4
The Connected Components of $\mathcal{L}_{\kappa_1}^{\kappa_2}$

The following theorem is the main result of this work. It presents a description of the components of $\mathcal{L}_{\kappa_1}^{\kappa_2}$ in terms of κ_1 and κ_2.

(4.1) Theorem. *Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$, $\rho_i = \operatorname{arccot} \kappa_i$ $(i = 1, 2)$ and $\lfloor x \rfloor$ denote the greatest integer smaller than or equal to x. Then $\mathcal{L}_{\kappa_1}^{\kappa_2}$ has exactly n connected components $\mathcal{L}_1, \dots, \mathcal{L}_n$, where*

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1 \tag{1}$$

and $\mathcal{L}_j$ contains circles traversed j times $(1 \leq j \leq n)$. The component $\mathcal{L}_{n-1}$ also contains circles traversed $(n-1) + 2k$ times, and $\mathcal{L}_n$ contains circles traversed $n + 2k$ times, for $k \in \mathbf{N}$. Moreover, each of $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$ is homotopy equivalent to $\mathbf{SO}_3$ $(n \geq 3)$.

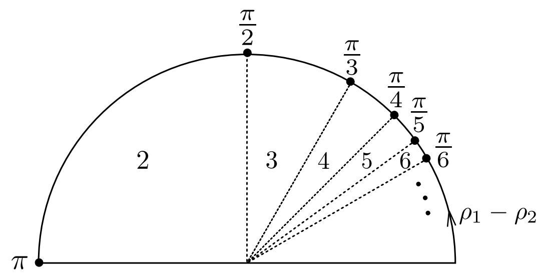

Figure 5: The number of connected components of $\mathcal{L}_{\kappa_1}^{\kappa_2}$, as $\rho_1 - \rho_2$ varies in $(0, \pi]$ (where $\rho_i = \operatorname{arccot} \kappa_i$). When $\rho_1 - \rho_2 = \frac{\pi}{n}$, $\mathcal{L}_{\kappa_1}^{\kappa_2}$ has $n + 1$ components.

If we replace $\mathcal{L}_{\kappa_1}^{\kappa_2}$ by $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ in the statement then the conclusion is the same, except that $\mathcal{L}_1(I), \dots, \mathcal{L}_{n-2}(I)$ are now contractible, and, of course, the circles are required to have initial and final frames equal to I. This is what will actually be proved; the theorem follows from this and the homeomorphism $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$, which was established in (2.15). We could also have replaced $\mathcal{L}_{\kappa_1}^{\kappa_2}$ by the space of all C^r closed curves $(r \geq 2)$ whose geodesic curvatures lie in the interval (κ_1, κ_2), with the C^r topology, since this space is homotopy equivalent to the former, by (2.10).

Examples. Let us first discuss some concrete cases of the theorem.

(a) We have already mentioned (on p. 22) that $\mathcal{L}_{-\infty}^{+\infty} = \mathcal{I} \simeq \mathbf{SO}_3 \times (\Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3)$ has two connected components $\mathcal{I}_+$ and $\mathcal{I}_-$, which are characterized by: $\gamma \in \mathcal{I}_+$ if and only if $\tilde{\Phi}_\gamma(1) = \tilde{\Phi}_\gamma(0)$ and $\gamma \in \mathcal{I}_-$ if and only if $\tilde{\Phi}_\gamma(1) = -\tilde{\Phi}_\gamma(0)$. This is consistent with (4.1).

(b) Suppose $\kappa_0 < 0$. Setting $\rho_2 = 0$ and $\rho_1 = \operatorname{arccot}\kappa_0$ in (4.1), we find that $\mathcal{L}_{\kappa_0}^{+\infty}$ also has two connected components. Since $\mathcal{L}_{\kappa_0}^{+\infty}$ can be considered a subspace of $\mathcal{L}_{-\infty}^{+\infty}$, these components have the same characterization in terms of $\tilde{\Phi}(1)$: two curves $\gamma, \eta \in \mathcal{L}_{\kappa_0}^{+\infty}$ are homotopic if and only if $\tilde{\Phi}_\gamma(1) = \pm\tilde{\Phi}_\gamma(0)$ and $\tilde{\Phi}_\eta(1) = \pm\tilde{\Phi}_\eta(0)$, with the same choice of sign for both curves.

(c) In contrast, $\mathcal{L}_{\kappa_0}^{+\infty}$ has at least three connected components when $\kappa_0 \geq 0$. It has exactly three components in case

$$0 \leq \kappa_0 < \frac{1}{\sqrt{3}}.$$

The case $\kappa_0 = 0$ is Little's theorem ([8], thm. 1). If

$$\frac{1}{\sqrt{3}} \leq \kappa_0 < 1$$

it has four connected components and so forth.

To sum up, as we impose starker restrictions on the geodesic curvatures, a homotopy which existed "before" may now be impossible to carry out. For instance, in any space $\mathcal{L}_{\kappa_0}^{+\infty}$ with $\kappa_0 < 0$, it is possible to deform a circle traversed once into a circle traversed three times. However, in $\mathcal{L}_0^{+\infty}$ this is not possible anymore, which gives rise to a new component.

The first part of theorem (4.1) is an immediate consequence of the following results.

(4.2) Theorem. *Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$. Every curve in $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) lies in the same component as a circle traversed k times, for some $k \in \mathbf{N}$ (depending on the curve).*

(4.3) Theorem. *Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ and let $\sigma_k \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) denote any circle traversed $k \geq 1$ times. Then σ_k, σ_{k+2} lie in the same component of $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) if and only if*

$$k \geq \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor \qquad (\rho_i = \operatorname{arccot}\kappa_i,\ i = 1, 2).$$

The following very simple result will be used implicitly in the sequel; it implies in particular that it does not matter which circle σ_k we choose in (4.2) and (4.3).

(4.4) Lemma. *Let $\sigma, \tilde{\sigma} \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) be parametrized circles traversed the same number of times. Then σ and $\tilde{\sigma}$ lie in the same connected component of $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$).*

Proof. By (2.15), it suffices to prove the result for $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$, since any circle in $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is obtained from a circle in the former space by a rotation and $\mathbf{SO}_3$ is connected. By (2.1), we can assume that both σ and $\tilde{\sigma}$ are parametrized by a multiple of arc-length. Let k be the common number of times that the circles are traversed, let $\rho, \tilde{\rho} \in (\rho_2, \rho_1)$ be their respective radii of curvature (where $\rho_i = \operatorname{arccot}(\kappa_i)$) and define $\rho(s) = (1-s)\rho + s\tilde{\rho}$ for $s \in [0,1]$. Then

$$\begin{aligned}(s,t) \mapsto &\cos\rho(s)(\cos\rho(s), 0, \sin\rho(s)) \\ &+ \sin\rho(s)\big(\sin\rho(s)\cos(2k\pi t)\,,\, \sin(2k\pi t)\,,\, -\cos\rho(s)\cos(2k\pi t)\big),\end{aligned}$$

where $s, t \in [0,1]$, yields the desired homotopy between σ and $\tilde{\sigma}$ in $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$. □

Next we introduce the main concepts and tools used in the proofs of the theorems listed above. From now on we shall work almost exclusively with spaces of type $\mathcal{L}_{\kappa_0}^{+\infty}$ and $\mathcal{L}_{\kappa_0}^{+\infty}(I)$; we are allowed to do so by (2.25).

The bands spanned by a curve

Let $\gamma\colon [0,1] \to \mathbf{S}^2$ be a C^2 regular curve. For $t \in [0,1]$, let $\chi(t)$ (or $\chi_\gamma(t)$) be the center, on $\mathbf{S}^2$, of the osculating circle to γ at $\gamma(t)$.[1] The point $\chi(t)$ will be called the *center of curvature* of γ at $\gamma(t)$, and the correspondence $t \mapsto \chi(t)$ defines a new curve $\chi\colon [0,1] \to \mathbf{S}^2$, the *caustic* of γ. In symbols,

$$\chi(t) = \cos\rho(t)\,\gamma(t) + \sin\rho(t)\,\mathbf{n}(t). \tag{2}$$

Here, as always, $\rho = \operatorname{arccot}\kappa$ is the radius of curvature and $\mathbf{n}$ the unit normal to γ. Note that the caustic of a circle degenerates to a single point, its center. This is explained by the following result.

(4.5) Lemma. *Let $r \geq 2$, $\gamma\colon [0,1] \to \mathbf{S}^2$ be a C^r regular curve and χ its caustic. Then χ is a curve of class C^{r-2}. When χ is differentiable, $\dot{\chi}(t) = 0$ if and only if $\dot{\kappa}(t) = 0$, where κ is the geodesic curvature of γ.*

Proof. If γ is C^r then ρ is a C^{r-2} function, hence χ is also of class C^{r-2}. The proof of the second assertion is a straightforward computation: Using the

[1] There are two possibilities for the center on $\mathbf{S}^2$ of a circle. To distinguish them we use the orientation of the circle, as in fig. 2. The radius of curvature $\rho(t)$ is the distance from $\gamma(t)$ to the center $\chi(t)$, measured along $\mathbf{S}^2$.

arc-length parameter s of γ instead of t, we find that

$$\begin{aligned}\chi'(s) &= \rho'(s)\big(-\sin\rho(s)\,\gamma(s) + \cos\rho(s)\,\mathbf{n}(s)\big) + \big(\cos\rho(s) - \kappa(s)\sin\rho(s)\big)\,\mathbf{t}(s)\\ &= \frac{\kappa'(s)}{1+\kappa(s)^2}\big(\sin\rho(s)\,\gamma(s) - \cos\rho(s)\,\mathbf{n}(s)\big),\end{aligned}$$

where we have used that

$$\cos\rho - \kappa\sin\rho = \sin\rho\,(\cot\rho - \kappa) = 0$$

together with $0 < \rho < \pi$. Therefore, $\chi'(s) = 0$ if and only if $\kappa'(s)$ vanishes. □

(4.6) Definitions. Let $\kappa_0 \in \mathbf{R}$, $\rho_0 = \operatorname{arccot}\kappa_0$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$. Define the *regular band* B_γ and the *caustic band* C_γ to be the maps

$$B_\gamma\colon [0,1]\times[\rho_0-\pi, 0] \to \mathbf{S}^2 \quad \text{and} \quad C_\gamma\colon [0,1]\times[0,\rho_0] \to \mathbf{S}^2$$

given by the same formula:

$$(t,\theta) \mapsto \cos\theta\,\gamma(t) + \sin\theta\,\mathbf{n}(t). \tag{3}$$

The image of C_γ will be denoted by C, and the geodesic circle orthogonal to γ at $\gamma(t)$ will be denoted by Γ_t. As a set,

$$\Gamma_t = \big\{\cos\theta\,\gamma(t) + \sin\theta\,\mathbf{n}(t) \,:\, \theta \in [-\pi,\pi)\big\}.$$

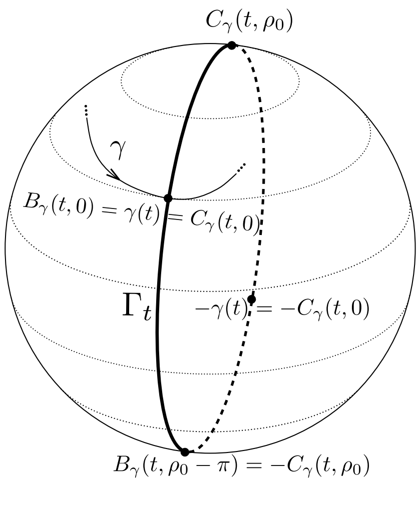

Figure 6:

For fixed t, the images of $\pm B_\gamma(t,\cdot)$ and $\pm C_\gamma(t,\cdot)$ divide the circle Γ_t in four parts. Note also that $\chi_\gamma(t) = C_\gamma(t,\rho(t))$.

(4.7) Lemma. *Let $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ and let $B_\gamma \colon [0,1] \times [\rho_0 - \pi, 0] \to \mathbf{S}^2$ be the regular band spanned by γ. Then:*

(a) *The derivative of B_γ is an isomorphism at every point.*

(b) *$\frac{\partial B_\gamma}{\partial \theta}(t,\theta)$ has norm 1 and is orthogonal to $\frac{\partial B_\gamma}{\partial t}(t,\theta)$. Moreover,*

$$\det\Big(B_\gamma\,,\ \frac{\partial B_\gamma}{\partial t}\,,\ \frac{\partial B_\gamma}{\partial \theta}\Big) > 0.$$

(c) *C_γ fails to be an immersion precisely at the points $(t,\rho(t))$ whose images form the caustic χ.*

Proof. We have:

$$\frac{\partial B_\gamma}{\partial \theta}(t,\theta) = -\sin\theta\,\gamma(t) + \cos\theta\,\mathbf{n}(t). \tag{4}$$

and

$$\frac{\partial B_\gamma}{\partial t}(t,\theta) = |\dot\gamma(t)|\,\big(\cos\theta - \kappa(t)\sin\theta\big)\,\mathbf{t}(t) \tag{5}$$

$$= \frac{|\dot\gamma(t)|}{\sin\rho(t)}\sin(\rho(t)-\theta)\,\mathbf{t}(t), \tag{6}$$

where $\rho(t) = \operatorname{arccot}\kappa(t)$ is the radius of curvature of γ at $\gamma(t)$. The inequality $\kappa_0 < \kappa < +\infty$ translates into $0 < \rho < \rho_0$, hence the factor multiplying $\mathbf{t}(t)$ in (6) is positive for θ satisfying $\rho_0 - \pi \leq \theta \leq 0$, and this implies (a) and (b). Part (c) also follows directly from (6), because C_γ and B_γ are defined by the same formula. □

Thus, B_γ is an immersion (and a submersion) at every point of its domain. It is merely a way of collecting the regular translations of γ (as defined on p. 24) in a single map.

If we fix t and let θ vary in $(0,\rho_0)$, the section $C_\gamma(t,\theta)$ of Γ_t describes the set of "valid" centers of curvature for γ at $\gamma(t)$, in the sense that the circle centered at $C_\gamma(t,\theta)$ passing through $\gamma(t)$, with the same orientation, has geodesic curvature greater than κ_0. This interpretation is important because it motivates many of the constructions that we consider ahead.

Condensed and diffuse curves

(4.8) Definition. Let $\kappa_0 \in \mathbf{R}$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$. We shall say that γ is *condensed* if the image C of C_γ is contained in a closed hemisphere, and *diffuse* if C contains antipodal points (i.e., if $C \cap -C \neq \emptyset$).

Examples. A circle in $\mathcal{L}_{\kappa_0}^{+\infty}$ is always condensed for $\kappa_0 \geq 0$, but when $\kappa_0 < 0$ it may or may not be condensed, depending on its radius. If a curve contains antipodal points then it must be diffuse, since $C_\gamma(t,0) = \gamma(t)$. By the same reason, a condensed curve is itself contained in a closed hemisphere.

There exist curves which are condensed and diffuse at the same time; an example is a geodesic circle in $\mathcal{L}_{\kappa_0}^{+\infty}$, with $\kappa_0 < 0$. There also exist curves which are neither condensed nor diffuse. To see this, let $\mathbf{S}^1$ be identified with the equator of $\mathbf{S}^2$ and let $\zeta \in \mathbf{S}^1$ be a primitive third root of unity. Choose small neighborhoods U_i of ζ^i $(i = 0, 1, 2)$ and V of the north pole in $\mathbf{S}^2$. Then the set G consisting of all geodesic segments joining points of $U_1 \cup U_2 \cup U_3$ to points of V does not contain antipodal points, nor is it contained in a closed hemisphere, by (11.2). By taking $\rho_0 = \operatorname{arccot} \kappa_0$ to be very small, we can construct a curve $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ for which $C = \operatorname{Im}(C_\gamma) \subset G$, but $\zeta^i \in C$ for each i, so that γ is neither condensed nor diffuse.

To sum up, a curve may be condensed, diffuse, neither of the two, or both simultaneously, but this ambiguity is not as important as it seems.

(4.9) Lemma. *Let $\kappa_0 \in \mathbf{R}$ and suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is condensed. Then the image of $\chi = \chi_\gamma$ is contained in an open hemisphere.*

Proof. Let $H = \{p \in \mathbf{S}^2 : \langle p, h\rangle \geq 0\}$ be a closed hemisphere containing the image of C_γ and suppose that $\langle \chi(t_0), h\rangle = 0$ for some $t_0 \in [0,1]$. At least one of $\gamma(t_0)$ or $\mathbf{n}(t_0)$ is not a multiple of $h \times \chi_\gamma(t_0)$. In either case,

$$C_\gamma\big((t_0 - \varepsilon, t_0 + \varepsilon) \times (\rho(t_0) - \varepsilon, \rho(t_0) + \varepsilon)\big) \not\subset H,$$

for sufficiently small $\varepsilon > 0$, a contradiction. $\square$

Let $\kappa_0 \in \mathbf{R}$ and let $\mathcal{O} \subset \mathcal{L}_{\kappa_0}^{+\infty}$ denote the subset of condensed curves. Define a map $h\colon \mathcal{O} \to \mathbf{S}^2$ by $\gamma \mapsto h_\gamma$, where h_γ is the image under gnomic (central) projection of the barycenter, in $\mathbf{R}^3$, of the set of closed hemispheres which contain $C = \operatorname{Im}(C_\gamma)$.

(4.10) Lemma. *The map $h\colon \mathcal{O} \to \mathbf{S}^2$, $\gamma \mapsto h_\gamma$, defined above is continuous.*

Proof. Consider first the subset $\mathcal{S} \subset \mathcal{L}_{\kappa_0}^{+\infty}$ consisting of all curves γ such that $\operatorname{Im}(C_\gamma)$ is contained in an open hemisphere. A minor modification in the proof of (3.1) shows that, in this case, the set $\mathcal{H}$ of closed hemispheres which contain γ is geodesically convex, open and contained in an open hemisphere. Thus, we may apply (3.3) and (3.4) to $\mathcal{H}$ and $\partial\mathcal{H}$, respectively. Using these, the proof of (3.2) goes through almost unchanged to establish that the restriction of h to $\mathcal{S}$ is continuous.

It remains to prove that h is continuous at any curve $\gamma \in \mathcal{O} \smallsetminus \mathcal{S}$. Note first that there exists exactly one closed hemisphere h_γ containing $\operatorname{Im}(C_\gamma)$ in this case. For if $C = \operatorname{Im}(C_\gamma)$ is contained in distinct closed hemispheres H_1 and H_2, then it is contained in the closed lune $H_1 \cap H_2$. The boundary of $\operatorname{Im}(C_\gamma)$ is contained in the union of the images of $\gamma = C_\gamma(\cdot, 0)$ and $\check{\gamma} = C_\gamma(\cdot, \rho_0)$; since these curves have a unit tangent vector at all points, they cannot pass through either of the points in $E_1 \cap E_2$ (where E_i is the equator corresponding to H_i). It follows that $\operatorname{Im}(C_\gamma)$ is contained in an open hemisphere, a contradiction. Furthermore, by (11.1), (11.2) and (11.5), we can find

$$z_i = C_\gamma(t_i, \theta_i) \in \operatorname{Im}(C_\gamma) \cap \big\{p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle = 0\big\} \quad (\theta_i \in \{0, \rho_0\},\ i = 1, 2, 3)$$

such that 0 lies in the simplex spanned by z_1, z_2, z_3; any hemisphere other than $\pm h_\gamma$ separates these three points. Let $z_0 = C_\gamma(t_0, \theta_0)$ be a point in $\operatorname{Im}(C_\gamma)$ satisfying $\langle z_0, h_\gamma \rangle > 0$. Then we may choose $\delta > 0$ and a sufficiently small neighborhood $\mathcal{U}$ of γ in $\mathcal{L}_{\kappa_0}^{+\infty}$ such that $\langle C_\eta(t_0, \theta_0), k \rangle < 0$ for any $\eta \in \mathcal{U}$ and $k \in \mathbf{S}^2$ satisfying $d(k, h_\gamma) \geq \pi - \delta$ (where d denotes the distance function on $\mathbf{S}^2$). By reducing $\mathcal{U}$ if necessary, we can also arrange that if $\delta \leq d(k, h_\gamma) \leq \pi - \delta$, then the hemisphere corresponding to k separates $\{C_\eta(t_i, \theta_i),\ i = 1, 2, 3\}$ whenever $\eta \in \mathcal{U}$. The conclusion is that if $k \in \mathbf{S}^2$ satisfies $\langle c, k \rangle \geq 0$ for all $c \in \operatorname{Im}(C_\eta)$ and $\eta \in \mathcal{U}$, then $d(k, h_\gamma) < \delta$. It follows that h is continuous at $\gamma \in \mathcal{O} \smallsetminus \mathcal{S}$. $\square$

An argument entirely similar to that given above can be used to modify (3.2) as follows.

(4.11) Lemma. *Let $\kappa_0 \in \mathbf{R}$ and $\mathcal{H} \subset \mathcal{L}_{\kappa_0}^{+\infty}$ be the subspace consisting of all γ whose image is contained in some c l o s e d hemisphere (depending on γ). Then the map $h\colon \mathcal{H} \to \mathbf{S}^2$, which associates to γ the barycenter h_γ on $\mathbf{S}^2$ of the set of closed hemispheres that contain γ, is continuous.* $\square$