $\bar{B}_s$ which makes the following diagram commute is injective:

$$\begin{tikzcd} \mathbf{R} \times [-\frac{\pi}{6}, \frac{\pi}{6}] \arrow[r, "B_s"] \arrow[d, "p\times\mathrm{id}"'] & \mathbf{S}^2 \\ \mathbf{S}^1 \times [-\frac{\pi}{6}, \frac{\pi}{6}] \arrow[ur, dotted, "\bar{B}_s"'] & \end{tikzcd}$$

Here $p(t) = \exp(2\pi i t)$. Now define

$$f \colon [0, 1] \times \mathbf{S}^1 \times [-\tfrac{\pi}{6}, \tfrac{\pi}{6}] \to \mathbf{S}^2 \times [0, 1], \qquad f(s, z, \theta) = \big(\bar{B}_s(z, \theta), s\big).$$

By (10.2 (a)), $\bar{B}_s$ is an immersion for all s, hence so is f. Suppose that there exists a sequence (s_k) with $s_k \to s_0$ and B_{s_k} not simple, and choose $z_k, z'_k \in \mathbf{S}^1$, $\theta_k, \theta'_k \in [-\frac{\pi}{6}, \frac{\pi}{6}]$ with

$$B_{s_k}(z_k, \theta_k) = B_{s_k}(z'_k, \theta'_k) \quad \text{and} \quad (z_k, \theta_k) \neq (z'_k, \theta'_k) \quad \text{for all } k \in \mathbf{N}.$$

By passing to a subsequence if necessary, we can assume that $(z_k, \theta_k) \to (z, \theta)$ and $(z'_k, \theta'_k) \to (z', \theta')$. If $(z, \theta) \neq (z', \theta')$ then $\bar{B}_{s_0}$ would not be injective, and if $(z, \theta) = (z', \theta')$ then f would not be an immersion. Thus, no such sequence (s_k) can exist, and this proves our claim.

Now suppose for the sake of obtaining a contradiction that there exists $s \in [0, 1]$ such that B_s is not simple, and let s_0 be the infimum of all such s. From what we have just proved, we know that $s_0 > 0$ and B_{s_0} is not simple. We claim that B_{s_0} is quasi-simple. If not, then there exist $z_1, z_2 \in \mathbf{S}^1$ and $\theta_1, \theta_2 \in (-\frac{\pi}{6}, \frac{\pi}{6})$ such that

$$f(s_0, z_1, \theta_1) = f(s_0, z_2, \theta_2) \quad \text{and} \quad (z_1, \theta_1) \neq (z_2, \theta_2).$$

Choose $\varepsilon > 0$, open sets $U_i \ni z_i$ in $\mathbf{S}^1$ and disjoint neighborhoods $V_i \ni (s_0, z_i, \theta_i)$ of the form

$$V_i = (s_0 - \varepsilon, s_0] \times U_i \times (\theta_i - \varepsilon, \theta_i + \varepsilon) \qquad (i = 1, 2)$$

restricted to which f is a diffeomorphism. (The fact that θ_i belongs to the open interval $(-\frac{\pi}{6}, \frac{\pi}{6})$ is essential for the definition of V_i.) Then $f(s, z, \theta) \in f(V_1)$ for all $(s, z, \theta) \in V_2$ sufficiently close to (s_0, z_2, θ_2), since $f(s_0, z_2, \theta_2) \in f(V_1)$. But this contradicts the fact that $\bar{B}_s$ is injective for all $s < s_0$.

Therefore, B_{s_0} must be quasi-simple, but not simple. The following lemma shows that this is impossible, which, in turn, allows us to conclude that B_s must be simple for all $s \in [0, 1]$. □

(10.14) Lemma. *Suppose that $\gamma \in \mathcal{A}$ and B_γ is quasi-simple. Then B_γ is simple.*

Proof. If $p = B_\gamma(t_1, \theta_1) = B_\gamma(t_2, \theta_2)$, $t_1 - t_2 \notin \mathbf{Z}$, is a point of self-intersection of B_γ, then $\theta_1 = \theta_2 \in \{\pm\frac{\pi}{6}\}$ and $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$, as guaranteed by (10.4).

For p as above, let C_i be the circle parametrized by

$$u \mapsto \cos u\, \gamma(t_i) + \sin u\, \mathbf{n}(t_i), \quad (u \in [0, 2\pi],\ i = 1, 2).$$

Then both circles are centered at the origin and pass through p in a direction orthogonal to $\mathbf{t}(t_2) = -\mathbf{t}(t_1)$. Hence $C_1 = C_2$, and we shall denote it by C from now on. Thus, by (10.10 (b)) and (10.11), B_γ has at least the following four crossing intervals, all degenerate: $\{t_1\}$, $\{t_2\}$, $\{t_1 + \frac{1}{2}\}$ and $\{t_2 + \frac{1}{2}\}$. Further, by (10.10 (b)), the number of crossing intervals of B_γ is even (or infinite).

Let $\tau_j \in [0,1)$, $j = 1, \dots, 4$, be the numbers t_i, $t_i + \frac{1}{2} \pmod 1$ arranged so that $\tau_j < \tau_{j'}$ if $j < j'$. By definition, $\tau_1, \tau_2 \in [0, \frac{1}{2})$ and $\tau_3 = \tau_1 + \frac{1}{2}$, $\tau_4 = \tau_2 + \frac{1}{2}$. Suppose that these are the only crossing intervals of B_γ. Then B_γ crosses from one of the disks D_1 bounded by C to the other one D_2 at $t = \tau_1$, from D_2 to D_1 at $t = \tau_2$ and from D_1 to D_2 at $t = \tau_3$. But the latter is incompatible with $\dot\gamma(\tau_3) = -\dot\gamma(\tau_1)$, which points towards D_1. We conclude that B_γ has at least six crossing intervals. Since C has total length 2π and B_γ is quasi-simple, (10.11) implies that there cannot be more than six crossing intervals, and that all six are degenerate.

Let us again rearrange the crossing intervals (or numbers) $\tau_j \in [0, 2)$, $j = 1, \dots, 6$, so that $\tau_j < \tau_{j'}$ if $j < j'$, and hence $\tau_i \in [0, \frac{1}{2})$ and $\tau_{i+3} = \tau_i + \frac{1}{2}$ for $i = 1, 2, 3$. The sets $C_j = B_\gamma(\{\tau_j\} \times [-\frac{\pi}{6}, \frac{\pi}{6}])$ fill out the circle C, hence B_γ intersects itself in exactly 6 points. Suppose that C_1 and C_2 are disjoint. The image of $[\tau_1, \tau_2]$ by γ separates the closed disk which contains it in two parts, and the image of $(\tau_2, \tau_1 + 1)$ by γ contains points in both of these parts. Since γ is a simple curve, this is a contradiction which shows that $C_j \cap C_{j+1} \neq \emptyset$ for all $j \pmod 6$.

Note that the intersection $C_j \cap C_{j+1}$ consists of a single point of the form $B_\gamma(t_j, \theta_{j,j+1}) = B_\gamma(t_{j+1}, \theta_{j,j+1})$, where $\theta_{j,j+1} \in \{\pm\frac{\pi}{6}\}$ by (10.4). We may assume without loss of generality that $\theta_{1,2} = \frac{\pi}{6}$. This forces $\theta_{3,4} = \theta_{5,6} = \frac{\pi}{6}$ also.

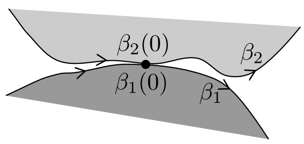

Figure 18: The darkly shaded region consists of points $B_\gamma(t, \theta)$ for (t, θ) close to $(t_1, \frac{\pi}{6})$, and the lightly shaded region consists of points $B_\gamma(t, \theta)$ for (t, θ) close to $(t_2, \frac{\pi}{6})$. Because B_γ is quasi-simple, the interiors of these regions cannot intersect.

Let ρ_j denote the radius of curvature of γ at $\gamma(\tau_j)$. Then the radius of curvature of β_+ at t_j is $\rho_j - \frac{\pi}{6}$, by (2.21). Choose a small $\varepsilon > 0$ and consider the curves

$$\beta_1, \beta_2 \colon (-\varepsilon, \varepsilon) \to \mathbf{S}^2, \quad \beta_1(u) = \beta_+(t_1 + u), \quad \beta_2(u) = \beta_+(t_2 - u).$$

Then $\beta_1(0) = \beta_2(0)$ and $\dot\beta_1(0)$ is a positive multiple of $\dot\beta_2(0)$ by (10.4).

Moreover, the radius of curvature of β_1 at 0 is $\rho_1 - \frac{\pi}{6}$ and that of β_2 is $\pi - (\rho_2 - \frac{\pi}{6}) = \frac{7\pi}{6} - \rho_2$ (the latter formula coming from the reversal of orientation). Because B_γ is quasi-simple, β_2 always lies to the right of β_1 (with respect to the common tangent unit vector at 0, cf. figure 18), hence the curvature of β_2 at 0 is greater than or equal to that of β_1 at 0. Or, in terms of the radii of curvature,

$$\rho_1 - \frac{\pi}{6} \geq \frac{7\pi}{6} - \rho_2, \quad \text{that is,} \quad \rho_1 + \rho_2 \geq \frac{4\pi}{3}$$

Similarly,

$$\rho_3 + \rho_4 \geq \frac{4\pi}{3} \quad \text{and} \quad \rho_5 + \rho_6 \geq \frac{4\pi}{3}.$$

Therefore, $\sum_{j=1}^{6} \rho_j \geq 4\pi$. On the other hand, the relation $\gamma(t+1) = -\gamma(t)$ yields $\rho_{i+3} = \pi - \rho_i$, $i = 1, 2, 3$. Hence $\sum_{i=1}^{6} \rho_i = 3\pi$. This contradiction shows that the assumption that B_γ has a point of self-intersection, i.e., that B_γ is not simple, must have been false. □

The inclusion $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}(I)$

We will show in this subsection that the inclusion

$$i\colon \mathcal{L}_{\kappa_1}^{\kappa_2}(I) \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}(I) \tag{7}$$

is not a homotopy equivalence when $0 < \rho_1 - \rho_2 \leq \frac{2\pi}{3}$, where $\rho_i = \operatorname{arccot}(\kappa_i)$ (prop. (10.18))

The proof separates into two cases: For $0 < \rho_1 - \rho_2 \leq \frac{\pi}{2}$, it is an easy consequence of Little's theorem that $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ has at least three connected components, so the map induced by i on π_0 is not a bijection. When $\frac{\pi}{2} < \rho_1 - \rho_2 \leq \frac{2\pi}{3}$, both spaces in (7) do have the same number of components, but we will exhibit a non-trivial element of $\pi_2\big(\mathcal{L}_{\kappa_1}^{\kappa_2}(I), \gamma_0\big)$ which lies in the kernel of the induced map i_* (the basepoint γ_0 is a circle traversed once). A very similar construction was previously used by Saldanha in [10] to obtain information on $\pi_2\big(\mathcal{L}_0^{+\infty}(I)\big)$ and $H^2\big(\mathcal{L}_0^{+\infty}(I)\big)$.

We conjecture, but do not prove, that the inclusion (7) is not a homotopy equivalence unless $\rho_1 - \rho_2 = \pi$ (when the inclusion i is simply the identity map of $\mathcal{L}_{-\infty}^{+\infty}(I)$). In order to show this directly it should be necessary to look at the induced map on π_{2n} for greater and greater n as $\rho_1 - \rho_2$ increases to π.

(10.15) Definition. Let $\mathcal{S} \subset \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ be the image of the map

$$G\colon (0,1) \times \mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\boldsymbol{k}}) \times \mathcal{L}_{-\kappa_1}^{+\kappa_1}(Q_{\boldsymbol{k}}) \to \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$$

which associates to a triple $(t_0, \gamma_1, \gamma_2)$ the curve γ obtained by concatenating γ_1 and γ_2 at $t = t_0$. More precisely, G is given by:

$$\gamma(t) = G(t_0, \gamma_1, \gamma_2)(t) = \begin{cases} \gamma_1\big(\frac{t}{t_0}\big) & \text{if } \ 0 \leq t \leq t_0 \\ Q_{\boldsymbol{k}}\gamma_2\big(\frac{t-t_0}{1-t_0}\big) & \text{if } \ t_0 \leq t \leq 1 \end{cases}$$

We start by showing that $\mathcal{S}$ is a submanifold of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$.

(10.16) Lemma. *Let $\mathcal{S}$ be as above. Then $\mathcal{S}$ is a closed submanifold of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ of codimension 2 which has trivial normal bundle.*

Proof. Let A be the arc of circle

$$A = \big\{(-\cos\theta\,,\, 0\,,\, \sin\theta) \in \mathbf{S}^2 \,:\, -\frac{\pi}{12} < \theta < \frac{\pi}{12}\big\}.$$

Let $\mathcal{U}$ be the subset of $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ consisting of all curves which intersect A exactly once, and transversally. Then $\mathcal{U} \supset \mathcal{S}$ and, although $\mathcal{U}$ is not open, it is a neighborhood of $\mathcal{S}$. Given $\gamma \in \mathcal{U}$, there exists exactly one $t_\gamma \in (0,1)$ such that $\gamma(t_\gamma) \in A$. Write

$$\Phi_\gamma(t_\gamma) = \begin{pmatrix} -\cos\theta_\gamma & * & * \\ 0 & * & * \\ \sin\theta_\gamma & z_\gamma & * \end{pmatrix},$$

so that θ_γ marks the point where γ crosses A and z_γ measures the slope of the crossing at this point. Define a map $F\colon \mathcal{U} \to \mathbf{R}^2$ by

$$F(\gamma) = (\theta_\gamma, z_\gamma).$$

Then $\mathcal{S} = F^{-1}(0,0)$, and it is easy to see that F is a submersion at any point of $\mathcal{S}$. Hence, lemma (2.7 (c)) applies. $\square$

(10.17) Lemma. *Let $1 < \kappa_1 \leq \sqrt{3}$. Then there exists $f\colon \mathbf{S}^2 \to \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ such that:*

(i) f intersects $\mathcal{S}$ only once and transversally;

(ii) f is null-homotopic in $\mathcal{L}_{-\infty}^{+\infty}(I) \supset \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$.

Proof. Let σ_α, $0 \leq \alpha \leq \pi$, be as described on pp. 83–84 and illustrated in figure 16. Since

$$\kappa_1 > 1 = \tan\Big(\frac{\pi}{4}\Big),$$

we may define a map $g\colon \mathbf{S}^2 \to \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ as follows: Set

$$g(N)(t) = (\cos 2\pi t, \sin 2\pi t, 0), \quad g(-N)(t) = (\cos 6\pi t, \sin 6\pi t, 0) \quad (t \in [0,1])$$

and, for $p \neq \pm N$, write $p = (\cos\theta\sin\alpha, \sin\theta\sin\alpha, \cos\alpha)$ with $\theta \in [0, 2\pi]$, $\alpha \in (0, \pi)$. Set

$$g(p)(t) = \left(\Phi_{\sigma_\alpha}\Big(t - \frac{\theta}{4\pi}\Big)\right)^{-1} \sigma_\alpha\Big(t - \frac{\theta}{4\pi}\Big) \quad (t \in [0,1],\ p \neq \pm N).$$

Thus, any longitude circle $\theta = \theta_0$ describes a homotopy between a circle traversed once and a circle traversed three times in $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$; as θ_0 varies, the only thing that changes is the starting point of the curves in homotopy, and we use multiplication by $\Phi_{\sigma_\alpha}^{-1}$ to ensure that all curves have the correct frames.

To define $f\colon \mathbf{S}^2 \to \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ as in the statement, let $r\colon \mathbf{S}^2 \to \mathbf{S}^2$ be reflection across the yz-plane, and let $\gamma_2 \in \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ be the equator traversed two times. Define

$$\bar{g}\colon \mathbf{S}^2 \to \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I), \quad \bar{g}(p) = \gamma_2 * \big((g \circ r)(p)\big),$$

where $*$ denotes the concatenation of paths. Then $[\bar{g}] = -[g]$ in $\pi_2(\mathcal{L}_{-\infty}^{+\infty}(I), \sigma_0)$, because $[g \circ r] = -[g]$ and concatenating with γ_2 has no effect on the homotopy class: For any map $h\colon K \to \mathcal{L}_{-\infty}^{+\infty}(I)$ with domain a compact set, h and $\gamma_2 * h$ are homotopic.

Therefore, if we define $f\colon \mathbf{S}^2 \to \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ to be the concatenation of g and $\bar{g}$ (as in the sum operation in π_2), then trivially $[f] = 0$. Moreover, it is immediate from the definition of $\mathcal{S}$ that $f(p) \in \mathcal{S}$ if and only if $p = N$. $\square$

(10.18) Proposition. *The inclusion*

$$i\colon \mathcal{L}_{\kappa_1}^{\kappa_2}(I) \hookrightarrow \mathcal{L}_{-\infty}^{+\infty}(I) \simeq \Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3 \tag{8}$$

is n o t a weak homotopy equivalence for $0 < \rho_1 - \rho_2 \leq \frac{2\pi}{3}$, *where* $\rho_i = \operatorname{arccot}\kappa_i$.

Proof. If $\kappa_0 \geq 0$, then $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ is a subspace of $\mathcal{L}_0^{+\infty}(I)$. Let $\sigma_j \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ be a circle traversed j times ($j = 1, 2, 3$). Little's theorem guarantees that σ_1, σ_2 and σ_3 are in pairwise distinct components of $\mathcal{L}_0^{+\infty}(I)$. Consequently, they must also be in different components of $\mathcal{L}_{\kappa_0}^{+\infty}(I)$. Together with (2.22), this implies that the map induced by (8) on π_0 is not a bijection for $0 < \rho_1 - \rho_2 \leq \frac{\pi}{2}$.

For the remaining cases we work instead with spaces of type $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ ($1 < \kappa_1 \leq \sqrt{3}$). It suffices to show that the map induced by (8) is not an isomorphism on π_2 in this case. Let $\mathcal{L} = \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$, and let $\mathcal{S}$ be its submanifold described in (10.15)

By (10.16), the normal bundle $N\mathcal{S}$ of $\mathcal{S}$ in $\mathcal{L}$ is trivial, hence orientable. Let τ be a 2-form representing the Thom class of this bundle. Using a tubular neighborhood $\mathcal{T}$ of $\mathcal{S}$ in $\mathcal{L}$, we can assume that τ is a 2-form defined on $\mathcal{T}$, extended by 0 to all of $\mathcal{L}$. Let $f\colon \mathbf{S}^2 \to \mathcal{L}$ be the map constructed in (10.17).

Then $f^*\tau$ is a 2-form on $\mathbf{S}^2$ which represents the Thom class of the normal bundle of $f^{-1}(\mathcal{S})$ in $\mathbf{S}^2$.

Now let S be a an oriented submanifold of an oriented, finite-dimensional manifold M. Then the Poincaré dual of S and the Thom class of the normal bundle of S in M are represented by the same form (see [1], pp. 66–67). Applying this to $M = \mathbf{S}^2$ and $S = f^{-1}(\mathcal{S})$, we obtain that $f^*\tau$ represents the Poincaré dual in $\mathbf{S}^2$ of a point. Therefore:

$$\int_{\mathbf{S}^2} f^*\tau = 1.$$

In particular, we conclude that f cannot be null-homotopic in $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$, otherwise $f^* = 0$. As we saw in (10.17), f is null-homotopic in $\mathcal{L}_{-\infty}^{+\infty}(I)$, whence

$$i_*\colon \pi_2\big(\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I), \gamma_0\big) \to \pi_2\big(\mathcal{L}_{-\infty}^{+\infty}(I), \gamma_0\big) \quad (1 < \kappa_1 \leq \sqrt{3})$$

is not injective, where γ_0 a circle traversed once, as in (10.17). $\square$