5
Grafting

(5.1) Definition. Let $\gamma\colon [a,b] \to \mathbf{S}^2$ be an admissible curve. The *total curvature* $\mathrm{tot}(\gamma)$ of γ is given by

$$\mathrm{tot}(\gamma) = \int_a^b K(t)\,|\dot\gamma(t)|\;dt,$$

where

$$K = \sqrt{1+\kappa^2} = \csc\rho \tag{1}$$

is the Euclidean curvature of γ. We say that $\gamma\colon [0,T] \to \mathbf{S}^2$, $u \mapsto \gamma(u)$, is a *parametrization of* γ *by curvature* if

$$\left|\Phi_\gamma'(u)\right| = \sqrt{2} \text{ or, equivalently, } \left|\tilde\Phi_\gamma'(u)\right| = \frac{1}{2} \text{ for a.e. } u \in [0,T].$$

The equivalence of the two equalities comes from (2.11). The next result justifies our terminology.

(5.2) Lemma. *Let* $\gamma\colon [0,T] \to \mathbf{S}^2$ *be an admissible curve. Then:*

(a) γ *is parametrized by curvature if and only if*

$$\mathrm{tot}\left(\gamma|_{[0,u]}\right) = u \textit{ for every } u \in [0,T].$$

(b) If γ *is parametrized by curvature then its logarithmic derivatives* $\Lambda = \Phi_\gamma^{-1}\Phi_\gamma'$ *and* $\tilde\Lambda = \tilde\Phi_\gamma^{-1}\tilde\Phi'$ *are given by:*

$$\Lambda(u) = \begin{pmatrix} 0 & -\sin\rho(u) & 0 \\ \sin\rho(u) & 0 & -\cos\rho(u) \\ 0 & \cos\rho(u) & 0 \end{pmatrix},$$
$$\tilde\Lambda(u) = \frac{1}{2}\big(\cos\rho(u)\boldsymbol{i} + \sin\rho(u)\boldsymbol{k}\big).$$

Here, as always, ρ is the radius of curvature of γ. In the expression for $\tilde\Lambda$ above and in the sequel we are identifying the Lie algebra $\tilde{\mathfrak{so}}_3 = T_{\mathbf{1}}\mathbf{S}^3$ (the tangent space to $\mathbf{S}^3$ at $\mathbf{1}$) with the vector space of all imaginary quaternions.

Also, it follows from (a) that if $\gamma\colon [0,T] \to \mathbf{S}^2$ is parametrized by curvature then $T = \operatorname{tot}(\gamma)$.

Proof. Let us denote differentiation with respect to u by $'$. Using (1), we deduce that

$$\Lambda(u) = |\gamma'(u)| \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\kappa(u) \\ 0 & \kappa(u) & 0 \end{pmatrix} \tag{2}$$

$$= K(u)\,|\gamma'(u)| \begin{pmatrix} 0 & -\sin\rho(u) & 0 \\ \sin\rho(u) & 0 & -\cos\rho(u) \\ 0 & \cos\rho(u) & 0 \end{pmatrix}, \tag{3}$$

hence $|\Phi'(u)| = |\Lambda(u)| = \sqrt{2}\,K(u)\,|\gamma'(u)|$. Therefore, γ is parametrized by curvature if and only if

$$K(u)\,|\gamma'(u)| = 1 \ \text{ for a.e. } u \in [0,T].$$

Integrating we deduce that this is equivalent to

$$\operatorname{tot}(\gamma|_{[0,u]}) = u \text{ for every } u \in [0,T],$$

which proves (a). The expression for $\tilde{\Lambda}$ is obtained from (2), using that under the isomorphism $\widetilde{\mathfrak{so}}_3 \to \mathfrak{so}_3$ induced by the projection $\mathbf{S}^3 \to \mathbf{SO}_3$, $\frac{\boldsymbol{i}}{2}$, $\frac{\boldsymbol{j}}{2}$ and $\frac{\boldsymbol{k}}{2}$ correspond respectively to

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \qquad \square$$

We now introduce the essential notion of grafting.

(5.3) Definition. Let $\gamma_i\colon [0,T_i] \to \mathbf{S}^2$ $(i = 0,1)$ be admissible curves parametrized by curvature.

(a) A *grafting function* is a function $\phi\colon [0,s_0] \to [0,s_1]$ of the form

$$\phi(t) = t + \sum_{x<t,\, x\in X^+} \delta^+(x) + \sum_{x\leq t,\, x\in X^-} \delta^-(x), \tag{4}$$

where $X^+ \subset [0,s_0)$ and $X^- \subset [0,s_0]$ are countable sets and $\delta^\pm\colon X^\pm \to (0,+\infty)$ are arbitrary functions.

(b) We say that γ_1 is *obtained from* γ_0 *by grafting*, denoted $\gamma_0 \preccurlyeq \gamma_1$, if there exists a grafting function $\phi\colon [0,T_0] \to [0,T_1]$ such that $\Lambda_{\gamma_0} = \Lambda_{\gamma_1} \circ \phi$.

(c) Let J be an interval (not necessarily closed). A *chain of grafts* consists of a homotopy $s \mapsto \gamma_s$, $s \in J$, and a family of grafting functions $\phi_{s_0,s_1}\colon [0,s_0] \to [0,s_1]$, $s_0 < s_1 \in J$, such that:

(i) $\Lambda_{\gamma_{s_0}} = \Lambda_{\gamma_{s_1}} \circ \phi_{s_0,s_1}$ whenever $s_0 < s_1$;

(ii) $\phi_{s_0,s_2} = \phi_{s_1,s_2} \circ \phi_{s_0,s_1}$ whenever $s_0 < s_1 < s_2$.

Here every curve is admissible and parametrized by curvature.

(5.4) *Remarks.*

(a) A function $\phi\colon [0,s_0] \to [0,s_1]$, $s_0 \leq s_1$, is a grafting function if and only if it is increasing and there exists a countable set $X \subset [0,s_0]$ such that $\phi(t) = t + c$ whenever t belongs to one of the intervals which form $(0,s_0) \smallsetminus X$, where $c \geq 0$ is a constant depending on the interval.

(b) Observe that in eq. (4), $x < t$ in the first sum, while $x \leq t$ in the second sum. We do not require X^+ and X^- to be disjoint, and they may be finite (or even empty).

(c) If $\phi\colon [0,s_0] \to [0,s_1]$ is a grafting function then it is monotone increasing and has derivative equal to 1 a.e.. Moreover, $\phi(t+h) - \phi(t) \geq h$ for any t and $h \geq 0$; in particular, $s_0 \leq s_1$.

(d) As the name suggests, $\gamma_0 \preccurlyeq \gamma_1$ if γ_1 is obtained by inserting a countable number of pieces of curves (e.g., arcs of circles) at chosen points of γ_0 (see fig. 9). This can be used, for instance, to increase the total curvature of a curve. The difficulty is that it is usually not clear how we can graft pieces of curves onto a closed curve so that the resulting curve is still closed and the restrictions on the geodesic curvature are not violated.

(e) Two curves $\gamma_0, \gamma_1 \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$ agree if and only if $\Lambda_{\gamma_0} = \Lambda_{\gamma_1}$ a.e. on $[0,1]$. Indeed, $\gamma_i = \Phi_{\gamma_i} e_1$, where Φ_{γ_i} is the unique solution to an initial value problem as in eq. (4) of §1. Of course, if the curves are parametrized by curvature instead, then the latter condition should be replaced by $T_0 = T_1$ and $\Lambda_{\gamma_0} = \Lambda_{\gamma_1}$ a.e. on $[0,T_0] = [0,T_1]$.

For a grafting function $\phi\colon [0,s_0] \to [0,s_1]$ and $t \in [0,s_0]$, define:

$$\omega^+(t) = \lim_{h \to 0^+} \phi(t+h) - \phi(t), \qquad \omega^-(t) = \lim_{h \to 0^+} \phi(t) - \phi(t-h).$$

We also adopt the convention that $\omega^+(s_0) = 0$, while $\omega^-(0) = \phi(0)$. Note that the limits above exist because ϕ is increasing.

(5.5) Lemma. *Let $\phi\colon [0,s_0] \to [0,s_1]$ be a grafting function, and let $X^\pm$ and $\delta^\pm$ be as in definition (5.3(a)).*

(a) $t \in X^{\pm}$ *if and only if* $\omega^{\pm}(t) > 0$. *In this case,* $\delta^{\pm}(t) = \omega^{\pm}(t)$.

(b) $X^{\pm}$ *and* $\delta^{\pm}$ *are uniquely determined by* ϕ.

(c) *If* $\phi_0 \colon [0, s_0] \to [0, s_1]$ *and* $\phi_1 \colon [0, s_1] \to [0, s_2]$ *are grafting functions then so is* $\phi = \phi_1 \circ \phi_0$. *Moreover,*

$$X_0^{\pm} \subset X^{\pm} \quad \textit{and} \quad \delta_0^{\pm} \leq \delta^{\pm}.$$

(Here $\delta_0^{\pm}$ *correspond to* ϕ_0, $\delta^{\pm}$ *correspond to* ϕ, *and so forth.)*

Proof. The proof will be split into parts.

(a) Firstly, $\omega^+(s_0) = 0$ by convention and $s_0 \notin X^+$ because $X^+ \subset [0, s_0)$. Secondly, $\omega^-(0) = \phi(0)$ by convention, and (4) tells us that $0 \in X^-$ if and only if $\phi(0) \neq 0$, in which case $\delta^-(0) = \phi(0)$. This proves the assertion for $t = 0$ (resp. $t = s_0$) and X^- (resp. X^+).

Since
$$\sum_{x \in X^+} \delta^+(x) + \sum_{x \in X^-} \delta^-(x) \leq s_1 - s_0,$$
given $\varepsilon > 0$ there exist finite subsets $F^{\pm} \subset X^{\pm}$ such that
$$\sum_{x \in X^+ \smallsetminus F^+} \delta^+(x) + \sum_{x \in X^- \smallsetminus F^-} \delta^-(x) < \varepsilon.$$

Suppose $t \notin X^+$, $t < s_0$. Then there exists η, $0 < \eta < \varepsilon$, such that $[t, t+\eta] \cap F^+ = \emptyset$ and $[t, t+\eta] \cap F^-$ is either empty or $\{x\}$. In any case,
$$\omega^+(t) \leq \phi(t+\eta) - \phi(t) < \eta + \varepsilon < 2\varepsilon,$$
which proves that $\omega^+(t) = 0$.

Conversely, suppose that $t \in X^+$. Then clearly $\omega^+(t) \geq \delta^+(t)$. Moreover, an argument entirely similar to the one above shows that $\omega^+(t) \leq \delta^+(t) + 2\varepsilon$ for any $\varepsilon > 0$, hence $\omega^+(t) = \delta^+(t) > 0$. The results for X^- (and $t > 0$) follow by symmetry.

(b) Since $\omega^{\pm}$ are determined by ϕ, the same must be true of $X^{\pm}$ and $\delta^{\pm}$, by part (a). The converse is an obvious consequence of the definition of grafting function in (4).

(c) Let ϕ_1, ϕ_0 be as in the statement and set $X_i = X_i^- \cup X_i^+$, $i = 0, 1$, and $X = X_0 \cup \phi_0^{-1}(X_1)$. Then X is countable since both X_0 and X_1 are countable and ϕ_0 is injective. Moreover, if $(a,b) \subset (0,s_0) \smallsetminus X$ then

$$\phi_1(\phi_0(t)) = \phi_1(t + c_0) = t + c_0 + c_1 \quad (t \in (a,b))$$

for some constants $c_0, c_1 \geq 0$. In addition, $\phi_1 \circ \phi_0$ is increasing, as ϕ_1 and ϕ_0 are both increasing. Thus, $\phi_1 \circ \phi_0$ is a grafting function by (5.4(e)).

For the second assertion, let $x \in X_0^+$ and $h > 0$ be arbitrary. Then

$$\phi_1(\phi_0(x+h)) - \phi_1(\phi_0(x)) \geq \phi_0(x+h) - \phi_0(x) \geq \omega_0^+(x),$$

hence $\omega^+(x) \geq \omega_0^+(x) > 0$. Similarly, if $x \in X_0^-$ then $\omega^-(x) \geq \omega_0^-(x) > 0$. Therefore, it follows from part (a) that $X_0^\pm \subset X^\pm$ and $\delta_0^\pm \leq \delta^\pm$. □

(5.6) Lemma. *The grafting relation $\preccurlyeq$ is a partial order over $\mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$.*

Proof. Suppose γ_0, γ_1 are as in (5.3), with $\gamma_0 \preccurlyeq \gamma_1$ and $\gamma_1 \preccurlyeq \gamma_0$. Let $\phi_0\colon [0,T_0] \to [0,T_1]$ and $\phi_1\colon [0,T_1] \to [0,T_0]$ be the corresponding grafting functions. By (5.4(d)), the existence of such functions implies that $T_0 = T_1$, which, in turn, implies that $\phi_0(t) = t = \phi_1(t)$ for all t. Hence $\Lambda_{\gamma_0} = \Lambda_{\gamma_1} \circ \phi_0 = \Lambda_{\gamma_1}$, and it follows that $\gamma_0 = \gamma_1$. This proves that $\preccurlyeq$ is antisymmetric.

Now suppose $\gamma_0 \preccurlyeq \gamma_1$, $\gamma_1 \preccurlyeq \gamma_2$ and let $\phi_i\colon [0,T_i] \to [0,T_{i+1}]$ be the corresponding grafting functions, $i = 0,1$. By (5.5(c)), $\phi = \phi_1 \circ \phi_0$ is also a grafting function. Furthermore,

$$\Lambda_{\gamma_0} = \Lambda_{\gamma_1} \circ \phi_0 = (\Lambda_{\gamma_2} \circ \phi_1) \circ \phi_0 = \Lambda_{\gamma_2} \circ \phi$$

by hypothesis, so $\gamma_0 \preccurlyeq \gamma_2$, proving that $\preccurlyeq$ is transitive.

Finally, it is clear that $\preccurlyeq$ is reflexive. □

(5.7) Lemma. *Let $\Gamma = (\gamma_s)_{s \in [a,b)}$, $\gamma_s \in \mathcal{L}_{\kappa_1}^{\kappa_2}(Q)$, be a chain of grafts. Then there exists a unique extension of Γ to a chain of grafts on $[a,b]$.*

Proof. For $s_0 < s_1 \in [a,b]$, let $\phi_{s_0,s_1}\colon [0,s_0] \to [0,s_1]$ be the grafting function corresponding to $\gamma_{s_0} \preccurlyeq \gamma_{s_1}$ and similarly for $X_{s_0,s_1}^\pm$, $\delta_{s_0,s_1}^\pm$, $\omega_{s_0,s_1}^\pm$.

Suppose $s_0 < s_1 < s_2$. By hypothesis, $\phi_{s_0,s_2} = \phi_{s_1,s_2} \circ \phi_{s_0,s_1}$. Therefore, by (5.5(c)),

$$X_{s_0,s_1}^\pm \subset X_{s_0,s_2}^\pm \quad \text{and} \quad \delta_{s_0,s_1}^\pm \leq \delta_{s_0,s_2}^\pm \quad (s_0 < s_1 < s_2). \tag{5}$$

Fix $s_0 \in [a,b)$ and set

$$X_{s_0,b}^\pm = \bigcup_{s_0 < s < b} X_{s_0,s}^\pm \quad \text{and} \quad \delta_{s_0,b}^\pm = \sup_{s_0 < s < b} \left\{\delta_{s_0,s}^\pm\right\}.$$

Since $(X^{\pm}_{s_0,s})$ is an increasing family of countable sets, $X^{\pm}_{s_0,b}$ must also be countable. Define $\phi_{s_0,b}\colon [0,s_0] \to [0,b]$ by

$$\phi_{s_0,b}(t) = t + \sum_{x<t,\, x\in X^{+}_{s_0,b}} \delta^{+}_{s_0,b}(x) + \sum_{x\leq t,\, x\in X^{-}_{s_0,b}} \delta^{-}_{s_0,b}(x).$$

Then $\phi_{s_0,b}$ is a grafting function for any s_0 by construction, and for $s_0 < s_1$ we have

$$\phi_{s_0,b} = \lim_{s\to b-} \phi_{s_0,s} = \lim_{s\to b-} \phi_{s_1,s} \circ \phi_{s_0,s_1} = \phi_{s_1,b} \circ \phi_{s_0,s_1}.$$

Before defining the curve γ_b, we construct its logarithmic derivative Λ. For each $s < b$, let

$$E_s = \phi_{s,b}\big([0,s]\big), \quad E = \bigcup_{s<b} E_s.$$

Then $\mu(E_s) = s$ for all s, hence $[0,b] \smallsetminus E$ has measure zero, which implies that E is measurable and $\mu(E) = b$. (Here μ denotes Lebesgue measure.) For $u \in E$, $u = \phi_{s,b}(t)$ for some $t \in [0,s]$ and $s \in [a,b)$, set

$$\Lambda(u) = \Lambda(\phi_{s,b}(t)) = \Lambda_s(t) \qquad (u \in E), \tag{6}$$

where Λ_s denotes the logarithmic derivative of γ_s. Observe that Λ is well-defined, for if $\phi_{s_0,b}(t_0) = u = \phi_{s_1,b}(t_1)$, with $s_0 < s_1$, then

$$\phi_{s_1,b}(t_1) = \phi_{s_0,b}(t_0) = \phi_{s_1,b} \circ \phi_{s_0,s_1}(t_0),$$

hence $t_1 = \phi_{s_0,s_1}(t_0)$ (because ϕ_{s_0,s_1} is increasing) and thus

$$\Lambda_{s_1}(t_1) = \Lambda_{s_1}(\phi_{s_0,s_1}(t_0)) = \Lambda_{s_0}(t_0).$$

Moreover, by (5.2),

$$\Lambda(u) = \begin{pmatrix} 0 & -\sin\rho(u) & 0 \\ \sin\rho(u) & 0 & -\cos\rho(u) \\ 0 & \cos\rho(u) & 0 \end{pmatrix}$$

where $\rho(u) = \rho_{s_0}(t)$ if $u = \phi_{s_0,b}(t)$. The measurability of ρ follows from that of each ρ_s. Thus, the entries of Λ belong to $L^2[0,b]$ and the initial value problem $\dot\Phi = \Phi\Lambda$, $\Phi(0) = I$, has a unique solution $\Phi\colon [0,b] \to \mathbf{SO}_3$. Naturally, we define $\gamma_b(t) = \Phi(t)e_1$.

Let $X_{s,b} = X^{+}_{s,b} \cup X^{-}_{s,b}$ and suppose that (α,β) is one of the intervals which form $(0,s) \smallsetminus X_{s,b}$. Then $\phi_{s,b}(\alpha,\beta) \subset E_s \subset [0,b]$ is an interval of measure

$\beta - \alpha$; we have $\Lambda(t) = \Lambda_s(t-c)$ for $t \in \phi_{s,b}(\alpha,\beta)$ and a constant $c \geq 0$, so that the restriction of γ_b to this interval is just $\gamma_s|[\alpha,\beta]$ composed with a rotation of $\mathbf{S}^2$. In particular, we deduce that the geodesic curvature κ of γ_b satisfies $\kappa_1 < \kappa < \kappa_2$ a.e. on $\phi_s(\alpha,\beta)$. Since $\lim_{s\to b}\mu(E_s) = b$, this argument shows that $\kappa_1 < \kappa < \kappa_2$ a.e. on $[0,b]$. We claim also that $\Phi(b) = Q$. To see this, let $\bar{\Lambda}_s\colon [0,b] \to \mathfrak{so}_3$ be the extension of Λ_s by zero to all of $[0,b]$. If $\bar{\Phi}_s$ is the solution to the initial value problem $\dot{\bar{\Phi}}_s = \bar{\Phi}_s\bar{\Lambda}_s$, $\bar{\Phi}_s(0) = I$, we have $\Phi_s(b) = \Phi_s(s) = Q$. Since $\bar{\Lambda}_s$ converges to Λ in the L^2-norm, it follows from continuous dependence on the parameters of a differential equation that

$$|\Phi(b) - Q| = \lim_{s\to b}|\Phi(b) - \Phi_s(b)| = 0.$$

The curve γ_b satisfies $\gamma_s \preccurlyeq \gamma_b$ for any $s \leq b$ by construction. Conversely, if this condition is satisfied then (6) must hold, showing that γ_b is the unique curve with this property. This completes the proof. □

Adding loops

This subsection presents adaptations of a few concepts and results contained in §5 of [12]. Let $\kappa_0 \in \mathbf{R}$, $\rho_0 = \operatorname{arccot}\kappa_0$ and $Q \in \mathbf{SO}_3$ be fixed throughout the discussion.

For arbitrary $\rho_1 \in (0,\rho_0)$, define σ^{ρ_1} to be the unique circle in $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ of radius of curvature ρ_1:

$$\begin{aligned}\sigma^{\rho_1}(t) =& \cos\rho_1(\cos\rho_1, 0, \sin\rho_1)\\ &+ \sin\rho_1\big(\sin\rho_1\cos(2\pi t), \sin(2\pi t), -\cos\rho_1\cos(2\pi t)\big),\end{aligned}$$

and let $\sigma_n^{\rho_1} \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ be σ^{ρ_1} traversed n times; in symbols, $\sigma_n^{\rho_1}(t) = \sigma^{\rho_1}(nt)$, $t \in [0,1]$. As we have seen in (4.4), if $\rho_1, \rho_2 < \rho_0$ then σ^{ρ_1} and σ^{ρ_2} are homotopic within $\mathcal{L}_{\kappa_0}^{+\infty}(I)$.

Now let $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$, $n \in \mathbf{N}$, $\varepsilon > 0$ be small and $t_0 \in (0,1)$. Let $\gamma^{[t_0\#n]}$ be the curve obtained by inserting (a suitable rotation of) $\sigma_n^{\rho_1}$ at $\gamma(t_0)$, as depicted in fig. 7. More explicitly,

$$\gamma^{[t_0\#n]}(t) = \begin{cases} \gamma(t) & \text{if } 0 \leq t \leq t_0 - 2\varepsilon \\ \gamma(2t - t_0 + 2\varepsilon) & \text{if } t_0 - 2\varepsilon \leq t \leq t_0 - \varepsilon \\ \Phi_\gamma(t_0)\sigma_n^{\rho_1}\left(\frac{t - t_0 + \varepsilon}{2\varepsilon}\right) & \text{if } t_0 - \varepsilon \leq t \leq t_0 + \varepsilon \\ \gamma(2t - t_0 - 2\varepsilon) & \text{if } t_0 + \varepsilon \leq t \leq t_0 + 2\varepsilon \\ \gamma(t) & \text{if } t_0 + 2\varepsilon \leq t \leq 1 \end{cases}$$

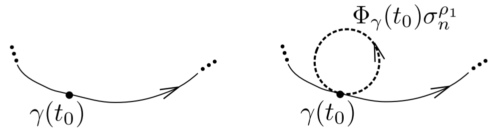

Figure 7: A curve $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$ and the curve $\gamma^{[t_0\#n]}$ obtained from γ by adding loops at $\gamma(t_0)$.

The precise values of ε and ρ_1 are not important, in the sense that different values of both parameters yield curves that are homotopic. For $t_0 \neq t_1 \in (0,1)$ and $n_0, n_1 \in \mathbf{N}$, the curve $\left(\gamma^{[t_0\#n_0]}\right)^{[t_1\#n_1]}$ will be denoted by $\gamma^{[t_0\#n_0;t_1\#n_1]}$.

We shall now explain how to spread loops along a curve, as in fig. 8; to do this, a special parametrization is necessary. Given $\gamma \in \mathcal{L}_{-\infty}^{+\infty}(Q)$, let $\Lambda_\gamma = (\Phi_\gamma)^{-1}\dot{\Phi}_\gamma \colon [0,1] \to \mathfrak{so}_3$ denote its logarithmic derivative. Since the entries of Λ_γ are L^2 functions and $[0,1]$ is bounded,

$$M = \int_0^1 |\Lambda_\gamma(t)|\,dt < +\infty. \tag{7}$$

Define a function $\tau \colon [0,1] \to [0,1]$ by

$$\tau(t) = \frac{1}{M}\int_0^t |\Lambda_\gamma(u)|\,du.$$

Then τ is a monotone increasing function, hence it admits an inverse. If we reparametrize γ by $\tau \mapsto \gamma(t(\tau))$, $\tau \in [0,1]$, then its logarithmic derivative with respect to τ satisfies

$$|\Lambda_\gamma(\tau)| = |\dot{\Phi}_\gamma(t(\tau))|\,\dot{t}(\tau) = |\Lambda_\gamma(t(\tau))|\,\frac{M}{|\Lambda_\gamma(t(\tau))|} = M.$$ [1]

Therefore, using (2.1), we may assume at the outset that all curves $\gamma \in \mathcal{L}_{-\infty}^{+\infty}(Q)$ are parametrized so that $|\dot{\Phi}_\gamma| = |\Lambda_\gamma|$ is constant (and finite). With this assumption in force, let $n \in \mathbf{N}$, $\rho_1 \in (0,\pi)$ and define a map $F_n \colon \mathcal{L}_{-\infty}^{+\infty}(Q) \to \mathcal{L}_{-\infty}^{+\infty}(Q)$ by:

$$F_n(\gamma)(t) = \Phi_\gamma(t)\sigma_n^{\rho_1}(t) \quad (\gamma \in \mathcal{L}_{-\infty}^{+\infty}(Q),\ t \in [0,1]). \tag{8}$$

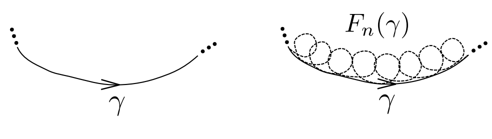

Figure 8:

[1]The parameter τ is a multiple of the curvature parameter considered in (5.1).

Using that $\dot{\Phi}_\gamma = \Phi_\gamma \Lambda_\gamma$ (where $\dot{}$ denotes differentiation with respect to t), we find that

$$\dot{F}_n(\gamma) = \Phi_\gamma\big(\Lambda_\gamma \sigma_n^{\rho_1} + \dot{\sigma}_n^{\rho_1}\big), \tag{9}$$

and this allows us to conclude that $\Phi_{F_n(\gamma)}(0) = \Phi_\gamma(0)$ and $\Phi_{F_n(\gamma)}(1) = \Phi_\gamma(1)$ for any admissible curve γ, so that F_n does indeed map $\mathcal{L}_{-\infty}^{+\infty}(Q)$ to itself. Moreover, $F_n(\gamma)$ is never homotopic to $F_m(\gamma)$ when $m \not\equiv n \pmod 2$. This is because the two curves have different final lifted frames: $\tilde{\Phi}_{F_n(\gamma)}(1) = (-1)^{n-m}\tilde{\Phi}_{F_m(\gamma)}(1)$ in $\mathbf{S}^3$.

(5.8) Lemma. *Let $\kappa_0 = \cot\rho_0 \in \mathbf{R}$, $Q \in \mathbf{SO}_3$, $\rho_1 \in (0,\rho_0)$, K be compact and $f\colon K \to \mathcal{L}_{-\infty}^{+\infty}(Q)$ be continuous. Then the image of $F_n \circ f$ is contained in $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$ for all sufficiently large n.*

Proof. In order to simplify the notation, we will prove the lemma when K consists of a single point. The proof still works in the more general case because all that we need is a uniform bound on $|\Lambda_{f(a)}|$ for $a \in K$. Denoting $\sigma_1^{\rho_1}$ simply by σ, we may rewrite (9) as:

$$\dot{F}_n(\gamma)(t) = n\,\Phi_\gamma(t)\big(\dot{\sigma}(nt) + O(\tfrac{1}{n})\big) \quad (t \in [0,1]), \tag{10}$$

where $O(\frac{1}{n})$ denotes a term such that $n\left|O(\frac{1}{n})\right|$ is uniformly bounded over $[0,1]$ as n ranges over all of $\mathbf{N}$. (In this case, $n\left|O(\frac{1}{n})\right| = |\Lambda_\gamma(t)| = M$ for all $t \in [0,1]$, with M as in (7).) Therefore,

$$F_n(\gamma)(t) \times \frac{\dot{F}_n(\gamma)(t)}{|\dot{F}_n(\gamma)(t)|} = \Phi_\gamma(t)\left(\sigma(nt) \times \frac{\dot{\sigma}(nt)}{|\dot{\sigma}(nt)|}\right) + O(\tfrac{1}{n}). \tag{11}$$

Let $\Phi_{F_n(\gamma)}$ (resp. Φ_σ) denote the frame of $F_n(\gamma)$ (resp. σ) and $\Lambda_{F_n(\gamma)}$ (resp. Λ_σ) its logarithmic derivative. It follows from (8), (10) and (11) that

$$\Phi_{F_n(\gamma)}(t) = \Phi_\gamma(t)\Phi_\sigma(nt) + O(\tfrac{1}{n}).$$

Differentiating both sides of this equality, we obtain that

$$\dot{\Phi}_{F_n(\gamma)}(t) = \dot{\Phi}_\gamma(t)\Phi_\sigma(nt) + n\,\Phi_\gamma(t)\dot{\Phi}_\sigma(nt) + O(1) = n\big(\Phi_\gamma(t)\dot{\Phi}_\sigma(nt) + O(\tfrac{1}{n})\big).$$

Multiplying on the left by the inverse of $\Phi_{F_n(\gamma)}$, we finally conclude that

$$\Lambda_{F_n(\gamma)}(t) = n\big(\Lambda_\sigma(nt) + O(\tfrac{1}{n})\big). \tag{12}$$

Recall that, by the definition of logarithmic derivative (eq. (2), §1),

$$\Lambda_{F_n(\gamma)} = \begin{pmatrix} 0 & -|\dot{F}_n(\gamma)| & 0 \\ |\dot{F}_n(\gamma)| & 0 & -|\dot{F}_n(\gamma)|\kappa_{F_n(\gamma)} \\ 0 & |\dot{F}_n(\gamma)|\kappa_{F_n(\gamma)} & 0 \end{pmatrix} \tag{13}$$

$$\text{and} \quad \Lambda_\sigma = \begin{pmatrix} 0 & -\,|\dot{\sigma}| & 0 \\ |\dot{\sigma}| & 0 & -\,|\dot{\sigma}|\,\kappa_1 \\ 0 & |\dot{\sigma}|\,\kappa_1 & 0 \end{pmatrix}, \tag{14}$$

where $\kappa_{F_n(\gamma)}$ (resp. $\kappa_1 = \cot\rho_1$) denotes the geodesic curvature of $F_n(\gamma)$ (resp. σ). Comparing the (3,2)-entries of (12) and (13), and using (10), we deduce that

$$n\big(\,|\dot{\sigma}(nt)| + O(\tfrac{1}{n})\big)\kappa_{F_n(\gamma)}(t) = n\big(\,|\dot{\sigma}(nt)|\,\kappa_1 + O(\tfrac{1}{n})\big).$$

Therefore $\lim_{n\to+\infty}\kappa_{F_n(\gamma)} = \kappa_1 > \kappa_0$ uniformly over $[0,1]$, as required. □

(5.9) Lemma. *Let* $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$, $t_0 \in (0,1)$. *Then* $\gamma^{[t_0\#n]} \simeq F_n(\gamma)$ *within* $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$ *for all sufficiently large* $n \in \mathbf{N}$.

Proof. Intuitively, the homotopy is obtained by pushing the loops in $F_n(\gamma)$ towards $\gamma(t_0)$. If n is large enough, then we can guarantee that the curvature remains greater than κ_0 throughout the deformation; the proof is similar to that of (5.8), so we will omit it. See lemma 5.4 in [12] for the details when $\kappa_0 = 0$. □

The next result states that after we add enough loops to a curve, it becomes so flexible that any condition on the curvature may be safely forgotten.

(5.10) Lemma. *Let* $\gamma_0, \gamma_1 \in \mathcal{L}_{\kappa_0}^{+\infty}(Q)$ *be two curves in the same component of* $\mathcal{I}(Q) = \mathcal{L}_{-\infty}^{+\infty}(Q)$. *Then* $F_n(\gamma_0)$ *and* $F_n(\gamma_1)$ *lie in the same component of* $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$ *for all sufficiently large* $n \in \mathbf{N}$.

Proof. Let γ_0, γ_1 be two curves in the same component of $\mathcal{L}_{-\infty}^{+\infty}(Q)$. Taking $K = [0,1]$ and $h\colon K \to \mathcal{L}_{-\infty}^{+\infty}(Q)$ to be a path joining γ_0 and γ_1, we conclude from (5.8) that $g = F_n \circ h$ is a path in $\mathcal{L}_{\kappa_0}^{+\infty}(Q)$ joining both curves if n is sufficiently large. □

Thus, if we can find a way to deform γ_i into $F_{2n}(\gamma_i)$ for large n, $i = 0,1$, then the question of deciding whether γ_0 and γ_1 are homotopic reduces to the easy verification of whether their final lifted frames $\tilde{\Phi}_{\gamma_0}(1)$ and $\tilde{\Phi}_{\gamma_1}(1)$ agree. One way to achieve this is to graft arbitrarily long arcs of circles onto such a curve; this is possible if it is diffuse (see fig. 9 below).

Grafting non-condensed curves

(5.11) Proposition. *Let $\kappa_0 \in \mathbf{R}$ and suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is diffuse. Then γ is homotopic to a circle traversed a number of times.*

Proof. Let $\gamma\colon [0,T] \to \mathbf{S}^2$ be parametrized by curvature and let $\tilde{\Lambda}\colon [0,T] \to \tilde{\mathfrak{so}}_3$ be its (lifted) logarithmic derivative. Since γ is diffuse, we can find $0 < t_1 < t_2 < T$ and $\rho_1, \rho_2 \in [0,\rho_0]$ such that $C_\gamma(t_1,\rho_1) = -C_\gamma(t_2,\rho_2)$. By deforming γ in a neighborhood of $\gamma(t_2)$ if necessary, we can actually assume that $\rho_1,\rho_2 \in (0,\rho_0)$. Set $z_i = \tilde{\Phi}(t_i)$,

$$\chi_i = C_\gamma(t_i,\rho_i) = \cos\rho_i\,\gamma(t_i) + \sin\rho_i\,\mathbf{n}(t_i) \text{ and } \lambda_i = \cos\rho_i\,\boldsymbol{i} + \sin\rho_i\,\boldsymbol{k} \quad (i=1,2).$$

Identifying $\mathbf{S}^2$ with the unit imaginary quaternions, we have

$$z_i\lambda_i z_i^{-1} = \chi_i \quad (i=1,2). \tag{15}$$

We will define a family of curves $s \mapsto \gamma_s$, $s \geq 0$, as follows: First, let $\tilde{\Lambda}_s\colon [0,T+2s] \to \tilde{\mathfrak{so}}_3$ be given by:

$$\tilde{\Lambda}_s(t) = \begin{cases} \tilde{\Lambda}(t) & \text{if} \quad 0 \leq t \leq t_1 \\ \frac{1}{2}\lambda_1 & \text{if} \quad t_1 \leq t \leq t_1 + s \\ \tilde{\Lambda}(t-s) & \text{if} \quad t_1 + s \leq t \leq t_2 + s \\ \frac{1}{2}\lambda_2 & \text{if} \quad t_2 + s \leq t \leq t_2 + 2s \\ \tilde{\Lambda}(t-2s) & \text{if} \quad t_2 + 2s \leq t \leq T + 2s \end{cases}$$

Next, let $\Lambda_s \in \mathfrak{so}_3$ correspond to $\tilde{\Lambda}_s \in \tilde{\mathfrak{so}}_3$ and define Φ_s to be the unique solution to the initial value problem $\Phi_s(0) = I$, $\dot{\Phi}_s = \Phi_s\Lambda_s$. Finally, set $\gamma_s = \Phi_s e_1$. Geometrically, when $s = 2\pi k$, γ_s is obtained from γ by grafting a circle of radius ρ_1 traversed k times at $\gamma(t_1)$ and another circle of radius ρ_2 traversed k times at $\gamma(t_2)$ (see fig. 9). We claim that $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ for all $s \geq 0$.

Indeed, we have

$$\tilde{\Phi}_s(t) = \begin{cases} \tilde{\Phi}(t) & \text{if} \quad 0 \leq t \leq t_0 \\ z_1 \exp\left(\frac{\lambda_1}{2}(t-t_1)\right) & \text{if} \quad t_1 \leq t \leq t_1 + s \\ \exp\left(\frac{\chi_1}{2}s\right)\tilde{\Phi}(t-s) & \text{if} \quad t_1 + s \leq t \leq t_2 + s \\ \exp\left(\frac{\chi_1}{2}s\right) z_2 \exp\left(\frac{\lambda_1}{2}(t-t_2-s)\right) & \text{if} \quad t_2 + s \leq t \leq t_2 + 2s \\ \exp\left(\frac{\chi_1}{2}s\right)\exp\left(\frac{\chi_2}{2}s\right)\tilde{\Phi}(t-2s) & \text{if} \quad t_2 + 2s \leq t \leq T + 2s \end{cases}$$

where we have used (15) to write

$$\left(z_1 \exp\left(\tfrac{s\lambda_1}{2}\right)\right)\left(z_1^{-1}\tilde{\Phi}(t-s)\right) = \exp\left(\tfrac{s\chi_1}{2}\right)\tilde{\Phi}(t-s),$$

which yields the expression for $\tilde{\Phi}(t)$ when $t \in [t_1, t_1+s]$, and similarly for the interval $[t_2+2s, T+2s]$. In particular, we deduce that the final lifted frame is:

$$\tilde{\Phi}_s(T+2s) = \exp\left(\tfrac{s\chi_1}{2}\right)\exp\left(\tfrac{s\chi_2}{2}\right)\tilde{\Phi}(T) = \tilde{\Phi}(T),$$

as $\chi_2 = -\chi_1$ by hypothesis. This proves that each γ_s has the correct final frame. The curvature κ^s of γ_s clearly satisfies $\kappa^s > \kappa_0$ almost everywhere in $[0,t_1]\cup[t_1+s, t_2+s]\cup[t_2+2s, T+2s]$, because, by construction, the restriction of γ_s to each of these intervals is the composition of a rotation of $\mathbf{S}^2$ with an arc of γ. Moreover, the restriction of γ_s to the interval $[t_1, t_1+s]$ is an arc of circle of radius of curvature $\rho_1 < \rho_0$; similarly, the restriction of γ_s to $[t_2+s, t_2+2s]$ is an arc of circle of radius of curvature $\rho_2 < \rho_0$. Therefore $\kappa^s > \kappa_0$ almost everywhere on $[0, T+2s]$, and we conclude that $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$.

We have thus proved that γ is homotopic to $\gamma^{[t_0\#n;t_1\#n]}$ for all $n \in \mathbf{N}$ when γ is diffuse. The proposition now follows from (5.9) and (5.10) combined. $\square$

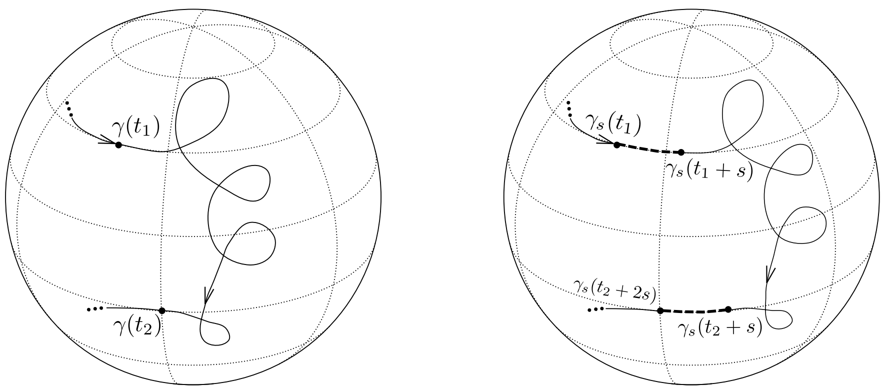

Figure 9: Grafting arcs of circles onto a diffuse curve, as described in (5.11).

The next result says that we can still graft small arcs of circle onto γ even when it is not diffuse, as long as it is also not condensed.

(5.12) Proposition. *Suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ is non-condensed. Then there exist $\varepsilon > 0$ and a chain of grafts (γ_s) such that $\gamma_0 = \gamma$, $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ and* $\operatorname{tot}(\gamma_s) = \operatorname{tot}(\gamma) + s$ *for all $s \in [0, \varepsilon)$.*

Proof. (In this proof the identification of $\mathbf{S}^2$ with the set of unit imaginary quaternions used in (5.11) is still in force.) Let $\gamma\colon [0,T] \to \mathbf{S}^2$ be parametrized by curvature and let $\tilde{\Lambda}\colon [0,T] \to \tilde{\mathfrak{so}}_3$ be its (lifted) logarithmic derivative.

Since γ is not condensed, 0 lies in the interior of the convex closure of the image C of C_γ by (11.2). Hence, by (11.5), we can find a 3-dimensional simplex with vertices in C containing 0 in its interior. In symbols, we can find $0 < t_1 < t_2 < t_3 < t_4 < T$ and $s_1, s_2, s_3, s_4 > 0$, $s_1 + s_2 + s_3 + s_4 = 1$, such that

$$0 = s_1\chi_1 + s_2\chi_2 + s_3\chi_3 + s_4\chi_4, \tag{16}$$

where $\chi_i = C_\gamma(t_i, \rho_i)$, for some $\rho_i \in (0, \rho_0)$, and the χ_i are in general position. Furthermore, these numbers s_i are the only ones which have these properties (for this choice of the χ_i). Define a function $G\colon \mathbf{R}^4 \to \mathbf{S}^3$ by

$$G(\sigma_1, \sigma_2, \sigma_3, \sigma_4) = \exp\left(\frac{\sigma_1\chi_1}{2}\right)\exp\left(\frac{\sigma_2\chi_2}{2}\right)\exp\left(\frac{\sigma_3\chi_3}{2}\right)\exp\left(\frac{\sigma_4\chi_4}{2}\right).$$

Then $G(0,0,0,0) = \mathbf{1}$ and

$$DG_{(0,0,0,0)}(a,b,c,d) = \frac{1}{2}\big(a\chi_1 + b\chi_2 + c\chi_3 + d\chi_4\big).$$

Since the χ_i are in general position by hypothesis, we can invoke the implicit function theorem to find some $\delta > 0$ and, without loss of generality, functions $\bar\sigma_2, \bar\sigma_3, \bar\sigma_4\colon (-\delta, \delta) \to \mathbf{R}$ of σ_1 such that

$$G\big(\sigma_1, \bar\sigma_2(\sigma_1), \bar\sigma_3(\sigma_1), \bar\sigma_4(\sigma_1)\big) = \mathbf{1} \quad (\sigma_1 \in (-\delta, \delta)).$$

Differentiating the previous equality with respect to σ_1 at 0 and comparing (16) we deduce that

$$\bar\sigma_i'(0) = \frac{s_i}{2s_1} > 0 \quad (i = 2, 3, 4).$$

Let $s(\sigma_1) = \sigma_1 + \bar\sigma_2(\sigma_1) + \bar\sigma_3(\sigma_1) + \bar\sigma_4(\sigma_1)$. Then $s'(\sigma_1) > 0$, hence we can write σ_1, σ_2, σ_3 and σ_4 as a function of s in a neighborhood of 0. The conclusion is thus that there exist $\varepsilon > 0$ and non-negative functions $\sigma_1, \sigma_2, \sigma_3, \sigma_4$ of s such that $\sigma_1(s) + \sigma_2(s) + \sigma_3(s) + \sigma_4(s) = s$ and

$$\exp\left(\frac{\sigma_1\chi_1}{2}\right)\exp\left(\frac{\sigma_2\chi_2}{2}\right)\exp\left(\frac{\sigma_3\chi_3}{2}\right)\exp\left(\frac{\sigma_4\chi_4}{2}\right) = \mathbf{1} \quad \text{for all } s \in [0, +\varepsilon).$$

We will now use these functions to obtain γ_s, $s \in [0, +\varepsilon)$.

Define $\tilde{\Lambda}_s\colon [0, T+s] \to \tilde{\mathfrak{so}}_3$ by:

$$\tilde{\Lambda}_s(t) = \begin{cases} \tilde{\Lambda}(t) & \text{if } \ 0 \leq t \leq t_1 \\ \frac{1}{2}\lambda_1 & \text{if } \ t_1 \leq t \leq t_1 + \sigma_1 \\ \tilde{\Lambda}(t-\sigma_1) & \text{if } \ t_1 + \sigma_1 \leq t \leq t_2 + \sigma_1 \\ \frac{1}{2}\lambda_2 & \text{if } \ t_2 + \sigma_1 \leq t \leq t_2 + \sigma_1 + \sigma_2 \\ \tilde{\Lambda}(t-\sigma_1-\sigma_2) & \text{if } \ t_2 + \sigma_1 + \sigma_2 \leq t \leq t_3 + \sigma_1 + \sigma_2 \\ \frac{1}{2}\lambda_3 & \text{if } \ t_3 + \sigma_1 + \sigma_2 \leq t \leq t_3 + \sigma_1 + \sigma_2 + \sigma_3 \\ \tilde{\Lambda}(t-\sigma_1-\sigma_2-\sigma_3) & \text{if } \ t_3 + \sigma_1 + \sigma_2 + \sigma_3 \leq t \leq t_4 + \sigma_1 + \sigma_2 + \sigma_3 \\ \frac{1}{2}\lambda_4 & \text{if } \ t_4 + \sigma_1 + \sigma_2 + \sigma_3 \leq t \leq t_4 + s \\ \tilde{\Lambda}(t-s) & \text{if } \ t_4 + s \leq t \leq T + s \end{cases}$$

where $\sigma_i = \sigma_i(s)$ $(i = 1,2,3,4)$ are the functions obtained above. Let $\tilde{\Phi}_s\colon [0, T+s] \to \mathbf{S}^3$ be the solution to the initial value problem $\tilde{\Phi}' = \tilde{\Phi}\tilde{\Lambda}$, $\tilde{\Phi}(0) = \mathbf{1}$ and let $\Phi\colon [0, T+s] \to \mathbf{SO}_3$ be its projection. Then using the relation $\chi_i = z_i\lambda_i z_i^{-1}$ one finds by a verification entirely similar to the one in the proof of (5.11) that

$$\tilde{\Phi}_s(T+s) = \exp\left(\frac{\sigma_1\chi_1}{2}\right)\exp\left(\frac{\sigma_2\chi_2}{2}\right)\exp\left(\frac{\sigma_3\chi_3}{2}\right)\exp\left(\frac{\sigma_4\chi_4}{2}\right)\Phi(\tilde{T}) = \tilde{\Phi}(T).$$

Hence, each $\gamma_s = \Phi_s e_1$ has the correct final frame. In addition, over each of the subintervals of $[0, T+s]$ listed above, γ_s is either the composition of a rotation of $\mathbf{S}^2$ with an arc of γ, or an arc of circle of radius $\rho_i \in (0, \rho_0)$ $(i = 1,2,3,4)$. We conclude from this that the geodesic curvature κ^s of γ_s satisfies $\kappa^s > \kappa_0$ almost everywhere on $[0, T+s]$, that is, $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ as we wished. Finally,

$$\operatorname{tot}(\gamma_s) = T + s = \operatorname{tot}(\gamma) + s$$

because γ_s is parametrized by curvature (see (5.2)), and (γ_s) is a chain of grafts by construction. □