2
Spaces of Curves of Bounded Geodesic Curvature

Basic definitions and notation

Let M denote either the euclidean space $\mathbf{R}^{n+1}$ or the unit sphere $\mathbf{S}^n \subset \mathbf{R}^{n+1}$, for some $n \geq 1$. By a *curve* γ in M we mean a continuous map $\gamma\colon [a,b] \to M$. A curve will be called *regular* when it has a continuous and nonvanishing derivative; in other words, a regular curve is a C^1 immersion of $[a,b]$ into M. For simplicity, the interval where γ is defined will usually be $[0,1]$.

Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be a regular curve and let $|\ |$ denote the usual Euclidean norm. The *arc-length parameter* s of γ is defined by

$$s(t) = \int_0^t |\dot\gamma(t)|\, dt,$$

and $L = \int_0^1 |\dot\gamma(t)|\, dt$ is called the *length* of γ. Since $\dot s(t) > 0$ for all t, s is an invertible function, and we may parametrize γ by $s \in [0,L]$. Derivatives with respect to t and s will be systematically denoted by a $\dot{}$ and a $'$, respectively; this convention extends, of course, to higher-order derivatives as well.

Up to homotopy, we can always assume that a family of curves is parametrized proportionally to arc-length.

(2.1) Lemma. *Let A be a topological space and let $a \mapsto \gamma_a$ be a continuous map from A to the set of all C^r regular curves $\gamma\colon [0,1] \to M$ $(r \geq 1)$ with the C^r topology. Then there exists a homotopy $\gamma_a^u\colon [0,1] \to M$, $u \in [0,1]$, such that for any $a \in A$:*

(i) $\gamma_a^0 = \gamma_a$ and γ_a^1 is parametrized so that $|\dot\gamma_a^1(t)|$ is independent of t.

(ii) γ_a^u is an orientation-preserving reparametrization of γ_a, for all $u \in [0,1]$.

Proof. Let $s_a(t) = \int_0^t |\dot\gamma_a(\tau)|\, d\tau$ be the arc-length parameter of γ_a, L_a its length and $\tau_a\colon [0, L_a] \to [0,1]$ the inverse function of s_a. Define $\gamma_a^u\colon [0,1] \to M$ by:

$$\gamma_a^u(t) = \gamma_a\big((1-u)t + u\tau_a(L_a t)\big) \qquad (u, t \in [0,1],\ a \in A).$$

Then γ_a^u is the desired homotopy. $\square$

The unit tangent vector to γ at $\gamma(t)$ will always be denoted by $\mathbf{t}(t)$. Set $M = \mathbf{S}^2$ for the rest of this section, and define the *unit normal vector* $\mathbf{n}$ to γ by

$$\mathbf{n}(t) = \gamma(t) \times \mathbf{t}(t),$$

where $\times$ denotes the vector product in $\mathbf{R}^3$. Equivalently, $\mathbf{n}(t)$ is the unique vector which makes $\big(\gamma(t), \mathbf{t}(t), \mathbf{n}(t)\big)$ a positively oriented orthonormal basis of $\mathbf{R}^3$.

Assume now that γ has a second derivative. By definition, the *geodesic curvature* $\kappa(s)$ at $\gamma(s)$ is given by

$$\kappa(s) = \langle \mathbf{t}'(s), \mathbf{n}(s) \rangle \,. \tag{1}$$

Note that the geodesic curvature is not altered by an orientation-preserving reparametrization of the curve, but its sign is changed if we use an orientation-reversing reparametrization. Since the sectional curvatures of the sphere are all equal to 1, the normal curvature of γ is 1 at each point. In particular, its *Euclidean curvature* K,

$$K(s) = \sqrt{1 + \kappa(s)^2},$$

never vanishes.

Closely related to the geodesic curvature of a curve $\gamma\colon [0,1] \to \mathbf{S}^2$ is the *radius of curvature* $\rho(t)$ of γ at $\gamma(t)$, which we define as the unique number in $(0, \pi)$ satisfying

$$\cot \rho(t) = \kappa(t).$$

Note that the sign of $\kappa(t)$ is equal to the sign of $\frac{\pi}{2} - \rho(t)$.

Example. A parallel circle of colatitude α, for $0 < \alpha < \pi$, has geodesic curvature $\pm \cot \alpha$ (the sign depends on the orientation), and radius of curvature α or $\pi - \alpha$ at each point. (Recall that the colatitude of a point measures its distance from the north pole along $\mathbf{S}^2$.) The radius of curvature $\rho(t)$ of an arbitrary curve γ gives the size of the radius of the osculating circle to γ at $\gamma(t)$, measured along $\mathbf{S}^2$ and taking the orientation of γ into account.

If we consider γ as a curve in $\mathbf{R}^3$, then its "usual" radius of curvature R is defined by $R(t) = \frac{1}{K(t)} = \sin \rho(t)$. We will rarely mention R or K again, preferring instead to work with ρ and κ, which are their natural intrinsic analogues in the sphere.

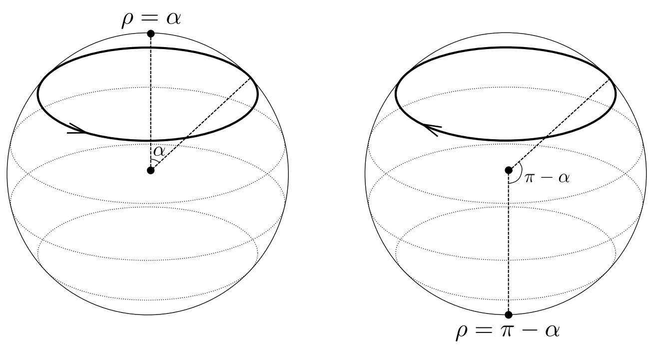

Figure 2: A parallel circle of colatitude α has radius of curvature α or $\pi - \alpha$, depending on its orientation. In the first figure the center of the circle on $\mathbf{S}^2$ is taken to be the north pole, and in the second, the south pole.

Spaces of curves

Given $p \in \mathbf{S}^2$ and $v \in T_p\mathbf{S}^2$ of norm 1, there exists a unique $Q \in \mathbf{SO}_3$ having $p \in \mathbf{R}^3$ as first column and $v \in \mathbf{R}^3$ as second column. We obtain thus a diffeomorphism between $\mathbf{SO}_3$ and the unit tangent bundle $UT\mathbf{S}^2$ of $\mathbf{S}^2$.

(2.2) Definition. For a regular curve $\gamma\colon [0,1] \to \mathbf{S}^2$, its *frame* $\Phi_\gamma\colon [0,1] \to \mathbf{SO}_3$ is the map given by

$$\Phi_\gamma(t) = \begin{pmatrix} | & | & | \\ \gamma(t) & \mathbf{t}(t) & \mathbf{n}(t) \\ | & | & | \end{pmatrix}.$$ [1]

In other words, Φ_γ is the curve in $UT\mathbf{S}^2$ associated with γ, under the identification of $UT\mathbf{S}^2$ with $\mathbf{SO}_3$. We emphasize that it is not necessary that γ have a second derivative for Φ_γ to be defined.

Now let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ and $Q \in \mathbf{SO}_3$. We would like to study the space $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ of all regular curves $\gamma\colon [0,1] \to \mathbf{S}^2$ satisfying:

(i) $\Phi_\gamma(0) = I$ and $\Phi_\gamma(1) = Q$;

(ii) $\kappa_1 < \kappa(t) < \kappa_2$ for each $t \in [0,1]$.

Here I is the 3×3 identity matrix and κ is the geodesic curvature of γ. Condition (i) says that γ starts at e_1 in the direction e_2 and ends at Qe_1 in the direction Qe_2.

[1]In the works of Saldanha this is denoted by $\mathfrak{F}_\gamma$ and called the *Frenet frame* of γ. We will not use this terminology to avoid any confusion with the usual Frenet frame of γ when it is considered as a curve in $\mathbf{R}^3$.

This definition is incomplete because we have not described the topology of $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$, nor explained what is meant by the geodesic curvature of a regular curve (which need not have a second derivative, according to our definition). The most natural choice would be to require that the curves in this space be of class C^2, and to give it the C^2 topology. The foremost reason why we will not follow this course is that we would like to be able to perform some constructions which yield curves that are not C^2. For instance, we may wish to construct a curve γ of positive geodesic curvature by concatenating two arcs of circles σ_1 and σ_2 of different radii, as in fig. 3 below. Even though the resulting curve is regular, it is not possible to assign any meaningful value to the curvature of γ at p. However, we may approximate γ as well as we like by a smooth curve which does have everywhere positive geodesic curvature. We shall adopt a more complicated definition precisely in order to avoid using convolutions or other tools all the time to smoothen such a curve.

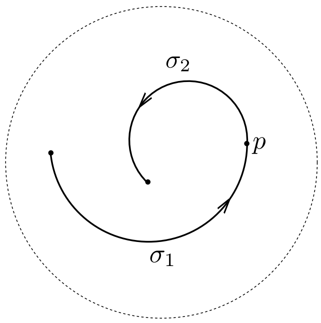

Figure 3: A curve on $\mathbf{S}^2$ obtained by concatenation of arcs of circles of different radii. The dashed line represents the equator.

(2.3) Definition. A function $f\colon [a,b] \to \mathbf{R}$ is said to be of class H^1 if it is an indefinite integral of some $g \in L^2[a,b]$. We extend this definition to maps $F\colon [a,b] \to \mathbf{R}^n$ by saying that F is of class H^1 if and only if each of its component functions is of class H^1.

Since $L^2[a,b] \subset L^1[a,b]$, an H^1 function is absolutely continuous (and differentiable almost everywhere).

We shall now present an explicit description of a topology on $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ which turns it into a Hilbert manifold. The definition is unfortunately not very natural. However, we shall prove the following two results relating this space to more familiar concepts: First, for any $r \in \mathbf{N}$, $r \geq 2$, the subset of $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ consisting of C^r curves will be shown to be dense in $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$. Second, we will see that the space of C^r regular curves satisfying conditions (i) and (ii) above, with the C^r topology, is (weakly) homotopy equivalent to $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$.[2]

Consider first a smooth regular curve $\gamma\colon [0,1] \to \mathbf{S}^2$. From the definition of Φ_γ we deduce that

$$\dot{\Phi}_\gamma(t) = \Phi_\gamma(t)\Lambda(t), \quad \text{where} \quad \Lambda(t) = \begin{pmatrix} 0 & -\left|\dot{\gamma}(t)\right| & 0 \\ \left|\dot{\gamma}(t)\right| & 0 & -\left|\dot{\gamma}(t)\right|\kappa(t) \\ 0 & \left|\dot{\gamma}(t)\right|\kappa(t) & 0 \end{pmatrix} \in \mathfrak{so}_3 \tag{2}$$

is called the *logarithmic derivative* of Φ_γ and κ is the geodesic curvature of γ.

Conversely, given $Q_0 \in \mathbf{SO}_3$ and a smooth map $\Lambda\colon [0,1] \to \mathfrak{so}_3$ of the form

$$\Lambda(t) = \begin{pmatrix} 0 & -v(t) & 0 \\ v(t) & 0 & -w(t) \\ 0 & w(t) & 0 \end{pmatrix}, \tag{3}$$

let $\Phi\colon [0,1] \to \mathbf{SO}_3$ be the unique solution to the initial value problem

$$\dot{\Phi}(t) = \Phi(t)\Lambda(t), \quad \Phi(0) = Q_0. \tag{4}$$

Define $\gamma\colon [0,1] \to \mathbf{S}^2$ to be the smooth curve given by $\gamma(t) = \Phi(t)(e_1)$. Then γ is regular if and only if $v(t) \neq 0$ for all $t \in [0,1]$, and it satisfies $\Phi_\gamma = \Phi$ if and only if $v(t) > 0$ for all t. (If $v(t) < 0$ for all t then γ is regular, but Φ_γ is obtained from Φ by changing the sign of the entries in the second and third columns.)

Equation (4) still has a unique solution if we only require that $v, w \in L^2[0,1]$ (cf. [3], p. 67). With this in mind, let $\mathbf{E} = L^2[0,1] \times L^2[0,1]$ and let $h\colon (0,+\infty) \to \mathbf{R}$ be the smooth diffeomorphism

$$h(t) = t - t^{-1}. \tag{5}$$

For each pair $\kappa_1 < \kappa_2 \in \mathbf{R}$, let $h_{\kappa_1,\kappa_2}\colon (\kappa_1,\kappa_2) \to \mathbf{R}$ be the smooth diffeomorphism

$$h_{\kappa_1,\kappa_2}(t) = (\kappa_1 - t)^{-1} + (\kappa_2 - t)^{-1}$$

and, similarly, set

$$\begin{aligned} &h_{-\infty,+\infty}\colon \mathbf{R} \to \mathbf{R} && h_{-\infty,+\infty}(t) = t \\ &h_{-\infty,\kappa_2}\colon (-\infty,\kappa_2) \to \mathbf{R} && h_{-\infty,\kappa_2}(t) = t + (\kappa_2 - t)^{-1} \\ &h_{\kappa_1,+\infty}\colon (\kappa_1,+\infty) \to \mathbf{R} && h_{\kappa_1,+\infty}(t) = t + (\kappa_1 - t)^{-1}. \end{aligned}$$

(2.4) Definition. Let κ_1, κ_2 satisfy $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$. A curve $\gamma\colon [0,1] \to$

[2]The definitions given here are straightforward adaptations of the ones in [13], where they are used to study spaces of locally convex curves in $\mathbf{S}^n$ (which correspond to the spaces $\mathcal{L}_0^{+\infty}(Q)$ when $n = 2$).

$\mathbf{S}^2$ will be called (κ_1,κ_2)-*admissible* if there exist $Q_0 \in \mathbf{SO}_3$ and a pair $(\hat{v},\hat{w}) \in \mathbf{E}$ such that $\gamma(t) = \Phi(t)\,e_1$ for all $t \in [0,1]$, where Φ is the unique solution to equation (4), with v, w given by

$$v(t) = h^{-1}(\hat{v}(t)), \quad w(t) = v(t)\,h^{-1}_{\kappa_1,\kappa_2}(\hat{w}(t)). \tag{6}$$

When it is not important to keep track of the bounds κ_1, κ_2, we shall say more simply that γ is *admissible*.

In vague but more suggestive language, an admissible curve γ is essentially an H^1 frame $\Phi\colon [0,1] \to \mathbf{SO}_3$ such that $\gamma = \Phi e_1\colon [0,1] \to \mathbf{S}^2$ has geodesic curvature in the interval (κ_1,κ_2). The unit tangent (resp. normal) vector $\mathbf{t}(t) = \Phi(t)e_2$ (resp. $\mathbf{n}(t) = \Phi(t)e_3$) of γ is thus defined everywhere on $[0,1]$, and it is absolutely continuous as a function of t. The curve γ itself is, like Φ, of class H^1. However, the coordinates of its velocity vector $\dot{\gamma}(t) = v(t)\,\Phi(t)e_2$ lie in $L^2[0,1]$, so the latter is only defined almost everywhere. The geodesic curvature of γ, which is also defined a.e., is given by

$$\kappa(t) = \frac{1}{v(t)}\left\langle \dot{\mathbf{t}}(t), \mathbf{n}(t)\right\rangle = h^{-1}_{\kappa_1,\kappa_2}(\hat{w}(t)) \in (\kappa_1,\kappa_2)$$

(cf. (2), (3) and (6)).

Remark. The reason for the choice of the specific diffeomorphism $h\colon (0,+\infty) \to \mathbf{R}$ in (5) (instead of, say, $h(t) = \log t$) is that we need $h^{-1}(t)$ to diverge linearly to $\pm\infty$ as $t \to 0, +\infty$ in order to guarantee that $v = h^{-1} \circ \hat{v} \in L^2[0,1]$ whenever $\hat{v} \in L^2[0,1]$. The reason for the choice of the other diffeomorphisms is analogous.

(2.5) Definition. Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$, $Q_0 \in \mathbf{SO}_3$. Define $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0,\cdot)$ to be the set of all (κ_1,κ_2)-admissible curves γ such that

$$\Phi_\gamma(0) = Q_0,$$

where Φ_γ is the frame of γ. This set is identified with $\mathbf{E}$ via the correspondence $\gamma \leftrightarrow (\hat{v},\hat{w})$, and this defines a (trivial) Hilbert manifold structure on $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0,\cdot)$.

In particular, this space is contractible by definition. We are now ready to define the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$, which constitute the main object of study of this work.

(2.6) Definition. Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$, $Q \in \mathbf{SO}_3$. We define $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ to be the subspace of $\mathcal{L}_{\kappa_1}^{\kappa_2}(I,\cdot)$ consisting of all curves γ in the latter space satisfying

$$\Phi_\gamma(0) = I \quad \text{and} \quad \Phi_\gamma(1) = Q. \tag{i}$$

Here Φ_γ is the frame of γ and I is the 3×3 identity matrix.[3]

Because $\mathbf{SO}_3$ has dimension 3, the condition $\Phi_\gamma(1) = Q$ implies that $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ is a closed submanifold of codimension 3 in $\mathbf{E} \equiv \mathcal{L}_{\kappa_1}^{\kappa_2}(I,\cdot)$. (Here we are using the fact that the map which sends the pair $(\hat{v},\hat{w}) \in \mathbf{E}$ to $\Phi(1)$ is a submersion; a proof of this when $\kappa_1 = 0$ and $\kappa_2 = +\infty$ can be found in §3 of [12], and the proof of the general case is analogous.) The space $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ consists of closed curves only when $Q = I$. Also, when $\kappa_1 = -\infty$ and $\kappa_2 = +\infty$ simultaneously, no restrictions are placed on the geodesic curvature. The resulting space (for arbitrary $Q \in \mathbf{SO}_3$) is known to be homotopy equivalent to $\Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3$; see the discussion after (2.13).

Note that we have natural inclusions $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \hookrightarrow \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(Q)$ whenever $\bar{\kappa}_1 \le \kappa_1 < \kappa_2 \le \bar{\kappa}_2$. More explicitly, this map is given by:

$$\gamma \equiv (\hat{v},\hat{w}) \mapsto \big(\hat{v}, h_{\bar{\kappa}_1,\bar{\kappa}_2} \circ h^{-1}_{\kappa_1,\kappa_2}(\hat{w})\big);$$

it is easy to check that the actual curve associated with the pair of functions in $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(Q)$ on the right side (via (3), (4) and (6)) is the original curve γ, so that the use of the term "inclusion" is justified. In fact, this map is an embedding, so that $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ can be considered a subspace of $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(Q)$ when $\bar{\kappa}_1 \le \kappa_1 < \kappa_2 \le \bar{\kappa}_2$.

The next lemma contains all results on Hilbert manifolds that we shall use.

(2.7) Lemma. *Let $\mathcal{M}$ be a Hilbert manifold. Then:*

(a) *$\mathcal{M}$ is locally path-connected. In particular, its connected components and path components coincide.*

(b) *If $\mathcal{M}$ is weakly contractible then it is contractible.*[4]

(c) *Assume that 0 is a regular value of $F\colon \mathcal{M} \to \mathbf{R}^n$. Then $\mathcal{P} = F^{-1}(0)$ is a closed submanifold which has codimension n and trivial normal bundle in $\mathcal{M}$.*

(d) *Let $\mathbf{E}$ and $\mathbf{F}$ be separable Banach spaces. Suppose $i\colon \mathbf{F} \to \mathbf{E}$ is a bounded, injective linear map with dense image and $M \subset \mathbf{E}$ is a smooth closed submanifold of finite codimension. Then $N = i^{-1}(M)$ is a smooth closed submanifold of $\mathbf{F}$ and $i\colon (\mathbf{F},N) \to (\mathbf{E},M)$ is a homotopy equivalence of pairs.*

[3]The letter 'L' in $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ is a reference to John A. Little, who determined the connected components of $\mathcal{L}_0^{+\infty}(I)$ in [8].

[4]Recall that a map $f\colon X \to Y$ between topological spaces X and Y is said to be a weak homotopy equivalence if $f_*\colon \pi_n(X,x_0) \to \pi_n(Y,f(x_0))$ is an isomorphism for any $n \ge 0$ and $x_0 \in X$. The space X is said to be weakly contractible if it is weakly homotopy equivalent to a point, that is, if all of its homotopy groups are trivial.

Proof. Part (a) is obvious and part (b) is a special case of thm. 15 in [9]. The first assertion of part (c) is a consequence of the implicit function theorem (for Banach spaces). The triviality of the normal bundle can be proved as follows: Let $p \in \mathcal{P}$ and $N\mathcal{P}_p$ be the fiber over p of the normal bundle $N\mathcal{P}$. Then

$$T\mathcal{M}_p = T\mathcal{P}_p \oplus N\mathcal{P}_p,$$

and $T\mathcal{P}_p$ lies in the kernel of the derivative TF_p by hypothesis, as F vanishes identically on $\mathcal{P}$. Since TF_p is surjective and $\dim N\mathcal{P}_p = n$, TF_p must be an isomorphism when restricted to $N\mathcal{P}_p$. This is valid for any $p \in \mathcal{P}$, so we can obtain a trivialization τ of $N\mathcal{P}$ by setting:

$$\tau(p,v) = \big((TF_p)|_{N\mathcal{P}_p}\big)^{-1}(v) \quad (p \in \mathcal{P},\ v \in \mathbf{R}^n).$$

Finally, part (d) is thm. 2 in [2]. □

(2.8) Lemma. *Let $r \in \{2,3,\dots,\infty\}$. Then the subset of all $\gamma\colon [0,1] \to \mathbf{S}^2$ of class C^r is dense in $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$.*

Proof. This follows from the fact that the set of smooth functions $f\colon [0,1] \to \mathbf{R}$ is dense in $L^2[0,1]$. □

(2.9) Definition. Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$, $Q \in \mathbf{SO}_3$ and $r \in \mathbf{N}$, $r \geq 2$. Define $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$ to be the set, furnished with the C^r topology, of all C^r regular curves $\gamma\colon [0,1] \to \mathbf{S}^2$ such that:

(i) $\Phi_\gamma(0) = I$ and $\Phi_\gamma(1) = Q$;

(ii) $\kappa_1 < \kappa(t) < \kappa_2$ for each $t \in [0,1]$.

The value of r is not important, as all of these spaces are homotopy equivalent. Because of this, after the next lemma, when we speak of $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$, we will implicitly assume that $r = 2$.

(2.10) Lemma. *Let $r \in \mathbf{N}$ $(r \geq 2)$, $Q \in \mathbf{SO}_3$ and $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$. Then the set inclusion $i\colon \mathcal{C}_{\kappa_1}^{\kappa_2}(Q) \hookrightarrow \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ is a homotopy equivalence.*

Proof. In this proof we will highlight the differentiability class by denoting $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)$ by $\mathcal{C}_{\kappa_1}^{\kappa_2}(Q)^r$. Let $\mathbf{E} = L^2[0,1] \times L^2[0,1]$, let $\mathbf{F} = C^{r-1}[0,1] \times C^{r-2}[0,1]$ (where $C^k[0,1]$ denotes the set of all C^k functions $[0,1] \to \mathbf{R}$, with the C^k norm) and let $i\colon \mathbf{E} \to \mathbf{F}$ be set inclusion. Setting $M = \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$, we conclude from (2.7(d)) that $i\colon N = i^{-1}(M) \hookrightarrow M$ is a homotopy equivalence. We claim that $N \approx \mathcal{C}_{\kappa_1}^{\kappa_2}(Q)^r$, where the homeomorphism is obtained by associating a pair $(\hat{v}, \hat{w}) \in N$ to the curve γ obtained by solving (4) (with Λ defined by (3) and (6) and $Q_0 = I$), and vice-versa.

Suppose first that $\gamma \in \mathcal{C}_{\kappa_1}^{\kappa_2}(Q)^r$. Then $|\dot{\gamma}|$ (resp. κ) is a function $[0,1] \to \mathbf{R}$ of class C^{r-1} (resp. C^{r-2}). Hence, so are $\hat{v} = h \circ |\dot{\gamma}|$ and $\hat{w} = h_{\kappa_1}^{\kappa_2} \circ \kappa$, since h and $h_{\kappa_1}^{\kappa_2}$ are smooth. Conversely, if $(\hat{v}, \hat{w}) \in N$, then $v = h^{-1}(\hat{v})$ is of class C^{r-1} and $w = (h_{\kappa_1}^{\kappa_2})^{-1} \circ \hat{w}$ of class C^{r-2}, and the frame Φ of the curve γ corresponding to that pair satisfies

$$\dot{\Phi} = \Phi\Lambda, \quad \Lambda = \begin{pmatrix} 0 & -|\dot{\gamma}| & 0 \\ |\dot{\gamma}| & 0 & -|\dot{\gamma}|\,\kappa \\ 0 & |\dot{\gamma}|\,\kappa & 0 \end{pmatrix} = \begin{pmatrix} 0 & -v & 0 \\ v & 0 & -w \\ 0 & w & 0 \end{pmatrix}.$$

Since the entries of Λ are of class (at least) C^{r-2}, the entries of Φ are functions of class C^{r-1}. Moreover, $\gamma = \Phi e_1$, hence

$$\dot{\gamma} = \dot{\Phi} e_1 = \Phi\Lambda e_1 = v\,\Phi e_2,$$

and the velocity vector of γ is seen to be of class C^{r-1}. It follows that γ is a curve of class C^r. Finally, it is easy to check that the correspondence $(\hat{v}, \hat{w}) \leftrightarrow \gamma$ is continuous in both directions. $\square$

Lifted frames

The (two-sheeted) universal covering space of $\mathbf{SO}_3$ is $\mathbf{S}^3$. Let us briefly recall the definition of the covering map $\pi\colon \mathbf{S}^3 \to \mathbf{SO}_3$.[5] We start by identifying $\mathbf{R}^4$ with the algebra $\mathbf{H}$ of quaternions, and $\mathbf{S}^3$ with the subgroup of unit quaternions. Given $z \in \mathbf{S}^3$, $v \in \mathbf{R}^4$, define a transformation $T_z\colon \mathbf{R}^4 \to \mathbf{R}^4$ by $T_z(v) = zvz^{-1} = zv\bar{z}$. One checks easily that T_z preserves the sum, multiplication and conjugation operations. It follows that, for any $v, w \in \mathbf{R}^4$,

$$\begin{aligned} 4\,\langle T_z(v), T_z(w)\rangle &= |T_z(v) + T_z(w)|^2 - |T_z(v) - T_z(w)|^2 \\ &= |v + w|^2 - |v - w|^2 = 4\,\langle v, w\rangle\,, \end{aligned}$$

where $\langle\ ,\ \rangle$ denotes the usual inner product in $\mathbf{R}^4$. Thus T_z is an orthogonal linear transformation of $\mathbf{R}^4$. Moreover, $T_z(\mathbf{1}) = \mathbf{1}$ (where $\mathbf{1}$ is the unit of $\mathbf{H}$), hence the three-dimensional vector subspace $\{0\} \times \mathbf{R}^3 \subset \mathbf{R}^4$ consisting of the purely imaginary quaternions is invariant under T_z. The element $\pi(z) \in \mathbf{SO}_3$ is the restriction of T_z to this subspace, where $(a, b, c) \in \mathbf{R}^3$ is identified with the quaternion $a\boldsymbol{i} + b\boldsymbol{j} + c\boldsymbol{k}$.

[5]See [4] for more details and further information on quaternions and rotations.

In what follows we adopt the convention that $\mathbf{S}^3$ (resp. $\mathbf{SO}_3$) is furnished with the Riemannian metric inherited from $\mathbf{R}^4$ (resp. $\mathbf{R}^9$).

(2.11) Lemma. *Let* $\langle\,,\,\rangle$ *denote the metric in* $\mathbf{S}^3$ *and* $\boldsymbol{\langle}\,,\,\boldsymbol{\rangle}$ *the metric in* $\mathbf{SO}_3$. *Then* $\pi^*\boldsymbol{\langle}\,,\,\boldsymbol{\rangle} = 8\,\langle\,,\,\rangle$, *where* $\pi^*\boldsymbol{\langle}\,,\,\boldsymbol{\rangle}$ *denotes the pull-back of* $\boldsymbol{\langle}\,,\,\boldsymbol{\rangle}$ *by* π.

Proof. It suffices to prove that if

$$z\colon (-1,1) \to \mathbf{S}^3, \qquad t \mapsto a(t)\mathbf{1} + b(t)\boldsymbol{i} + c(t)\boldsymbol{j} + d(t)\boldsymbol{k}$$

is a regular curve and $Q = \pi \circ z$ then $\big|\dot{Q}(0)\big|^2 = 8\,|\dot{z}(0)|^2$. Let us assume first that $z(0) = \mathbf{1}$, so that $\dot{a}(0) = 0$. From the definition of Q, we have

$$Q(t)e_1 = z(t)\boldsymbol{i}\bar{z}(t)$$

and similarly for $\boldsymbol{j}, \boldsymbol{k}$, where, as above, we identify $\mathbf{R}^3$ with the imaginary quaternions. Hence

$$\begin{aligned}\big|\dot{Q}(0)e_1\big|^2 &= |z(0)\boldsymbol{i}\dot{\bar{z}}(0) + \dot{z}(0)\boldsymbol{i}\bar{z}(0)|^2 = 2\,|\dot{z}(0)|^2 - \big(\dot{z}(0)\boldsymbol{i}\big)^2 - \big(\boldsymbol{i}\dot{\bar{z}}(0)\big)^2 \\ &= 2\,|\dot{z}(0)|^2 - 2\,\mathrm{Re}\,\big((\dot{z}(0)\boldsymbol{i})^2\big)\end{aligned}$$

Therefore

$$\big|\dot{Q}(0)\big|^2 = 6\,|\dot{z}(0)|^2 - 2\,\mathrm{Re}\,\big((\dot{z}(0)\boldsymbol{i})^2\big) - 2\,\mathrm{Re}\,\big((\dot{z}(0)\boldsymbol{j})^2\big) - 2\,\mathrm{Re}\,\big((\dot{z}(0)\boldsymbol{k})^2\big)$$

Since $\mathrm{Re}(w^2) = \alpha^2 - \beta^2 - \gamma^2 - \delta^2$ if $w = \alpha + \beta\boldsymbol{i} + \gamma\boldsymbol{j} + \delta\boldsymbol{k}$ and $\dot{a}(0) = 0$, we deduce that

$$-2\,\mathrm{Re}\,\big((\dot{z}(0)\boldsymbol{i})^2\big) = 2\dot{c}(0)^2 + 2\dot{d}(0)^2 - 2\dot{b}(0)^2 = 2\,|\dot{z}(0)|^2 - 4\dot{b}(0)^2$$

and analogously for $\boldsymbol{j}$, $\boldsymbol{k}$. Thus $\big|\dot{Q}(0)\big|^2 = 8\,|\dot{z}(0)|^2$ as claimed, provided $z(0) = \mathbf{1}$.

Now consider any regular curve $w\colon (-1,1) \to \mathbf{S}^3$, let $P = \pi \circ w$ and set

$$z(t) = w(0)^{-1}w(t), \quad Q(t) = \pi(z(t)) = P(0)^{-1}P(t).$$

Then $z(0) = \mathbf{1}$, hence

$$\big|\dot{P}(0)\big|^2 = \big|P(0)\dot{Q}(0)\big|^2 = \big|\dot{Q}(0)\big|^2 = 8\,|\dot{z}(0)|^2 = 8\,|w(0)\dot{z}(0)|^2 = 8\,|\dot{w}(0)|^2. \quad \square$$

(2.12) Definition. Let $\Phi\colon [0,1] \to \mathbf{SO}_3$ be a frame (of class H^1) and let $z \in \mathbf{S}^3$ satisfy $\pi(z) = \Phi_\gamma(0)$. We define the *lifted frame* $\tilde{\Phi}^z\colon [0,1] \to \mathbf{S}^3$ to be the lift of Φ to $\mathbf{S}^3$, starting at z. When $\Phi(0) = I$ we adopt the convention that $z = \mathbf{1}$, and we denote the lifted frame simply by $\tilde{\Phi}$.

Here is a simple but important application of this concept.

(2.13) Lemma. *Let $\gamma_0, \gamma_1 \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$, for some $Q \in \mathbf{SO}_3$, and suppose that γ_0, γ_1 lie in the same connected component of this space. Then $\tilde{\Phi}_{\gamma_0}(1) = \tilde{\Phi}_{\gamma_1}(1)$.*

Proof. Since $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ is a Hilbert manifold, its path and connected components coincide. Therefore, to say that γ_0, γ_1 lie in the same connected component of $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ is the same as to say that there exists a continuous family of curves $\gamma_s \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ joining γ_0 and γ_1, $s \in [0, 1]$. The family Φ_{γ_s} yields a homotopy between the paths Φ_{γ_0} and Φ_{γ_1} in $\mathbf{SO}_3$. (Recall that each of the frames Φ_{γ_s} is (absolutely) continuous.) By the homotopy lifting property of covering spaces, the paths $\tilde{\Phi}_{\gamma_0}$ and $\tilde{\Phi}_{\gamma_1}$ are also homotopic in $\mathbf{S}^3$ (fixing the endpoints). $\square$

The role of the initial and final frames

We will now study how the topology of $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ changes if we consider variations of condition (i) in (2.6); by the end of the section it should be clear that our original definition is sufficiently general. A summary of all the definitions considered here is given in table form on p. 28.

For fixed $z \in \mathbf{S}^3$, let $\Omega_z\mathbf{S}^3$ denote the set of all continuous paths $\omega\colon [0, 1] \to \mathbf{S}^3$ such that $\omega(0) = \mathbf{1}$ and $\omega(1) = z$, furnished with the compact-open topology. It can be shown (see [1], p. 198) that $\Omega_z\mathbf{S}^3 \simeq \Omega\mathbf{S}^3$ for any $z \in \mathbf{S}^3$, where $\Omega\mathbf{S}^3$ is the space of paths in $\mathbf{S}^3$ which start and end at $\mathbf{1} \in \mathbf{S}^3$.[6] The topology of this space is well understood; we refer the reader to [1], §16, for more information.

Now let $\kappa_1 < \kappa_2$, $z \in \mathbf{S}^3$ be arbitrary and $Q = \pi(z)$. Define

$$F\colon \mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \to \Omega_z\mathbf{S}^3 \cup \Omega_{-z}\mathbf{S}^3 \simeq \Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3 \quad \text{by} \quad F(\gamma) = \tilde{\Phi}_\gamma. \tag{7}$$

In the special case $\kappa_1 = -\infty$, $\kappa_2 = +\infty$, it follows from the Hirsch-Smale theorem that this map is a homotopy equivalence. In the general case this is false, however. For instance, $\Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3$ has two connected components, while Little has proved ([8], thm. 1) that $\mathcal{L}_0^{+\infty}(I)$ has three connected components. We take this opportunity to recall the precise statement of Little's theorem and to introduce a new class of spaces.

[6] The notation $X \simeq Y$ (resp. $X \approx Y$) means that X is homotopy equivalent (resp. homeomorphic) to Y.

(2.14) Definition. Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$. Define $\mathcal{L}_{\kappa_1}^{\kappa_2}$ to be the space of all (κ_1, κ_2)-admissible curves $\gamma\colon [0,1] \to \mathbf{S}^2$ such that

$$\Phi_\gamma(0) = \Phi_\gamma(1).$$

Note that the only difference between $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ and $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is that curves in the latter space may have arbitrary initial and final frames, as long as they coincide. An argument analogous to the one given for the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ shows that $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is also a Hilbert manifold. In fact, we have the following relationship between the two classes.

(2.15) Proposition. *The space $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is homeomorphic to $\mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$.*

Proof. For $Q \in SO_3$ and $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$, let $Q\gamma$ be the curve defined by $(Q\gamma)(t) = Q(\gamma(t))$. Because Q is an isometry, the geodesic curvatures of $Q\gamma$ at $(Q\gamma)(t)$ and of γ at $\gamma(t)$ coincide. Define $F\colon \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I) \to \mathcal{L}_{\kappa_1}^{\kappa_2}$ by $F(Q,\gamma) = Q\gamma$; clearly, F is continuous. Since it has the continuous inverse $\eta \mapsto (\Phi_\eta(0), \Phi_\eta(0)^{-1}\eta)$, F is a homeomorphism. $\square$

Let us temporarily denote by $\mathcal{L}$ the space $\mathcal{L}_{-\infty}^{0} \sqcup \mathcal{L}_0^{+\infty}$ studied by Little. We have $\mathcal{L}_{-\infty}^{0} \approx \mathcal{L}_0^{+\infty}$, since the map which takes a curve in $\mathcal{L}$ to the same curve with reversed orientation is a (self-inverse) homeomorphism mapping $\mathcal{L}_{-\infty}^{0}$ onto $\mathcal{L}_0^{+\infty}$. What is proved in [8] is that $\mathcal{L}$ has six connected components.[7] Using prop. (2.15) and the fact that $\mathbf{SO}_3$ is connected, we see that Little's theorem is equivalent to the assertion that $\mathcal{L}_0^{+\infty}(I)$ has three connected components, as was claimed immediately above (2.14).

A natural generalization of the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ is obtained by modifying condition (i) of (2.6) as follows.

(2.16) Definition. Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ and $Q_0, Q_1 \in \mathbf{SO}_3$. Define $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1)$ to be the space of all (κ_1, κ_2)-admissible curves $\gamma\colon [0,1] \to \mathbf{S}^2$ such that

$$\Phi_\gamma(0) = Q_0 \quad \text{and} \quad \Phi_\gamma(1) = Q_1. \tag{i$'$}$$

Thus, the only difference between condition (i) on p. 17 and condition (i$'$) is that the latter allows arbitrary initial frames.

(2.17) Proposition. $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1) \approx \mathcal{L}_{\kappa_1}^{\kappa_2}(PQ_0, PQ_1)$ *for any* $P, Q_0, Q_1 \in \mathbf{SO}_3$*. Then. In particular,* $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1) \approx \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$*, where* $Q = Q_0^{-1}Q_1$.

Proof. The proof is similar to that of (2.15). The map $\gamma \mapsto P\gamma$ takes $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1)$ into $\mathcal{L}_{\kappa_1}^{\kappa_2}(PQ_0, PQ_1)$ and is continuous. The map $\gamma \mapsto P^{-1}\gamma$, which is likewise continuous, is its inverse. $\square$

[7] Little works with C^2 curves, but, as we have seen, this is not important.

Of course, we could also consider the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$, consisting of all (κ_1, κ_2)-admissible curves γ having final frame $\Phi_\gamma(1) = Q \in \mathbf{SO}_3$ (but arbitrary initial frame). Like $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$, this space is contractible. To see this, one can go through the definition to check that it is indeed diffeomorphic to $\mathbf{E}$, or, alternatively, one can observe that the map $\gamma \mapsto \bar{\gamma}$, $\bar{\gamma}(t) = \gamma(1-t)$, establishes a homeomorphism

$$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q) \approx \mathcal{L}_{\kappa_1}^{\kappa_2}(QR, \cdot),$$

where

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Finally, we could study the space $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ of all (κ_1, κ_2)-admissible curves, with no conditions placed on the frames. The argument given in the proof of (2.15) shows that

$$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot) \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I, \cdot).$$

Hence, $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ is homeomorphic to $\mathbf{E} \times \mathbf{SO}_3$, and has the homotopy type of $\mathbf{SO}_3$.

Thus, the topology of the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$, $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$ and $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ is uninteresting. We will have nothing else to say about these spaces.

The role of the bounds on the curvature

Having analyzed the significance of condition (i) on p. 14, let us examine next condition (ii). Notice that we have allowed the bounds κ_1, κ_2 on the curvature to be infinite. The definition of radius of curvature is extended accordingly by setting $\operatorname{arccot}(+\infty) = 0$ and $\operatorname{arccot}(-\infty) = \pi$. We can then rephrase (ii) as:

(ii) $\rho(t) \in (\rho_2, \rho_1)$ for each $t \in [0, 1]$.

Here ρ is the radius of curvature of γ and $\rho_i = \operatorname{arccot} \kappa_i \in [0, \pi]$, $i = 1, 2$. The main result of this section relates the topology of $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ to the size $\rho_1 - \rho_2$ of the interval (ρ_2, ρ_1). Its proof relies on the following construction.

Given $-\pi < \theta < \pi$ and an admissible curve $\gamma \colon [0, 1] \to \mathbf{S}^2$, define the *translation* $\gamma_\theta \colon [0, 1] \to \mathbf{S}^2$ of γ by θ to be the curve given by

$$\gamma_\theta(t) = \cos\theta\, \gamma(t) + \sin\theta\, \mathbf{n}(t) \qquad (t \in [0, 1]). \tag{8}$$

Example. Let $0 < \alpha < \frac{\pi}{2}$ and let C be the circle of colatitude α. Depending on the orientation, the translation of C by θ, $0 \le \theta \le \alpha$, is either the circle of colatitude $\alpha + \theta$ or the circle of colatitude $\alpha - \theta$. In particular, taking $\theta = \alpha$ and a suitable orientation of C, the translation degenerates to a single point (the north pole).

This example shows that some care must be taken in the choice of θ for the resulting curve to be admissible.

(2.18) Lemma. *Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be an admissible curve and ρ its radius of curvature. Suppose*

$$\rho_2 < \rho(t) < \rho_1 \ \textit{for a.e. } t \in [0,1] \quad \textit{and} \quad \rho_1 - \pi \leq \theta \leq \rho_2. \tag{9}$$

Then γ_θ is an admissible curve and its frame is given by:

$$\Phi_{\gamma_\theta} = \Phi_\gamma R_\theta\,, \quad \textit{where} \quad R_\theta = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{10}$$

Proof. Let $\Psi = \Phi_\gamma R_\theta$. Since Φ_γ satisfies the differential equation (2), Ψ satisfies

$$\dot{\Psi} = \Psi\,(R_\theta^{-1}\Lambda\, R_\theta).$$

A direct calculation shows that

$$R_\theta^{-1}\Lambda\, R_\theta = \begin{pmatrix} 0 & -\big(\cos\theta\, v - \sin\theta\, w\big) & 0 \\ \cos\theta\, v - \sin\theta\, w & 0 & -\big(\cos\theta\, w + \sin\theta\, v\big) \\ 0 & \cos\theta\, w + \sin\theta\, v & 0 \end{pmatrix},$$

where $v = v(t) = |\dot{\gamma}(t)|$ and $w = w(t) = v(t)\kappa(t)$. Also, $\Psi e_1 = \gamma_\theta$ by construction. To show that γ_θ is admissible, it is thus only necessary to show that

$$\cos\theta\, v(t) - \sin\theta\, w(t) = v(t)\big(\cos\theta - \sin\theta\,\cot\rho(t)\big) = \frac{v(t)}{\sin\rho(t)}\sin(\rho(t)-\theta) > 0$$

for almost every $t \in [0,1]$, and this is true by our choice of θ and the fact that $v > 0$. $\square$

Thus, for θ satisfying (9), we obtain from (10) that the unit tangent vector $\mathbf{t}_\theta$ and unit normal vector $\mathbf{n}_\theta$ to the translation γ_θ of γ are given by:

$$\mathbf{t}_\theta(t) = \mathbf{t}(t) \quad \text{and} \quad \mathbf{n}_\theta(t) = -\sin\theta\,\gamma(t) + \cos\theta\,\mathbf{n}(t) \tag{11}$$

for almost every $t \in [0,1]$.

(2.19) Lemma. *Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be an admissible curve and suppose that* (9) *holds. Then $(\gamma_\theta)_\varphi = \gamma_{\theta+\varphi}$ for any $\varphi \in (-\pi,\pi)$. In particular, $(\gamma_\theta)_{-\theta} = \gamma$.*

Proof. Note that $(\gamma_\theta)_\varphi$ is defined because γ_θ is admissible, as we have just seen. Using (8) and (11) we obtain that

$$(\gamma_\theta)_\varphi = \cos\varphi\big(\cos\theta\,\gamma + \sin\theta\,\mathbf{n}\big) + \sin\varphi\big(-\sin\theta\,\gamma + \cos\theta\,\mathbf{n}\big) = \gamma_{\theta+\varphi}. \qquad \square$$

Given three distinct points on $\mathbf{S}^2$, there is a unique circle passing through them; this circle is also contained in the sphere, for it is the intersection of the unique plane containing the points with $\mathbf{S}^2$. Now consider a C^2 regular curve $\gamma\colon [0,1] \to \mathbf{S}^2$. Fix $t \in [0,1]$, and take distinct $t_1, t_2, t_3 \in [0,1]$. Because the Euclidean curvature $K(t) \neq 0$, the osculating circle to γ at $\gamma(t)$ exists and is equal to the limit position, as t_1, t_2, t_3 approach t, of the unique circle through $\gamma(t_1)$, $\gamma(t_2)$ and $\gamma(t_3)$. Therefore, being a limit of circles contained in the sphere, the osculating circle at any point of γ is also contained in the sphere.

(2.20) Lemma. *Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be C^2 regular and let θ satisfy* (9)*. Then the osculating circle to the translation γ_θ at $\gamma_\theta(t)$ is the translation of the osculating circle to γ at $\gamma(t)$ by θ.*

Proof. Let γ be parametrized by arc-length and let σ be a parametrization, also by arc-length, of the osculating circle to γ at $\gamma(0)$. By definition, the osculating circle is the unique circle in $\mathbf{R}^3$ which has contact of order 3 with γ at $\gamma(0)$; that is, σ must satisfy:

$$\sigma(0) = \gamma(0), \qquad \sigma'(0) = \gamma'(0), \qquad \sigma''(0) = \gamma''(0).$$

In particular, the geodesic curvatures of γ and σ at the point $\gamma(0) = \sigma(0)$ coincide. From these relations and (8) we deduce that $\sigma_\theta(0) = \gamma_\theta(0)$, $\dot\sigma_\theta(0) = \dot\gamma_\theta(0)$. Another calculation shows that

$$\begin{aligned}\ddot\gamma_\theta(0) &= \big(\kappa(0)\sin\theta - \cos\theta\big)\big(\gamma(0) - \kappa(0)\,\mathbf{n}(0)\big) - \kappa'(0)\sin\theta\,\mathbf{t}(0),\\ \ddot\sigma_\theta(0) &= \big(\kappa(0)\sin\theta - \cos\theta\big)\big(\sigma(0) - \kappa(0)\,\mathbf{n}(0)\big).\end{aligned}$$

(Here γ_θ (resp. σ_θ) is parametrized with respect to the arc-length parameter of γ (resp. σ).) This shows that the vector subspaces of $\mathbf{R}^3$ spanned by the two pairs $\{\dot\gamma_\theta(0), \ddot\gamma_\theta(0)\}$ and $\{\dot\sigma_\theta(0), \ddot\sigma_\theta(0)\}$ coincide. Consequently, the image of σ_θ is a circle in the sphere contained in the plane parallel to $\dot\gamma_\theta(0)$ and $\ddot\gamma_\theta(0)$ through $\gamma_\theta(0)$. But there is only one such circle, viz., the osculating circle to γ_θ at $\gamma_\theta(0)$. Since 0 could have been replaced by any $s_0 \in [0,1]$ in this argument, the proof is complete. $\square$

(2.21) Corollary. *Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be an admissible curve and let θ satisfy* (9)*. Then the radius of curvature $\bar\rho$ of γ_θ is given by $\bar\rho = \rho - \theta$.*

Proof. If γ is C^2 regular we can, by (2.20), actually assume that it is a circle. Then an easy direct verification shows that the formula $\bar{\rho} = \rho - \theta$ holds regardless of which orientation we choose. The general case where γ is only admissible can be deduced from this by applying (2.8). $\square$

(2.22) Theorem. *Let $Q \in \mathbf{SO}_3$, $\kappa_1 < \kappa_2$, $\bar{\kappa}_1 < \bar{\kappa}_2$, $\rho_i = \operatorname{arccot} \kappa_i$, $\bar{\rho}_i = \operatorname{arccot} \bar{\kappa}_i$. Suppose that $\rho_1 - \rho_2 = \bar{\rho}_1 - \bar{\rho}_2$. Then $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_{-\theta} Q R_\theta)$, where $\theta = \rho_2 - \bar{\rho}_2$ and*

$$R_\theta = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}.$$

We recall that the bounds κ_i, $\bar{\kappa}_i$ may take on infinite values, and we adopt the conventions that $\operatorname{arccot}(+\infty) = 0$ and $\operatorname{arccot}(-\infty) = \pi$.

Proof. Let $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ and let ρ be its radius of curvature. We have:

$$\rho_2 < \rho(t) < \rho_1 \text{ for a.e. } t \in [0, 1].$$

Set $\theta = \rho_2 - \bar{\rho}_2$. Then (9) is satisfied, so γ_θ is and admissible curve. By (2.21), the radius of curvature $\bar{\rho}$ of γ_θ is given by $\bar{\rho} = \rho - \theta$. Thus,

$$\bar{\rho}_2 < \bar{\rho}(t) < \bar{\rho}_1 \text{ for a.e. } t \in [0, 1].$$

Together with (2.18), this says that $F\colon \gamma \mapsto \gamma_\theta$ maps $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ into $\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta)$. Similarly, translation by $-\theta$ is a map $G\colon \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta) \to \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$. By (2.19), the maps F and G are inverse to each other, hence

$$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta).$$

Finally, because $R_\theta^{-1} = R_{-\theta}$, (2.17) guarantees that

$$\mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_\theta, QR_\theta) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(R_{-\theta} Q R_\theta). \qquad \square$$

(2.23) *Remark.* Taking $Q = I$ we obtain from (2.22) that $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \approx \mathcal{L}_{\bar{\kappa}_1}^{\bar{\kappa}_2}(I)$ (κ_i, $\bar{\kappa}_i$ as in the hypothesis of the theorem). It will also be important to us that under the homeomorphisms of (2.22) and the following corollaries, the image of any circle traversed k times is another circle traversed k times.

(2.24) Corollary. *Let $Q \in \mathbf{SO}_3$ and $\kappa_1 < \kappa_2$. Then $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{-\kappa_0}^{+\kappa_0}(P)$ for suitable $\kappa_0 > 0$, $P \in \mathbf{SO}_3$. Moreover, if $Q = I$ then $P = I$ also.*

Proof. Let $\rho_i = \operatorname{arccot} \kappa_i$, $i = 1, 2$, and set

$$\bar{\rho}_1 = \frac{\pi}{2} + \frac{\rho_1 - \rho_2}{2}, \quad \bar{\rho}_2 = \frac{\pi}{2} - \frac{\rho_1 - \rho_2}{2} \quad \text{and} \quad \kappa_0 = \cot(\bar{\rho}_2).$$

The interval $(\bar{\rho}_2, \bar{\rho}_1)$ has the same size as (ρ_2, ρ_1) by construction. Since $\cot(\bar{\rho}_1) = -\kappa_0$, (2.22) yields that $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{-\kappa_0}^{+\kappa_0}(R_{-\theta}QR_\theta)$, where $\theta = \frac{\rho_1+\rho_2-\pi}{2}$. □

(2.25) Corollary. *Let $Q \in \mathbf{SO}_3$ and $\kappa_1 < \kappa_2$. Then $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\kappa_0}^{+\infty}(P)$ for suitable $\kappa_0 \in [-\infty, +\infty)$ and $P \in \mathbf{SO}_3$. Moreover, if $Q = I$ then $P = I$ also.*

Proof. Let $\rho_i = \operatorname{arccot} \kappa_i$, $i = 1, 2$. Then the interval (ρ_2, ρ_1) has the same size as the interval $(0, \rho_1 - \rho_2)$. Hence, by (2.22), $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q) \approx \mathcal{L}_{\kappa_0}^{+\infty}(R_{-\theta}QR_\theta)$, where

$$\kappa_0 = \cot(\rho_1 - \rho_2) = \frac{1 + \kappa_1\kappa_2}{\kappa_2 - \kappa_1} \quad \text{and} \quad \theta = \rho_2. \qquad \square$$

Corollaries (2.24) and (2.25) both express the fact that, for fixed $Q \in \mathbf{SO}_3$, the topology of the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ depends essentially on one parameter, not two. The spaces of type $\mathcal{L}_{-\kappa_0}^{+\kappa_0}(Q)$ and $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$ have been singled out merely because they are more convenient to work with. For spaces of closed curves we have the following result relating the two classes, which is another simple consequence of (2.24).

(2.26) Corollary. *Let $\kappa_0 \in [-\infty, +\infty)$, $\kappa_1 \in (0, +\infty]$ and $\rho_i = \operatorname{arccot}(\kappa_i)$, $i = 0, 1$. If $\rho_0 = \pi - 2\rho_1$ then $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$.* □

For convenience, we list in table 2.1 all the spaces considered thus far, together with some of the results that we have proved about their topology. As we have already remarked, the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$, $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$ and $\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$ will not be mentioned again.

Space	Definition	Condition on Frames	Topology
$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$	p. 17, (2.6)	$\Phi(0) = I$, $\Phi(1) = Q$	depends on $\rho_1 - \rho_2$, Q
$\mathcal{L}_{\kappa_1}^{\kappa_2}$	p. 22, (2.14)	$\Phi(0) = \Phi(1)$ arbitrary	$\approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$
$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0, Q_1)$	p. 23, (2.16)	$\Phi(0) = Q_0$, $\Phi(1) = Q_1$	$\approx \mathcal{L}_{\kappa_1}^{\kappa_2}(Q_0^{-1}Q_1)$
$\mathcal{L}_{\kappa_1}^{\kappa_2}(Q, \cdot)$	p. 17, (2.5)	$\Phi(0) = Q$, $\Phi(1)$ arbitrary	contractible
$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, Q)$	p. 24	$\Phi(0)$ arbitrary, $\Phi(1) = Q$	contractible
$\mathcal{L}_{\kappa_1}^{\kappa_2}(\cdot, \cdot)$	p. 24	none	$\simeq \mathbf{SO}_3$

Table 2.1: Spaces of spherical curves of bounded geodesic curvature. Here $Q \in \mathbf{SO}_3$, $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ and $\rho_i = \operatorname{arccot}(\kappa_i)$. The notation $X \approx Y$ (resp. $X \simeq Y$) means that X is homeomorphic (resp. homotopy equivalent) to Y.