3
Curves Contained in a Hemisphere

There exists a two-way correspondence between the unit sphere $\mathbf{S}^n$ in $\mathbf{R}^{n+1}$ and the set consisting of its open hemispheres; namely, with $h \in \mathbf{S}^n$ we can associate

$$H = \big\{ p \in \mathbf{S}^n : \langle h, p \rangle > 0 \big\}.$$

Thus the set of open hemispheres of $\mathbf{S}^n$ carries a natural topology. For convenience, we will often identify H with h. In the sequel all hemispheres shall be open, save explicit mention to the contrary, and we will assume throughout that $n \geq 2$.

Let $\gamma\colon [0,1] \to \mathbf{S}^n$ be a (continuous) curve contained in the hemisphere H. As a consequence of the compactness of $[0,1]$, if $\tilde{h} \in \mathbf{S}^n$ is sufficiently close to h, then γ is also contained in the hemisphere $\tilde{H}$ corresponding to $\tilde{h}$. It is desirable to be able to select, in a natural way, a distinguished hemisphere among those which contain γ.

(3.1) Lemma. *Let $\gamma\colon [0,1] \to \mathbf{S}^n$ be contained in a hemisphere. Then the set $\mathcal{H} \subset \mathbf{S}^n$ of hemispheres that contain γ is open, geodesically convex and itself contained in a hemisphere.*[1]

Proof. The hemisphere determined by $\gamma(0)$ contains $\mathcal{H}$ since $\langle h, \gamma(0) \rangle > 0$ for each $h \in \mathcal{H}$. Suppose that the hemispheres $H, \tilde{H}$ corresponding respectively to $h, \tilde{h} \in \mathbf{S}^n$ belong to $\mathcal{H}$. We lose no generality in assuming that

$$h = e_1, \quad \tilde{h} = e^{i\theta_0} = \cos\theta_0\, e_1 + \sin\theta_0\, e_2, \quad \text{where } 0 < \theta_0 < \pi\,.$$[2]

Any k in the shortest geodesic through $h, \tilde{h}$ has the form

$$k = e^{i\theta}, \quad \text{where } 0 \leq \theta \leq \theta_0,$$

while any $p \in \mathbf{S}^n$ satisfying both $\langle p, h \rangle > 0$ and $\langle p, \tilde{h} \rangle > 0$ is of the form

$$p = e^{i\phi} + \nu, \quad \text{where } \theta_0 - \pi/2 < \phi < \pi/2 \text{ and } \nu \text{ is normal to } e_1 \text{ and } e_2.$$

[1] See the appendix for the definition and basic properties of geodesically convex sets.
[2] The use of complex numbers here is made only to simplify the notation.

The bounds on θ and ϕ give $|\theta - \phi| < \pi/2$, hence $\langle p, k\rangle = \cos(\theta - \phi) > 0$. Thus $p \in K$ (the hemisphere determined by k) whenever $p \in H, \tilde{H}$, that is, $\mathcal{H}$ is geodesically convex. Finally, we have already remarked above that $\mathcal{H}$ is open. □

From (3.1) we deduce that the barycenter (in $\mathbf{R}^{n+1}$) of the set $\mathcal{H}$ of hemispheres containing γ is not the origin. Its image under gnomic (i.e., central) projection on the sphere, to be denoted by h_γ, will be our choice of distinguished hemisphere containing γ.

(3.2) Lemma. *Let $r \geq 0$, let $\mathcal{A}$ denote the space of arcs $\gamma\colon [0,1] \to \mathbf{S}^n$, with the C^r topology, and let $\mathcal{S} \subset A$ be the subspace consisting of all γ whose image is contained in some open hemisphere (depending on γ). Then the map $\mathcal{S} \to \mathbf{S}^n$, $\gamma \mapsto h_\gamma$, defined in the preceding paragraph, is continuous.*

Before proving this, we record two results which we will use.

(3.3) Lemma. *Let $C \subset \mathbf{S}^n$ be geodesically convex with non-empty interior. Then there exists a homeomorphism $F\colon \mathbf{S}^{n-1} \to \partial C$ which is bi-Lipschitz.*[3]

Proof. We may assume without loss of generality that C contains $N = e_{n+1}$ in its interior. Let

$$\left\{(p^1, \dots, p^{n+1}) \in \mathbf{S}^n : p^{n+1} = 0\right\}$$

be the equator of $\mathbf{S}^n$, which we identify with $\mathbf{S}^{n-1}$. Because $N \in \mathrm{Int}(C)$, there exists δ, $0 < \delta < 1$, such that the open disk

$$U = \left\{(p^1, \dots, p^{n+1}) \in \mathbf{S}^n :\ (1-\delta) < p^{n+1} \leq 1\right\} \tag{1}$$

is contained in C. In particular, $\partial C \cap U = \emptyset$. Since C cannot contain antipodal points, ∂C is also disjoint from $-U$ (the image of U under the antipodal map). Because $N \in C$ and $-N \notin C$, any semicircle containing them, say, the one that also contains $\sigma \in \mathbf{S}^{n-1}$, intersects ∂C at some point $F(\sigma)$.

Let $p \in \partial C$, $u \in U$. We assert that the semicircle through p, u and $-u$ cannot contain another point $q \in \partial C$ (see fig. 4). If we take $u = N$ then this shows that the definition of $F\colon \mathbf{S}^{n-1} \to \partial C$ is unambiguous. Assume for a contradiction that the assertion is false, and suppose further that q lies between p and u (if it lies between $-u$ and p instead, the argument is analogous). Consider the union of all geodesic segments joining points of U to p. This set contains q in its interior by hypothesis. The same is true of the union of all

[3] This means that there exist $k_1, k_2 > 0$ such that

$$k_1\,|\sigma - \tau| \leq |F(\sigma) - F(\tau)| \leq k_2\,|\sigma - \tau| \ \text{ for any } \sigma, \tau \in \mathbf{S}^{n-1}.$$

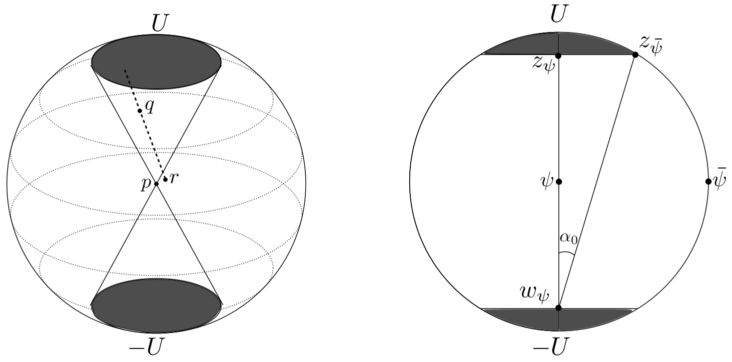

Figure 4: An illustration of part of the proof of (3.3) when $n = 2$.

geodesic segments joining points of U to r, whenever r is sufficiently close to p. Since $p \in \partial C$, we can choose $r \in C$ to conclude from the convexity of C that $q \in \mathrm{Int}(C)$, a contradiction.

Let $\sigma \neq \tau \in \mathbf{S}^{n-1}$. Then

$$\frac{|F(\sigma) - F(\tau)|}{|\sigma - \tau|} \geq \frac{\left|\left(\sqrt{\delta(2-\delta)}\sigma\,,\,1-\delta\right) - \left(\sqrt{\delta(2-\delta)}\tau\,,\,1-\delta\right)\right|}{|\sigma - \tau|}$$
$$\geq \sqrt{\delta(2-\delta)} > 0.$$

Let d denote the distance function on $\mathbf{S}^n$. To establish a reverse Lipschitz condition for F, it suffices to prove that

$$\frac{d\big(F(\sigma), F(\tau)\big)}{d(\sigma, \tau)} = \frac{F(\sigma)F(\tau)}{\sphericalangle F(\sigma)NF(\tau)}$$

admits an upper bound independent of the pair $\sigma \neq \tau$.[4] Since $F(\sigma)F(\tau)$ is bounded by π and $\lim_{x\to 0} \frac{\sin x}{x} = 1$, it actually suffices to establish a bound on

$$\frac{\sin\big(F(\sigma)F(\tau)\big)}{\sin\big(\sphericalangle F(\sigma)NF(\tau)\big)} = \frac{\sin\big(NF(\tau)\big)}{\sin\big(\sphericalangle NF(\sigma)F(\tau)\big)}, \tag{2}$$

where the equality follows from the law of sines (for spherical triangles) applied to $\triangle F(\sigma)NF(\tau)$. For arbitrary $\psi \in \mathbf{S}^{n-1}$, define $z_\psi \in \partial U$ and $w_\psi \in \partial(-U)$ to be the points where the great circle through N and ψ meets ∂U (resp. $\partial(-U)$); more explicitly,

$$z_\psi = \left(\sqrt{\delta(2-\delta)}\,\psi, 1-\delta\right), \qquad w_\psi = \left(\sqrt{\delta(2-\delta)}\,\psi, -1+\delta\right).$$

Let ψ, $\bar\psi \in \mathbf{S}^{n-1}$ satisfy $d(\psi, \bar\psi) = \frac{\pi}{2}$ and let α_0 be the angle at w_ψ in $\triangle w_\psi z_\psi z_{\bar\psi}$. Clearly, this angle is independent of $\psi, \bar\psi$. We claim that $\sphericalangle NF(\sigma)F(\tau) > \alpha_0$.

[4] AB denotes the geodesic segment joining A to B and $\sphericalangle ABC$ the angle at B in the spherical triangle ABC.

Otherwise, the geodesic through $F(\sigma)$ and $F(\tau)$ meets U, and so does the geodesic through p and $F(\tau)$ for p close to $F(\sigma)$, for U is open. Since $F(\sigma) \in \partial C$, we can choose $p \in C$ with this property, which, using the convexity of C, contradicts the fact that $F(\tau) \notin \text{Int}(C)$. Hence, we can complete (2) to

$$\frac{\sin\big(F(\sigma)F(\tau)\big)}{\sin\big(\sphericalangle F(\sigma)NF(\tau)\big)} = \frac{\sin\big(NF(\tau)\big)}{\sin\big(\sphericalangle NF(\sigma)F(\tau)\big)} < \frac{\pi}{\sin\alpha_0},$$

finishing the proof that F is bi-Lipschitz. □

(3.4) Lemma. *Let $A \subset \mathbf{S}^n$ be a closed set of Hausdorff dimension less than n. If B_ε consists of all points at distance less than ε from A, then $\lim_{\varepsilon\to 0} V(B_\varepsilon) = 0$, where V denotes the volume in $\mathbf{S}^n$.*

Proof. Let $\delta(S)$ denote the diameter of a set $S \subset \mathbf{S}^n$ and $\Gamma_\alpha(S)$ its Hausdorff measure of dimension $\alpha > 0$. Since $\Gamma_n(A) = 0$, given any $\eta > 0$ we can cover A by a countable collection of sets $A_k \subset \mathbf{S}^n$ such that $\sum_k \delta(A_k)^n < \eta$. Each A_k can be enclosed in an open ball U_k of diameter $3\delta(A_k)$, and since A is compact, $\bigcup_k U_k$ contains some B_ε. Therefore, the conclusion follows from the estimate

$$V(B_\varepsilon) \leq \sum_k V(U_k) \leq C\sum_k \delta(U_k)^n < 3^n C\eta,$$

where C is the constant, depending only on n, which relates the Hausdorff measure in dimension n to the usual measure (volume). □

Proof of (3.2).. It suffices to prove the result when $\mathcal{A}$ has the C^0 topology, since it is coarser than the C^r topology for any $r \geq 1$.

Let $\gamma \in \mathcal{S}$ and $\mathcal{H}$ (regarded as a subset of $\mathbf{S}^n$) be the set of all open hemispheres containing $\gamma([0,1])$. Let $\varepsilon > 0$ and define

$$B_\varepsilon = \bigcup_{q\in\partial\mathcal{H}} B(q;\varepsilon), \quad \mathcal{H}_0 = \mathcal{H} \smallsetminus B_\varepsilon \quad \text{and} \quad \mathcal{H}_1 = \mathcal{H} \cup B_\varepsilon. \tag{3}$$

Then $\overline{\mathcal{H}}_0 \subset \mathcal{H} \subset \overline{\mathcal{H}} \subset \mathcal{H}_1$. As a consequence of the compactness of $[0,1]$, $\overline{\mathcal{H}}_0$ and $\mathbf{S}^n \smallsetminus \mathcal{H}_1$, there exists $\delta > 0$ for which

$$\langle\gamma(t), u\rangle \geq \delta \text{ if } u \in \mathcal{H}_0 \quad \text{and} \quad \langle\gamma(t), v\rangle \leq -\delta \text{ for } v \notin \mathcal{H}_1 \quad \text{for all } t \in [0,1].$$

Consequently, there exists a neighborhood $\mathcal{U} \subset \mathcal{A}$ of γ such that if $\eta \in \mathcal{U}$ then

$$\langle\eta(t), u\rangle \geq \delta/2 \text{ if } u \in \mathcal{H}_0 \quad \text{and} \quad \langle\eta(t), v\rangle \leq -\delta/2 \text{ for } v \notin \mathcal{H}_1 \quad \text{for all } t \in [0,1].$$

Thus, if $\mathcal{K}$ is the set of hemispheres containing η, we have $\overline{\mathcal{H}}_0 \subset \mathcal{K} \subset \overline{\mathcal{K}} \subset \mathcal{H}_1$.

Without loss of generality, we may assume that the barycenter h_γ of $\mathcal{H}$ is e_{n+1}. Let h_η^j denote the j-th coordinate of the barycenter h_η of $\mathcal{K}$. By definition $h_\eta^j \int_{\mathcal{K}} dx = \int_{\mathcal{K}} x_j\, dx$, and the latter term satisfies

$$\begin{aligned}\int_{\mathcal{K}} x_j dx &= \int_{\mathcal{H}} x_j\, dx + \int_{\mathcal{K}\smallsetminus\mathcal{H}} x_j\, dx - \int_{\mathcal{H}\smallsetminus\mathcal{K}} x_j\, dx \\ &\leq \int_{\mathcal{H}} x_j\, dx + \int_{\mathcal{H}_1\smallsetminus\mathcal{H}_0} 1\, dx - \int_{\mathcal{H}_1\smallsetminus\mathcal{H}_0} (-1)\, dx \\ &= \int_{\mathcal{H}} x_j\, dx + 2\int_{\mathcal{H}_1\smallsetminus\mathcal{H}_0} dx\end{aligned}$$

Since the j-th coordinate h_γ^j of h_γ is non-negative for each j, it follows that

$$h_\eta^j \leq \left(\frac{\int_{\mathcal{H}} dx}{\int_{\mathcal{H}_0} dx}\right) h_\gamma^j + 2\left(\frac{\int_{\mathcal{H}_1-\mathcal{H}_0} dx}{\int_{\mathcal{H}_0} dx}\right) ;$$

similarly,

$$h_\eta^j \geq \left(\frac{\int_{\mathcal{H}} dx}{\int_{\mathcal{H}_1} dx}\right) h_\gamma^j - 2\left(\frac{\int_{\mathcal{H}_1-\mathcal{H}_0} dx}{\int_{\mathcal{H}_0} dx}\right) .$$

The set $\partial\mathcal{H}$ has Hausdorff dimension $n-1$, for it is the image of $\mathbf{S}^{n-1}$ under a Lipschitz map (by (3.1) and (3.3)). We also have:

$$\int_{\mathcal{H}_1} dx \leq \int_{\mathcal{H}} dx + V(B_\varepsilon), \quad \int_{\mathcal{H}_0} dx \geq \int_{\mathcal{H}} dx - V(B_\varepsilon) \quad \text{and} \quad \int_{\mathcal{H}_1\smallsetminus\mathcal{H}_0} dx \leq V(B_\varepsilon).$$

Therefore, according to (3.4), we can make h_η arbitrarily close to h_γ for all $\eta \in \mathcal{U}$ by an adequate choice of ε in (3). In other words, $\gamma \mapsto h_\gamma$ is continuous. □

The following result (for C^1 curves) is quite old; see [5], §1.

(3.5) Lemma. *Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be an admissible closed curve, and let $\mathbf{t}(t)$ denote its unit tangent vector at $\gamma(t)$. Then the curve $\mathbf{t}\colon [0,1] \to \mathbf{S}^2$ intersects any great circle.*

Proof. Let L be the length of γ and $h \in \mathbf{S}^2$ any fixed vector. Since γ is a closed curve,

$$\int_0^L \langle \mathbf{t}(s), h\rangle\, ds = \int_0^L \langle \gamma'(s), h\rangle\, ds = \langle \gamma(L) - \gamma(0), h\rangle = 0.$$

In particular, the function $\langle \mathbf{t}(s), h\rangle$ must vanish for some $s_0 \in [0, L]$. This means that $\mathbf{t}$ intersects the great circle $C = \big\{p \in \mathbf{S}^2 : \langle p, h\rangle = 0\big\}$ at $\mathbf{t}(s_0)$. □