# 11
# Basic Results on Convexity

In this section we collect some results on convexity, none of which is new, that are used throughout the work.

Let $C \subset \mathbf{R}^{n+1}$. We say that $C$ is *convex* if it contains the line segment $[p, q]$ joining $p$ to $q$ whenever $p, q \in C$. The *convex hull* $\hat{X}$ of a subset $X \subset \mathbf{R}^{n+1}$ is the intersection of all convex subsets of $\mathbf{R}^{n+1}$ which contain $X$. It may be characterized as the set of all points $q$ of the form

$$q = \sum_{k=1}^{m} s_k p_k, \quad \text{where} \quad \sum_{k=1}^{m} s_k = 1,\ s_k > 0 \text{ and } p_k \in X \text{ for each } k. \tag{1}$$

**(11.1) Lemma.** *Let $X \subset \mathbf{S}^n$ and consider the conditions:*

*(i)* $0$ *does not belong to the closure of* $\hat{X}$*.*

*(ii) There exists an open hemisphere containing* $X$*.*

*(iii)* $0$ *does not belong to* $\hat{X}$*.*

*(iv)* $X$ *does not contain any pair of antipodal points.*

*Then* (i) → (ii) → (iii) → (iv)*, but none of the implications is reversible. If* $X$ *is closed then (ii) and (iii) are equivalent.*

*Proof.*

(i) → (ii) This is a special case of the Hahn-Banach theorem, since $\{0\}$ is a compact convex set and the closure of $\hat{X}$ is a closed convex set.

(ii) $\not\to$ (i) For $X \subset \mathbf{S}^n$ the open upper hemisphere, we have

$$\hat{X} = \big\{(x_1, \ldots, x_{n+1}) \in \mathbf{D}^{n+1} : x_{n+1} > 0\big\}.$$

Hence the closure of $\hat{X}$ contains the origin, even though $X$ is (contained in) an open hemisphere.

(ii) → (iii) Let $H = \big\{p \in \mathbf{S}^n : \langle p, h\rangle > 0\big\}$ be an open hemisphere containing $X$ and $U = \big\{p \in \mathbf{R}^{n+1} : \langle p, h\rangle > 0\big\}$. Then $U$ is convex, $X \subset U$ and $0 \notin U$. Thus, $0 \notin \hat{X}$.

(iii) $\not\to$ (ii) Let $X$ be the image of $[0, \pi)$ under $t \mapsto \exp(it)$.

(iii) $\to$ (iv) If $p$ and $-p$ both belong to $X$, then $0 \in [-p, p] \subset \hat{X}$.

(iv) $\not\to$ (iii) Let $X = \{1, \zeta, \zeta^2\} \subset \mathbf{S}^1$, where $\zeta = \exp(\frac{2}{3}\pi i)$ is a primitive third root of unity. Then $X$ does not contain antipodal points, but $0 = \frac{1}{3}\big(1 + \zeta + \zeta^2\big)$. $\square$

The last assertion is the combination of (i) $\to$ (ii) and (ii) $\to$ (iii), together with the fact that $\hat{X}$ is closed if $X$ is closed, as shown in (11.6) below (its proof does not rely on the present lemma).

**(11.2) Lemma.** *Let $X \subset \mathbf{S}^n$. Then $0$ belongs to the interior of $\hat{X}$ if and only if $X$ is not contained in any closed hemisphere of $\mathbf{S}^n$.*

*Proof.* Suppose first that $0 \notin \operatorname{Int} \hat{X}$. If $0$ does not belong to the closure of $\hat{X}$ then, as above, we can use the Hahn-Banach theorem to find a hyperplane separating $0$ and $X$. If $0 \in \partial\hat{X}$ then there exists a supporting hyperplane for $\hat{X}$ through $0$. One of the closed hemispheres determined by this hyperplane contains $X$.

Conversely, if $X$ is contained in a closed hemisphere

$$H = \big\{ p \in \mathbf{S}^n : \langle p, h \rangle \geq 0 \big\}$$

then $\hat{X}$ is contained in the "dome"

$$D = \big\{ p \in \mathbf{R}^n : |p| \leq 1 \text{ and } \langle p, h \rangle \geq 0 \big\},$$

which contains $0$ in its boundary. Hence, $\hat{X}$ cannot contain $0$ in its interior. $\square$

Let $A \subset \mathbf{S}^n$, $n \geq 1$. We say that $A$ is *geodesically convex* if it contains no antipodal points and if for any $p, q \in A$, the shortest geodesic joining $p$ to $q$ is also contained in $A$. The *convexification* $\check{X}$ of a subset $X \subset \mathbf{S}^n$ is defined to be the intersection of all geodesically convex subsets of $\mathbf{S}^n$ which contain $X$; if no such subset exists, then we set $\check{X} = \mathbf{S}^n$.

In what follows let $\operatorname{pr}\colon \mathbf{R}^{n+1} \smallsetminus \{0\} \to \mathbf{S}^n$ denote the gnomic projection $x \mapsto \frac{x}{|x|}$.

**(11.3) Lemma.** *Let $X \subset \mathbf{S}^n$.*

*(a) If $0 \notin \hat{X}$ then $\check{X} = \operatorname{pr}(\hat{X})$.*

*(b) $0 \in \hat{X}$ if and only if $\check{X} = \mathbf{S}^n$.*

*Proof.* We may assume that $X \neq \emptyset$ since (a) and (b) are trivially true if $X = \emptyset$.

(a) Assume that $0 \notin \hat{X}$. If $p = \mathrm{pr}(p_0)$ and $-p = \mathrm{pr}(p_0')$ for $p_0, p_0' \in \hat{X}$, then $0 \in [p_0, p_0'] \subset \hat{X}$, a contradiction. Hence, $\mathrm{pr}(\hat{X})$ does not contain any antipodal points.

Let $q \in \hat{X}$ be as in (1). We shall prove by induction on $m$ that $\mathrm{pr}(q) \in \check{X}$. This is obvious for $m = 1$, so assume $m > 1$, and set $\sigma = s_1 + \cdots + s_{m-1}$. Then

$$q = (1-\sigma)p_1 + \sigma\Big(\sum_{k=2}^{m} \frac{s_k}{\sigma} p_k\Big) = (1-\sigma)p_1 + \sigma p.$$

Both $p_1$ and $p$ belong to $\hat{X}$. Moreover, by the induction hypothesis, $\mathrm{pr}(p) \in \check{X}$. Since $\check{X}$ is geodesically convex, it contains the shortest geodesic joining $p_1$ to $\mathrm{pr}(p)$, which is precisely the image of the line segment $[p_1, p]$ under pr. Hence $\mathrm{pr}(q) \in \check{X}$, and $\mathrm{pr}(\hat{X}) \subset \check{X}$ is established.

Now let $p = \mathrm{pr}(p_0)$, $q = \mathrm{pr}(q_0)$, with $p_0, q_0 \in \hat{X}$. Then $(1-s)p_0 + sq_0 \in \hat{X}$ for all $s \in [0,1]$, whence $\mathrm{pr}[p_0, q_0] \subset \mathrm{pr}(\hat{X})$. Since $\mathrm{pr}[p_0, q_0]$ is the shortest geodesic joining $p$ to $q$, we conclude that $\mathrm{pr}(\hat{X})$ is geodesically convex. Therefore the reverse inclusion $\check{X} \subset \mathrm{pr}(\hat{X})$ also holds.

(b) Suppose first that $0 \in \hat{X}$ and write 0 as a convex combination

$$0 = \sum_{k=1}^{m} s_k p_k, \quad \text{where} \quad \sum_{k=1}^{m} s_k = 1, \quad p_k \in X \quad \text{and} \quad s_k > 0 \text{ for each } k,$$

with $m$ as small as possible; clearly, $m > 1$. Set $\sigma = s_2 + \cdots + s_m$. Then

$$0 = (1-\sigma)p_1 + \sigma\Big(\sum_{k=2}^{m} \frac{s_k}{\sigma} p_k\Big) = (1-\sigma)p_1 + \sigma p.$$

Let $S = \{p_2, \ldots, p_m\} \subset \mathbf{S}^n$. If $0 \in \hat{S}$, then we would be able to write 0 as a convex combination of $m-1$ points in $X$, a contradiction. Thus, applying part (a) to $S$, we deduce that $\mathrm{pr}(p) \in \check{S} \subset \check{X}$. Because $0 \in [p_1, p]$, $p_1$ and $\mathrm{pr}(p)$ must be antipodal to each other, whence $\check{X}$ contains a pair of antipodal points. Therefore $\check{X} = \mathbf{S}^n$.

Finally, suppose that $0 \notin \hat{X}$. By part (a), $\check{X} = \mathrm{pr}(\hat{X})$. Further, as we saw in the first paragraph of the proof, $\mathrm{pr}(\hat{X})$ does not contain antipodal points. Therefore $\check{X} \neq \mathbf{S}^n$. □

**(11.4) Lemma.** *A convex set $C \subset \mathbf{R}^n$ has empty interior if and only if it is contained in a hyperplane.*

*Proof.* Suppose that $C$ is not contained in a hyperplane and let $x_0 \in C$. Then we can find $x_1, \ldots, x_n \in C$ such that $\{x_i - x_0\}_{i=1,\ldots,n}$ forms a basis for $\mathbf{R}^n$.

Being convex, $C$ must contain the simplex $[x_0, \dots, x_n]$, which has nonempty interior since it is homeomorphic to the standard $n$-simplex. The converse is obvious. □

**(11.5) Lemma.** *Let $X \subset \mathbf{R}^n$ be any set. If $p \in \hat{X}$, then there exists a $k$-dimensional simplex which has vertices in $X$ and contains $p$, for some $k \leq n$.*

Another way to formulate this result is the following: If $X \subset \mathbf{R}^n$ and $p \in \hat{X}$, then it is possible to write $p$ as a convex combination of $k+1$ points in $X$ which are in general position, where $k$ is at most equal to $n$.

*Proof.* If $p \in \hat{X}$ then $p$ can be written as a finite convex combination of points in $X$. Hence, we may always assume that $X$ is finite. The proof will be by induction on $m+n$, where $m$ is the cardinality of $X$. If $m = 1$ or $n = 1$ the result is trivial.

Let $X = \{x_0, \dots, x_m\}$ and $X_0 = X \smallsetminus \{x_0\}$. If $p \in \hat{X}_0$ then we can use the induction hypothesis on $X_0$, so we may suppose that $p \notin \hat{X}_0$. There exist $q \in \hat{X}_0$ and $t \in [0,1]$ such that $p = (1-t)x_0 + tq$. Let $t_0$ be the infimum of all $u \geq t$ such that $(1-u)x_0 + uq \in \hat{X}_0$, and let $q_0 = (1-t_0)x_0 + t_0 q$ be the corresponding point. Note that $q_0 \in \hat{X}_0$ since the latter set is closed, by (11.6), and that $t_0 > t$, since $p \notin \hat{X}_0$

If $X_0$ is contained in some hyperplane, then we can apply the induction hypothesis to $X_0 \subset \mathbf{R}^{n-1}$ to conclude that there exists some $(k-1)$-dimensional simplex $\Delta_0$ with vertices in $X_0$ containing $q_0$, for some $k \leq n$. Then $p$ belongs to the $k$-dimensional simplex which is the cone on $\Delta_0$ with vertex $x_0$.

If $X_0$ is not contained in a hyperplane then $\hat{X}_0$ has nonempty interior in $\mathbf{R}^n$, by (11.4). Suppose that it is not possible to write $q_0$ as a combination of fewer than $m$ points in $X_0$. Then $q_0 \in \operatorname{Int} \hat{X}_0$, so that $(1-t)x_0 + tq \in \hat{X}_0$ for all $t$ sufficiently close to $t_0$. This contradicts the choice of $t_0$. Hence, we may write $q_0$ as a convex combination of $m-1$ points in $X_0$, and $p$ as a convex combination of $m$ points in $X$. By the induction hypothesis, we conclude that $p$ lies in some $k$-dimensional simplex with vertices in $X$, $k \leq n$. □

Let $Y \subset \mathbf{R}^2$ the graph of the function $f(t) = (1+t^2)^{-1}$, for $t \in \mathbf{R}$. Then any point on the $x$-axis belongs to the closure of $\hat{Y}$, but not to $\hat{Y}$. Thus, even though $Y$ is closed, $\hat{Y}$ is not. When $X$ is compact, however, the situation is different.

**(11.6) Lemma.** *If $X \subset \mathbf{R}^n$ is compact, then $\hat{X}$ is compact. In particular, if $X \subset \mathbf{S}^n$ is closed, then $\hat{X}$ is compact.* □

*Proof.* Let

$$\Delta^n = \big\{(s_1, \ldots, s_{n+1}) \in \mathbf{R}^{n+1} : s_1 + s_2 + \cdots + s_{n+1} = 1,\ \ s_i \in [0,1] \text{ for all } i\big\}$$

and define $f\colon \Delta^n \times X^{n+1} \to \mathbf{R}^n$ by

$$f(s_1, \ldots, s_{n+1}, x_1, \ldots, x_{n+1}) = s_1 x_1 + s_2 x_2 + \cdots + s_{n+1} x_{n+1}.$$

By (11.5), the image of $f$ is exactly the convex closure $\hat{X}$ of $X$. Since $\Delta^n \times X^{n+1}$ is compact and $f$ is continuous, $\hat{X}$ must also be compact. □