6
Condensed Curves

The *rotation number* $N(\eta)$ of a regular closed plane curve $\eta\colon [0,1] \to \mathbf{R}^2$ is simply the degree of the unit tangent vector $\mathbf{t}\colon \mathbf{S}^1 \to \mathbf{S}^1$ (we may consider γ and $\mathbf{t}$ to be defined on $\mathbf{S}^1$ since γ is closed). Suppose now that $\eta\colon [0,L] \to \mathbf{R}^2$ is parametrized by arc-length, and write

$$\mathbf{t}(s) = \exp(i\theta(s)),$$

for some angle-function $\theta\colon [0,L] \to \mathbf{R}$. Then the curvature κ of η is given by

$$\kappa(s) = \theta'(s); \tag{1}$$

furthermore, the rotation number of η is given by $2\pi N(\eta) = \theta(L) - \theta(0)$. These facts are explained in any textbook on differential geometry. The Whitney-Graustein theorem ([17], thm. 1) states that two regular closed plane curves are homotopic through regular closed curves if and only if they have the same rotation number.

Now suppose $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ has image contained in some closed hemisphere. Let h_γ be the barycenter, on $\mathbf{S}^2$, of the set of closed hemispheres which contain $\operatorname{Im}(\gamma)$ (cf. (4.11)), and let $\operatorname{pr}\colon \mathbf{S}^2 \to \mathbf{R}^2$ denote stereographic projection from $-h_\gamma$. Define the *rotation number* $\nu(\gamma)$ of γ by $\nu(\gamma) = -N(\eta)$, where $\eta = \operatorname{pr} \circ \gamma$. Recall that a curve $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is called *condensed* if the image C of its caustic band $C_\gamma\colon [0,1] \times [0,\rho_0] \to \mathbf{S}^2$ is contained in some closed hemisphere. Because $C_\gamma(t,0) = \gamma(t)$, any condensed curve is contained in a closed hemisphere, hence we may speak of its rotation number.

Remark. It is natural to ask why this notion of rotation number is not extended to a larger class of curves. For instance, if γ is any admissible curve then, by Sard's theorem, there exists some point $p \in \mathbf{S}^2$ not in the image of γ. We could use stereographic projection from p to define the rotation number of γ. The trouble is that it is not clear how p can be chosen so that the resulting number is locally constant (i.e., continuous) on $\mathcal{L}_{\kappa_0}^{+\infty}$: A different choice of p yields a different rotation number (although its parity remains the same). In fact, the class of spherical curves for which a meaningful notion of rotation number exists must be restricted, since it is always possible to deform a circle traversed ν times into a circle traversed $\nu+2$ times in $\mathcal{L}_{\kappa_0}^{+\infty}$ if ν is sufficiently large.

(6.1) Proposition. *Let A be a connected compact space, $\kappa_0>0$ and $f\colon A\to \mathcal{L}_{\kappa_0}^{+\infty}(I)$ be such that $f(a)$ is condensed for all $a\in A$. Then there exists $\nu\in\mathbf{N}$ such that f is homotopic in $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ to the constant map $a\mapsto\sigma_\nu$, σ_ν a circle traversed ν times.*

The idea of the proof is to use Möbius transformations to make the curves $\eta_a = f(a)$ so small that they become approximately plane curves. The hypothesis that the curves are condensed guarantees that the geodesic curvature does not decrease during the deformation. A slight variation of the Whitney-Graustein theorem is then used to deform the curves to a circle traversed ν times, where ν is the common rotation number of the curves.

We will also need the following technical result, which is a corollary of the proof of (6.1).

(6.2) Corollary. *Let $\kappa_0>0$ and $\gamma\in\mathcal{L}_{\kappa_0}^{+\infty}$ be a condensed curve. Then there exists a homotopy $s\mapsto\gamma_s\in\mathcal{L}_{\kappa_0}^{+\infty}$ $(s\in[0,1])$ such that $\gamma_1=\gamma$, γ_0 is a parametrized circle and $\operatorname{Im}(C_{\gamma_s})$ is contained in an open hemisphere for each $s\in[0,1)$.*

We start by defining spaces of closed curves in $\mathbf{R}^2$ which are analogous to the spaces $\mathcal{L}_{\kappa_1}^{\kappa_2}$ of curves on $\mathbf{S}^2$.[1] Let $-\infty\le\kappa_1<\kappa_2\le+\infty$. A (κ_1,κ_2)-*admissible plane curve* is an element $(c,z,\hat{v},\hat{w})$ of $\mathbf{R}^2\times\mathbf{S}^1\times L^2[0,1]\times L^2[0,1]$. With such a 4-tuple we associate the unique curve $\gamma\colon[0,1]\to\mathbf{R}^2$ satisfying

$$\gamma(t)=c+\int_0^t v(\tau)\mathbf{t}(\tau)\,d\tau,\quad \mathbf{t}(0)=z,\quad \mathbf{t}'(t)=w(t)i\mathbf{t}(t)\quad (t\in[0,1]),$$

where v and w are given by eq. (6) on p. 17 and $i=(0,1)$ is the imaginary unit. The space of all (κ_1,κ_2)-admissible plane curves is thus given the structure of a Hilbert manifold, and we define $\mathcal{W}_{\kappa_1}^{\kappa_2}$ to be its subspace consisting of all closed curves.

Although $\dot\gamma$ is defined only almost everywhere for a curve $\gamma\in\mathcal{W}_{\kappa_1}^{\kappa_2}$, its unit tangent vector $\mathbf{t}$ is defined over all of $[0,1]$, and if we parametrize γ by a multiple of arc-length instead, then $\dot\gamma$ is defined and nonzero everywhere. More importantly, since $\mathbf{t}$ is (absolutely) continuous, we may speak of the rotation number of γ and (1) still holds a.e..

[1] These spaces of plane curves will only be considered in this section.

(6.3) Lemma. *Let A be compact and connected, $\kappa_0 \geq 0$ and $A \to \mathcal{W}_{\kappa_0}^{+\infty}$, $a \mapsto \eta_a$, be a continuous map. Then there exists a homotopy $[0,1] \times A \to \mathcal{W}_{\kappa_0}^{+\infty}$, $(s,a) \mapsto \eta_a^s$, such that $\eta_a^0 = \eta_a$ and*

$$\eta_a^1 = \sigma_N(t + t_a) \quad \textit{for all } a \in A,$$

where $\sigma_N(t) = R_0 \exp(2\pi i N t)$ is a circle traversed $N > 0$ times. In addition, if the image of η_a is contained in some ball $B(0;R)$ for all $a \in A$, then we can arrange that η_a^s have the same property for all $s \in [0,1]$ and $a \in A$.

Thus, given a family of curves in $\mathcal{W}_{\kappa_0}^{+\infty}$ indexed by a compact connected set, we may deform all of them to the same parametrized circle σ_N, except for the starting point of the parametrization.

Proof. Since A is connected, all the curves η_a have the same rotation number N. Moreover, $N > 0$ because of (1) and the fact that $\kappa_0 \geq 0$.

For $\eta \in \mathcal{W}_{\kappa_0}^{+\infty}$, let $z_\eta = \mathbf{t}_\eta(0)$, where $\mathbf{t}_\eta$ is the unit tangent vector to η. The homotopy $g\colon [0,1] \times A \to \mathcal{W}_{\kappa_0}^{+\infty}$ by translations,

$$g(s,a)(t) = \eta_a(t) - s\big(iz_{\eta_a} + \eta_a(0)\big) \qquad (s,\, t \in [0,1],\ a \in A),$$

preserves the curvature and, for any $a \in A$, $g(1,a)$ has the property that it starts at some $z \in \mathbf{S}^1$ in the direction iz. Thus, we may assume without loss of generality that the original curves η_a have this property.

Let $\rho_0 = \frac{1}{\kappa_0}$, $L(\eta_a)$ denote the length of η_a, $L_0 = \min_{a \in A} \{L(\eta_a)\}$ and let $R_1 > 0$ satisfy

$$R_1 < \min\Big\{\frac{L}{2\pi N}, \rho_0\Big\}.^2 \tag{2}$$

Define $f\colon [0,1] \times A \to \mathcal{W}_{\kappa_0}^{+\infty}$ to be the homotopy given by

$$f(s,a)(t) = \eta_a(0) + \Big((1-s) + s\frac{R_1 2\pi N}{L(\eta_a)}\Big)\big(\eta_a(t) - \eta_a(0)\big) \qquad (s,\, t \in [0,1],\ a \in A).$$

Then $f(1,a)$ has length $L = 2\pi N R_1$ for all $a \in A$. In addition, the curvature of $f(s,a)$ is bounded from below by κ_0 for all $s \in [0,1]$, $a \in A$ and almost every $t \in [0,1]$, as an easy calculation using (2) shows.

The conclusion is that we lose no generality in assuming that the curves η_a all have the same length $L = 2\pi N R_0$. Further, by (2.1), we can assume that they are all parametrized by a multiple of arc-length. This implies that $\dot\eta_a$ takes values on the circle $L\mathbf{S}^1$ of radius L. Using angle-functions θ_a with

[2] If $\kappa_0 = 0$ then we adopt the convention that $\rho_0 = +\infty$.

$\theta_a(0) = 0$ and $\theta_a(1) = 2\pi N$, we can write:

$$\dot{\eta}_a(t) = Lz_a \exp\big(i\theta_a(t)\big) \qquad (t \in [0,1]),$$

where $z_a = \mathbf{t}_{\eta_a}(0)$. Let $\theta(t) = 2\pi N t$, $t \in [0,1]$, and define

$$\theta_a^s(t) = (1-s)\theta_a(t) + s\theta(t), \qquad \bar{\tau}_a^s(t) = Lz_a \exp(i\theta_a^s(t)) \quad (s,\, t \in [0,1],\ a \in A).$$

Then $\theta_a^s(0) = 0$ and $\theta_a^s(1) = 2\pi N$ for all $s \in [0,1]$, $a \in A$. The idea is that $\bar{\tau}_a^s$ should be the tangent vector to a curve; the problem is that this curve need not be closed. We can fix this by defining instead

$$\tau_a^s(t) = \bar{\tau}_a^s(t) - \int_0^1 \bar{\tau}_a^s(v)\, dv, \qquad \eta_a^s(t) = -iz_a + \int_0^t \tau_a^s(v)\, dv.$$

The conditions $\int_0^1 \tau_a^s(t)\, dt = 0$ and $\tau_a^s(0) = \tau_a^s(1)$ then guarantee that η_a^s is a closed curve. Because $\theta_a^s(1) = 2\pi N$ and $N > 0$, $\bar{\tau}_a^s$ must traverse all of $L\mathbf{S}^1$, so that $\int_0^1 \bar{\tau}_a^s(v)\, dv$ lies in the interior of the disk bounded by this circle for any $s \in [0,1]$, $a \in A$. Consequently, $\tau_a^s(t)$ never vanishes. Moreover,

$$\eta_a^0 = \eta_a \quad \text{and} \quad \eta_a^1(t) = -iz_{\eta_a} \exp(2\pi N i t) \quad \text{for all } a \in A.$$

Finally, η_a^s has positive curvature for all $s \in [0,1]$ and $a \in A$. Although it is easier to see this using a geometrical argument, the following computation suffices: The curvature κ_a^s of η_a^s is given by

$$\kappa_a^s(t) = \frac{\det\big(\tau_a^s(t), \dot{\tau}_a^s(t)\big)}{|\tau_a^s(t)|^3} = \frac{L^2 \dot{\theta}_a^s(t)}{|\tau_a^s(t)|^3}\Big[1 - \det\Big(\int_0^1 \exp(i\theta_a^s(v))\, dv,\ i\exp(i\theta_a^s(t))\Big)\Big].$$

Because $\theta_a^s = (1-s)\theta_a + s\theta$ is monotone increasing (recall that $\theta_a' = \kappa_a > \kappa_0 \geq 0$ a.e. by hypothesis), the map $t \mapsto \exp(i\theta_a^s(t))$ runs over all of $\mathbf{S}^1$ for any s and a. As a consequence, the integral above has norm strictly less than 1, hence so does the determinant. In fact, since A is compact, we can find a constant $C > 0$, independent of a and s, such that

$$\kappa_a^s > C\kappa_0.$$

For $\lambda > 0$ and an admissible plane curve γ, the curve $\lambda\gamma$ has curvature given by $\frac{\kappa}{\lambda}$, where κ is the curvature of γ. Again using compactness of A, we may find a smooth function $\lambda\colon [0,1] \to (0,1]$ such that $\lambda(0) = 1$ and $\lambda(s)$ is as small as necessary for $s \in (0,1]$ to guarantee that $\kappa_a^s > \kappa_0$ for all $s \in [0,1]$ and $a \in A$ if we replace η_a^s by $\lambda(s)\bar{\eta}_a^s$. In addition, we can choose λ so that the image of $\lambda(s)\eta_a^s$ is contained in the ball $B_R(0)$ if this is the case for each η_a. This establishes the lemma with $R_0 = \lambda(1)$. □

The next result states that the geodesic curvature of a curve $\gamma\colon [0,1] \to \mathbf{S}^2$ and the curvature of the plane curve obtained by projecting γ orthogonally on $T_p\mathbf{S}^2$ are roughly the same, as long as the curve is contained in a small neighborhood of p.

(6.4) Lemma. *Let $\kappa_0 < \kappa_1 < \kappa_2$ and $p \in \mathbf{S}^2$ be given. Identifying $T_p\mathbf{S}^2$ with $\mathbf{R}^2$, with p corresponding to the origin, let $P\colon \mathbf{S}^2 \to \mathbf{R}^2$ be the orthogonal projection. Then there exists $\varepsilon > 0$ such that:*

(a) If $\gamma \in \mathcal{L}_{\kappa_2}^{+\infty}$ satisfies $d(\gamma(t), p) < \varepsilon$ for all $t \in [0,1]$, then $\eta = P \circ \gamma \in \mathcal{W}_{\kappa_1}^{+\infty}$.

(b) If $\eta \in \mathcal{W}_{\kappa_1}^{+\infty}$ satisfies $|\eta(t)| < \varepsilon$ for all $t \in [0,1]$, then $\gamma = P^{-1} \circ \eta \in \mathcal{L}_{\kappa_0}^{+\infty}$.

In part (a), d denotes the distance function on $\mathbf{S}^2$ and the transformation P^{-1} in part (b) is to be understood as the inverse of P when restricted to the hemisphere $\{q \in \mathbf{S}^2 : \langle q, p\rangle > 0\}$.

Proof. Since the subset of smooth curves is dense in the space of all admissible (plane or spherical) curves, it suffices to prove the lemma for C^2 curves. Let $\gamma \in \mathcal{L}_{\kappa_2}^{+\infty}$ be a C^2 curve such that $d(\gamma(t), p) < \varepsilon$ for all $t \in [0,1]$. If $0 < \varepsilon < \frac{\pi}{2}$ then η will also be a C^2 regular curve. Let $UT\mathbf{S}^2$ denote the unit tangent bundle of $\mathbf{S}^2$ and $U \subset UT\mathbf{S}^2$ the open set consisting of all vectors in the fibers of those $q \in \mathbf{S}^2$ with $d(p,q) < \frac{\pi}{2}$ (d being the distance on $\mathbf{S}^2$). Define $f, g\colon U \to \mathbf{R}$ by

$$f(u) = \frac{\det(P(u), P(q \times u))}{|P(u)|^3} \quad \text{and} \quad g(u) = \frac{\det(P(u), P(q))}{|P(u)|^3} \quad \text{for } u \in T_q\mathbf{S}^2,$$

where $\times$ denotes the vector product in $\mathbf{R}^3$.

Note that we may identify P with its derivative $dP_q\colon T_q\mathbf{S}^2 \to \mathbf{R}^2$ at any $q \in \mathbf{S}^2$, because P is the restriction of a linear transformation $\mathbf{R}^3 \to \mathbf{R}^2$. With this observation in mind, a straightforward calculation yields the following expression for the curvature κ_η of $\eta = P \circ \gamma$:

$$\kappa_\eta(t) = f(\mathbf{t}(t))\,\kappa_\gamma(t) - g(\mathbf{t}(t)) \qquad (t \in [0,1]).$$

Here $\mathbf{t}$ is the unit tangent vector to γ.

Since f, g are continuous and $f(u) = 1$, $g(u) = 0$ for all unit vectors $u \in T_p\mathbf{S}^2$, it follows that there exists ε such that if $d(p,q) < \varepsilon$, then

$$\kappa_0 < f(v)\,\kappa_1 - g(v) \quad \text{for any } v \in T_q\mathbf{S}^2,\ |v| = 1.$$

Hence, if $d(\gamma(t), p) < \varepsilon$ for all $t \in [0,1]$ then κ_η satisfies the conclusion of (a).

A similar reasoning shows that, by reducing ε if necessary, we can also arrange for (b) to hold. □

Remark. An analogous result to (6.4), with a similar proof, holds for upper bounds on the curvature, or even lower and upper bounds simultaneously. However, since we need neither of these versions, we will not formulate them carefully.

(6.5) Lemma. *Let $h \in \mathbf{S}^2$, $H = \big\{q \in \mathbf{S}^2 : \langle q, h\rangle \geq 0\big\}$, let $\mathrm{pr}\colon \mathbf{S}^2 \to \mathbf{R}^2$ denote stereographic projection from $-h$. Let $\kappa_0 > 0$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ be such that $\mathrm{Im}(C_\gamma) \subset H$. Define $T_r\colon \mathbf{S}^2 \to \mathbf{S}^2$ to be the Möbius transformation (dilatation) given by*

$$T_r(p) = \mathrm{pr}^{-1}\big(r\,\mathrm{pr}(p)\big) \quad (r \in (0,1],\ p \in \mathbf{S}^2).$$

Then, given $\kappa_1 > \kappa_0$, there exists $r_0 > 0$, depending only on κ_0 and κ_1, such that the geodesic curvature κ^r of $T_r(\gamma)$ satisfies $\kappa^r > \kappa_1$ a.e. for any $r \in (0, r_0)$.

Proof. Suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is parametrized by its arc-length and let σ be a parametrization, also by arc-length, of an arc of the osculating circle to γ at $\gamma(s_0)$, i.e., let σ satisfy:

$$\sigma(s_0) = \gamma(s_0), \qquad \sigma'(s_0) = \gamma'(s_0), \qquad \sigma''(s_0) = \gamma''(s_0).$$

(It makes sense to speak of γ'' (as an L^2 map) because $\gamma' = \mathbf{t}$ is H^1 by hypothesis.) Then $T_r \circ \sigma$ has contact of order 3 with $T_r \circ \gamma$ at s_0, hence their geodesic curvatures at the corresponding point agree. Therefore, it suffices to prove the result for a circle C whose center χ lies in H. Let $\rho_i = \operatorname{arccot} \kappa_i$, $i = 0, 1$, and ρ be the radius of curvature of C, $\rho < \rho_0 < \frac{\pi}{2}$. If d denotes the distance function on $\mathbf{S}^2$, then $C \subset B_d\big(h; \frac{\pi}{2} + \rho_0\big)$ (where the latter denotes the set of $q \in \mathbf{S}^2$ such that $d(h, q) < \frac{\pi}{2} + \rho_0$). Choose r_0 such that

$$T_r\big(B_d\big(h; \tfrac{\pi}{2} + \rho_0\big)\big) \subset B_d\big(h; \rho_1\big) \text{ for all } r \in (0, r_0).$$

Then $T_r(C)$ is a circle, for a Möbius transformation such as T_r maps circles to circles, and its diameter is at most $2\rho_1$. Thus, its geodesic curvature must be greater than κ_1. Moreover, it is clear that the choice of r_0 does not depend on h or on C. □

Proof of (6.1). Let γ_a denote $f(a)$ and let h_a be the barycenter of the set of closed hemispheres which contain $\mathrm{Im}(C_{\gamma_a})$; by (4.10), the map $h\colon A \to \mathbf{S}^2$ so defined is continuous.

Let pr_a denote stereographic projection $\mathbf{S}^2 \to \mathbf{C}$ from $-h_a$, so that h_a is projected to the origin, and define a family $T_a^s\colon \mathbf{S}^2 \to \mathbf{S}^2$ of Möbius transformations by:

$$T_a^s(q) = \mathrm{pr}_a^{-1}(s\,\mathrm{pr}_a(q)) \qquad (q \in \mathbf{S}^2,\ s \in (0,1],\ a \in A).$$

Set $\gamma_a^s = T_a^s \gamma_a$. From (6.5) it follows that we can choose $\delta > 0$ so small that the geodesic curvature of γ_a^δ is greater than $\kappa_0 + 2$ a.e. for any $a \in A$.

Now choose $\varepsilon > 0$ as in (6.4), with $\kappa_1 = \kappa_0 + 1$, $\kappa_2 = \kappa_0 + 2$. By reducing δ if necessary, we can guarantee that the curves γ_a^δ have image contained in $B_d(h_a;\varepsilon)$, for each a. Let η_a be the orthogonal projection of γ_a^δ onto $T_{h_a}\mathbf{S}^2$. We are then in the setting of (6.3). The conclusion is that we can deform all η_a to a single circle σ_ν, modulo the starting point of the parametrization, in such a way that the curves have image contained in $B(0;\varepsilon)$ and curvature greater than $\kappa_0 + 1$ throughout the deformation. By (6.4) again, when we project this homotopy back to $\mathbf{S}^2$, the geodesic curvature of the curves is always greater than κ_0.

To sum up, we have described a homotopy $H\colon [0,1] \times A \to \mathbf{S}^2$ such that $H(0,a) = \gamma_a$ and $H(1,a)$ is a circle traversed ν times for all $a \in A$; further, the geodesic curvature κ_a^s of $H(s,a)$ satisfies $\kappa_a^s(t) > \kappa_0$ for all $s,t \in [0,1]$. These curves $H(a,s)$ do not satisfy $\Phi(0) = I = \Phi(1)$, but we can correct this by setting

$$\bar{\gamma}_a^s = \Phi_{H(a,s)}(0)^{-1} H(a,s)$$

and using $\bar{\gamma}_a^s$ instead; this has no effect on the geodesic curvature and finishes the proof that f is null-homotopic, since $\bar{\gamma}_a^1$ is the same parametrized circle for all a. □

We now provide a proof of (6.2). This result will be used to show that a notion of rotation number for non-diffuse curves, which will be introduced in the next section, coincides with the one presented at the beginning of this section.

Proof of (6.2). Let h_γ be the barycenter of the set of closed hemispheres which contain $\mathrm{Im}(C_\gamma)$ and, as in the proof of (6.1), define $\gamma_s = T^s \circ \gamma$, where

$$T^s(q) = \mathrm{pr}^{-1}(s\,\mathrm{pr}(q)) \quad (q \in \mathbf{S}^2,\ s \in (0,1]) \tag{3}$$

and pr denotes stereographic projection from $-h_\gamma$. Let $H = \big\{p \in \mathbf{S}^2 : \langle p, h_\gamma\rangle > 0\big\}$. We claim that $\mathrm{Im}(C_{\gamma_s}) \subset H$ for all $s \in (0,1)$. This follows from the following two assertions:

(i) If $\mathrm{Im}(C_{\gamma_s}) \subset \bar{H}$, then there exists $\varepsilon > 0$ such that $\mathrm{Im}(C_{\gamma_\sigma}) \subset H$ for all $\sigma \in (s - \varepsilon, s)$;

(ii) If $\mathrm{Im}(C_{\gamma_s}) \not\subset H$, then there exists $\varepsilon > 0$ such that $\mathrm{Im}(C_{\gamma_\sigma}) \not\subset \bar{H}$ for all $\sigma \in (s, s + \varepsilon)$.

For any s, the boundary of $\mathrm{Im}(C_{\gamma_s})$ is contained in the union of the images of $\gamma_s = C_{\gamma_s}(\cdot, 0)$ and $\check{\gamma}_s = C_{\gamma_s}(\cdot, \rho_0)$. Moreover, γ has positive geodesic curvature by hypothesis, and a straightforward calculation shows that $\check{\gamma}$ also does (the details may be found in (8.6)).

If $\mathrm{Im}(C_{\gamma_s}) \subset H$ then (i) is obviously true, since H is an open hemisphere; similarly, (ii) holds if $\mathrm{Im}(C_{\gamma_s}) \not\subset \bar{H}$. Suppose then that $\mathrm{Im}(C_{\gamma_s}) \subset \bar{H}$, but $\mathrm{Im}(C_{\gamma_s}) \not\subset H$ for some $s > 0$. This means that there exists $t_0 \in [0, 1]$ such that either γ_s or $\check{\gamma}_s$ is tangent to ∂H at $\gamma_s(t_0)$ or $\check{\gamma}_s(t_0)$, respectively. In the first case, $\mathbf{n}_{\gamma_s}(t_0) = h_\gamma$, and in the second $\mathbf{n}_{\gamma_s}(t_0) = -h_\gamma$. In either case, $C_{\gamma_s}\big(\{t_0\} \times [0, \rho_0]\big)$ is an arc of the geodesic through $\gamma_s(t_0)$ and h_γ. Such geodesics through h_γ are mapped to lines through the origin by pr, hence (3) implies that there exists $\varepsilon > 0$ such that $C_\gamma(t, \sigma) \subset H$ for any $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$ and $\sigma \in (s - \varepsilon, s)$ and $C_\gamma(t_0, \sigma) \not\subset \bar{H}$ for any $\sigma \in (s, s+\varepsilon)$. Furthermore, since the geodesic curvatures of γ, $\check{\gamma}$ are positive and ∂H is a geodesic, the set of $t_0 \in [0, 1]$ where they are tangent to ∂H must be finite. This implies (i) and (ii).

Now let $S = \big\{s \in (0, 1) : \mathrm{Im}(C_{\gamma_s}) \not\subset H\big\}$. Assume that $S \neq \emptyset$ and let $s_0 = \sup S$. Applying (i) to $\gamma_1 = \gamma$ we conclude that there exists $\varepsilon > 0$ with $S \cap (1 - \varepsilon, 1) = \emptyset$. Hence, $s_0 < 1$ and $\mathrm{Im}(C_{\gamma_{s_0}}) \not\subset H$ by construction. An application of (ii) yields a contradiction. Thus, $S = \emptyset$.

Let $\rho_0 = \operatorname{arccot} \kappa_0$ and $r = \frac{\pi}{2} - \rho_0$. Choosing $\delta > 0$ so that $\mathrm{Im}(\gamma_\delta) \subset B_d(h_\gamma; r)$, and proceeding as in the proof of (6.1), we can extend $s \mapsto \gamma_s$ ($s \in [\delta, 1]$) to all of $[0, 1]$ so that γ_0 is a parametrized circle and $\mathrm{Im}(\gamma_s) \subset B_d(h_\gamma; r)$ for all $s \in [0, \delta]$ (where d denotes the distance function on $\mathbf{S}^2$). The inequality $d(\eta(t), C_\eta(t, \theta)) = \theta < \rho_0$, which holds for any $\eta \in \mathcal{L}_{\kappa_0}^{+\infty}$, implies that

$$d(h_\gamma, C_{\gamma_s}(t, \theta)) < \frac{\pi}{2} \quad \text{for any } t \in [0, 1],\ \theta \in [0, \rho_0] \text{ and } s \in [0, \delta].$$

Hence $\mathrm{Im}(C_{\gamma_s}) \subset H$ for all $s \in [0, \delta]$. The same inclusion for $s \in [\delta, 1)$ was established above, so the proof is complete. □

(6.6) Corollary. *Let* $\kappa_0 > 0$ *and* $1 \leq \nu \in \mathbf{N}$.

(a) The set $\mathcal{O}$ *(resp.* $\mathcal{O}_\nu$*) of all condensed curves (resp. all condensed curves having rotation number* ν*) in* $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ *is the closure of an open set.*

(b) If $\gamma \in \mathcal{O}_\nu$ *and* $\mathcal{U} \subset \mathcal{L}_{\kappa_0}^{+\infty}(I)$ *is any open set containing* γ*, then* γ *is homotopic to a smooth curve within* $\mathcal{O}_\nu \cap \mathcal{U}$.

Proof. Let $\mathcal{S} \subset \mathcal{O}$ be the subset consisting of all curves $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ such that $\mathrm{Im}(C_\gamma)$ is contained in an open hemisphere. Then $\mathcal{S}$ is open, because if the compact set $C = \mathrm{Im}(C_\gamma)$ is such that $\langle c, h \rangle > 0$ for some $h \in \mathbf{S}^2$ and all $c \in C$, then the same inequality holds for all $c \in \mathrm{Im}(C_\eta)$ whenever $\eta \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ is sufficiently close to γ. Similarly, $\mathcal{O}$ is closed. For if $\gamma \notin \mathcal{O}$, then, by (11.2) and (11.5), we can find a 3-dimensional simplex Δ_γ with vertices in $\mathrm{Im}(C_\gamma)$ containing $0 \in \mathbf{R}^3$ in its interior. If $\eta \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ is sufficiently close to γ then we can also find a simplex Δ_η with vertices in $\mathrm{Im}(C_\eta)$ such that $0 \in \mathrm{Int}\,\Delta_\eta$. It follows that $\bar{\mathcal{S}} \subset \mathcal{O}$.

Let $\gamma \in \mathcal{O}$. Define a family $T^s \colon \mathbf{S}^2 \to \mathbf{S}^2$ of Möbius transformations by (3), where $\mathrm{pr} \colon \mathbf{S}^2 \to \mathbf{R}^2$ denotes stereographic projection from $-h_\gamma$, and h_γ is the barycenter of the set of closed hemispheres which contain $C = \mathrm{Im}(C_\gamma)$ (cf. (4.10)). Then $\gamma_s = T^s \circ \gamma \in \mathcal{S}$ for all $s \in (0,1)$ by (6.2), which shows that $\bar{\mathcal{S}} \supset \mathcal{O}$. The proof of the assertion about $\mathcal{O}_\nu$ is analogous and will be omitted.

To prove (b), let $\varepsilon > 0$ be such that $\gamma_s = T^s \circ \gamma \in \mathcal{U}$ for all $s \in [1-\varepsilon, 1]$. Choose a path-connected neighborhood $\mathcal{V} \subset \mathcal{S} \cap \mathcal{U}$ of $\gamma_{1-\varepsilon}$, and, for $s \in [0, 1-\varepsilon]$, let γ_s be a path in $\mathcal{V}$ joining a smooth curve γ_0 to $\gamma_{1-\varepsilon}$. As each γ_s is condensed ($s \in [0,1]$), $\nu(\gamma_s)$ is defined for all s; since it can only take on integral values, it must be independent of s. Thus, $s \mapsto \gamma_s$ ($s \in [0,1]$) is the desired path. □

Condensed curves in $\mathcal{L}_{\kappa_0}^{+\infty}$ for $\kappa_0 < 0$

The purpose of this subsection is to prove the following result.

(6.7) Proposition. *Let $\kappa_0 < 0$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ be a condensed curve. Then γ lies in the same connected component of $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ as a circle traversed a number of times.*

Let $1 \le \nu \in \mathbf{N}$ and let $\mathbf{S}^2_\nu$ denote the ν-sheeted connected covering of $\mathbf{S}^2 \smallsetminus \{\pm\text{point}\}$, where we may assume that the point is the north pole N. We will identify $\mathbf{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$ with $\mathbf{S}^2 \smallsetminus \{\pm N\}$ through the homeomorphism h given by $h(z, \phi) = (\cos\phi\, z, \sin\phi)$. This, in turn, yields an identification of $\mathbf{S}^2_\nu$ with $\mathbf{S}^1_\nu \times (-\frac{\pi}{2}, \frac{\pi}{2})$, where $\mathbf{S}^1_\nu$ is the ν-sheeted connected covering space of $\mathbf{S}^1$. We will prefer to work with the space $\mathbf{S}^1_\nu \times (-\frac{\pi}{2}, \frac{\pi}{2})$ instead of $\mathbf{S}^2_\nu$, but its Riemannian metric is the one induced on the latter space by $\mathbf{S}^2$ through the covering map.

(6.8) Definition.[3] Let $0 < R < \frac{\pi}{2}$. An *acceptable band* $A \colon [0,1] \times [0,1] \to \mathbf{S}^1_\nu \times (-\frac{\pi}{2}, \frac{\pi}{2}) \equiv \mathbf{S}^2_\nu$ is a map given by

[3]These notions will only be used in this subsection.

$$A(t,u) = \big(\exp(2\pi\nu i t)\,,\,(1-u)\theta_-(t) + u\theta_+(t)\big) \quad (t,\,u \in [0,1]) \qquad (4)$$

and satisfying the following conditions:

(i) $\theta_\pm\colon [0,1] \to (-\frac{\pi}{2}, \frac{\pi}{2})$ are continuous, $0 \le \theta_+ \le R$ and $-R \le \theta_- \le 0$.

(ii) Let ∂A_+ (resp. ∂A_-) denote the image of $[0,1] \times \{1\}$ (resp. $[0,1] \times \{0\}$) under A. Then $d(p, \partial A_-) \ge R$ and $d(q, \partial A_+) \ge R$ for every $p \in \partial A_+$ and every $q \in \partial A_-$.[4]

The *interior* $\mathring{A}$ of A is simply the interior of the image of A. The set of all acceptable bands (for fixed R) will be denoted by $\mathcal{A}$ and furnished with the C^0 (uniform) topology. Finally, we denote by $\mathcal{G}$ the subspace of $\mathcal{A}$ consisting of all acceptable bands A such that $d(p, \partial A_-) = R = d(q, \partial A_+)$ for any $p \in \partial A_+$ and $q \in \partial A_-$. Such a band will be called *good* and R its *width*.

The motivation for this definition comes from the following lemma.

(6.9) Lemma. *Let $\kappa_0 = \cot\rho_0 < 0$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ be a condensed curve having rotation number ν. Then the image of the lift of the regular band $B_\gamma\colon [0,1] \times [\rho_0 - \pi, 0] \to \mathbf{S}^2$ of γ to $\mathbf{S}^2_\nu$ is the image of a good band of width $\pi - \rho_0$.*

Proof. By hypothesis, the image of the caustic band C_γ is contained in a hemisphere, say,

$$H = \big\{p \in \mathbf{S}^2 : \langle p, N\rangle \ge 0\big\}.$$

Let $\hat\gamma$ be the other boundary curve of B_γ, $\hat\gamma(t) = B_\gamma(t, \rho_0 - \pi)$. Then $\hat\gamma(t) = -C_\gamma(t, \rho_0) \in -H$ for all $t \in [0,1]$. Since $d(\gamma(t), \hat\gamma(t)) = \pi - \rho_0 < \frac{\pi}{2}$, $\operatorname{Im}(\gamma) \subset H$ and $\operatorname{Im}(\hat\gamma) \subset -H$, the image of the regular band is actually contained in $\mathbf{S}^1 \times [\rho_0 - \pi, \pi - \rho_0]$ (where we are identifying $\mathbf{S}^2 \smallsetminus \{\pm N\}$ with $\mathbf{S}^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$).

Let $\tilde B_\gamma\colon [0,1] \times [\rho_0 - \pi, 0] \to \mathbf{S}^2_\nu$ be the lift of B_γ to $\mathbf{S}^2_\nu \equiv \mathbf{S}^1_\nu \times (-\frac{\pi}{2}, \frac{\pi}{2})$. For each $z \in \mathbf{S}^1_\nu$, let the *meridian* μ_z be the geodesic parametrized by $\mu_z(t) = (z, t)$, $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$. By what we have just proved and the fact that γ has rotation number ν, we may define continuous functions $\theta_\pm\colon \mathbf{S}^1_\nu \to (-\frac{\pi}{2}, \frac{\pi}{2})$ by the relations

$$\mu_z(\theta_+(z)) \in \tilde B_\gamma([0,1] \times \{0\}) \quad \text{and} \quad \mu_z(\theta_-(z)) \in \tilde B_\gamma([0,1] \times \{\rho_0 - \pi\}).$$

Then the map $A\colon [0,1] \times [0,1] \to \mathbf{S}^1_\nu \times (-\frac{\pi}{2}, \frac{\pi}{2}) \equiv \mathbf{S}^2_\nu$ given by

$$A(t,u) = \big(\exp(2\pi\nu i t)\,,\,(1-u)\theta_-(t) + u\theta_+(t)\big) \quad (t,\,u \in [0,1])$$

[4]Here and in what follows, d denotes the distance function on $\mathbf{S}^2_\nu$ (or on $\mathbf{S}^2$).

defines an acceptable band whose image coincides with that of $\tilde{B}_\gamma$. Furthermore, the equality $d(\gamma(t), \hat{\gamma}(t)) = \pi - \rho_0$ implies that $d(p, \partial A_\pm) \leq \pi - \rho_0$ for any $p \in \partial A_\mp$. We claim that A is a good band of width $\pi - \rho_0$. To see this, suppose $\eta\colon [0,1] \to \mathbf{S}^2_\nu$ is a piecewise C^1 curve of length less than $\pi - \rho_0$ joining ∂A_- to ∂A_+ and write $\eta(u) = \tilde{B}_\gamma(t(u), \theta(u))$. Then the length is minimized when θ is monotone and $\dot{t}(u) = 0$ for all $u \in [0,1]$, hence the minimal length is $\pi - \rho_0$; we omit the details since an entirely similar argument is presented in the proof of (10.5). □

(6.10) Lemma. *The space $\mathcal{A}$ is contractible.*

Proof. Let $A \in \mathcal{A}$ be given by (4) and let $s \in [0,1]$. Define a family of acceptable bands A_s by

$$A_s(t,u) = \big(\exp(2\pi\nu i t),\, (1-u)\theta_-^s(t) + u\theta_+^s(t)\big),$$

where

$$\theta_+^s(t) = (1-s)\theta_+(t) + sR \quad \text{and} \quad \theta_-^s(t) = (1-s)\theta_-(t) - sR$$

Then the map $\mathcal{A} \times [0,1] \to \mathcal{A}$ given by $(A,s) \mapsto A_s$ is a contraction of $\mathcal{A}$. □

(6.11) Lemma. *The subspace $\mathcal{G}$ is a retract of $\mathcal{A}$.*

Proof. Let $A \in \mathcal{A}$ be given by (4). Define $A^1 = \operatorname{Im}(A)$, $\theta^1_\pm = \theta_\pm$ and

$$A^2 = \big\{p \in A^1 : d(p, \partial A^1_-) \leq R + \tfrac{1}{2}\big\}.$$

We will call a geodesic μ_z in $\mathbf{S}^2_\nu \equiv \mathbf{S}^1_\nu \times (-\frac{\pi}{2}, \frac{\pi}{2})$ of the form $\{z\} \times (-\frac{\pi}{2}, \frac{\pi}{2})$ a *meridian*, and parametrize it by $\mu_z(t) = (z,t)$. We begin by establishing the following facts:

(a) Each meridian μ_z intersects ∂A^2 at exactly two points $\mu_z(\theta^2_-(z))$ and $\mu_z(\theta^2_+(z))$, with $\theta^2_+ \geq 0$ and $\theta^2_- \leq 0$. We define $\partial A^2_\pm$ as the set of all $\mu_z(\theta^2_\pm(z))$ for $z \in \mathbf{S}^1_\nu$.

(b) $\partial A^2_- = \partial A^1_-$.

(c) $p \in \partial A^2_+$ if and only if one of the following holds:

$$p \in \partial A^1_+ \quad \text{and} \quad d(p, \partial A^1_-) \leq R + \tfrac{1}{2}, \quad \text{or}$$
$$p \in \mathring{A}^1 \quad \text{and} \quad d(p, \partial A^1_-) = R + \tfrac{1}{2}.$$

(d) The boundary ∂A^2 of A^2 is the disjoint union of ∂A^2_+ and ∂A^2_-. Moreover,

$$R \leq d(p, \partial A^2_-) \leq R + \frac{1}{2} \quad \text{and} \quad R \leq d(q, \partial A^2_+) \leq d(q, \partial A^1_+)$$

for any $p \in \partial A^2_+$ and $q \in \partial A^2_-$.

(e) A^2 is the (image of) an acceptable band, and the functions in (6.8 (i)) corresponding to A^2 are $\theta^2_\pm$. Moreover,

$$0 \leq \theta^2_+ \leq \min\{R + \tfrac{1}{2}, \theta^1_+\} \quad \text{and} \quad -R \leq \theta^2_- = \theta^1_- \leq 0. \tag{5}$$

The inclusion $\partial A^1_- \subset \mathbf{S}^1_\nu \times [-R, 0]$ implies, firstly, that

$$A^2 \cap \big(\mathbf{S}^1_\nu \times [-R, 0]\big) = A^1 \cap \big(\mathbf{S}^1_\nu \times [-R, 0]\big), \tag{6}$$

as every point of $A^1 \cap \big(\mathbf{S}^1_\nu \times [-R, 0]\big)$ lies at a distance less than or equal to R from ∂A^1_-. Secondly, it implies that

$$t \mapsto d(\mu_z(t), \partial A^1_-)$$

is a monotone decreasing function of t when $t \geq 0$.

It follows from (6) and the properties of A^1 that, for any $z \in \mathbf{S}^1_\nu$, there exists a unique $\theta^2_-(z) \in [-R, 0]$ such that $\mu_z(\theta^2_-(z)) \in \partial A^2$, unless $\mu_z(0) \in \partial A^1_+$. In the latter case, $d(\mu_z(0), \partial A^1_-) = R$, $\theta^2_-(z) = -R$ and $\theta^2_+(z) = 0$. If $\mu_z(0) \notin \partial A^1_+$, let $\theta^2_+(z) > 0$ be the smallest $t \in (0, R]$ such that either $\mu_z(t) \in \partial A^1_+$ or $d(\mu_z(t), \partial A^1_-) = R + \frac{1}{2}$. Suppose $\mu_z(\theta^2_+(z)) \in \partial A^1_+$. Then $\mu_z(\theta^2_+(z)) \in A^2$ (because it lies a distance $\leq R + \frac{1}{2}$ from ∂A^1_-), while $\mu_z(t) \notin A^1 \supset A^2$ for $t > \theta^2_+(z)$. Thus, $\mu_z(\theta^2_+(z)) \in \partial A^2$. If $d(\mu_z(\theta^2_+(z)), \partial A^1_-) = R + \frac{1}{2}$, then again $\mu_z(\theta^2_+(z)) \in A^2$ while $\mu_z(t) \notin A^2$ for $t > \theta^2_+(z)$, since, for such t, $d(\mu_z(t), \partial A^1_-) > R + \frac{1}{2}$ by the second consequence. Moreover, in both cases $\mu_z(t)$ does not intersect ∂A^2 again for $t > 0$. This proves (a), (b), (c) and also establishes (5).

Since

$$\partial A^2 = \bigcup_{z \in \mathbf{S}^1_\nu} \mu_z \cap \partial A^2,$$

(a) implies the first assertion of (d). In turn, (b) and (c) together immediately imply that

$$R \leq d(p, \partial A^2_-) = d(p, \partial A^1_-) \leq R + \frac{1}{2}$$

for any $p \in \partial A^2_+$. That $d(q, \partial A^2_+) \leq d(q, \partial A^1_+)$ for any $q \in \partial A^2_-$ follows from the fact that ∂A^2_+ lies below ∂A^1_+, in the sense that any geodesic joining ∂A^1_- to ∂A^1_+ must first intersect a point of ∂A^2_+. Indeed, $\theta^2_+(z) \leq \theta^1_+(z)$ for any

$z \in \mathbf{S}^1_\nu$, as we have already seen in (5). Thus, (d) holds.

By construction,

$$A^2 = \{p \in \mathbf{S}^2_\nu \equiv \mathbf{S}^1_\nu \times (-\tfrac{\pi}{2}, \tfrac{\pi}{2}) : p = (z, \theta) \text{ for some } \theta \in [\theta^2_-(z), \theta^2_+(z)]\}.$$

Hence, A^2 is the image of the acceptable band given by

$$(t, u) \mapsto \big(\exp(2\pi\nu i t),\, (1-u)\theta^2_-(t) + u\theta^2_+(t)\big) \quad (t,\, u \in [0,1]).$$

Using induction and the corresponding versions of items (a)–(e) (whose proofs are the same in the general case), define

$$A^{n+1} = \{p \in A^n : d(p, \partial A^n_{(-1)^n}) \le R + 2^{-n}\} \qquad (n \in \mathbf{N}).$$

Finally, let $B = \bigcap_{n=1}^{+\infty} A^n$. We claim that B is the image of a good band.

Given $N \in \mathbf{N}$ and $m, n > N$, we have

$$\big|\theta^n_\pm(z) - \theta^m_\pm(z)\big| \le 2^{-N+1} \ \text{ for any } z \in \mathbf{S}^1_\nu$$

by construction. Therefore, $\theta^n_+ \searrow \theta_+$ and $\theta^n_- \nearrow \theta_-$ for some functions $\theta_\pm \colon \mathbf{S}^1_\nu \to [-R, R]$, which are continuous as the uniform limit of continuous functions. Moreover, B is the image of the map

$$(t, u) \mapsto \big(\exp(2\pi\nu i t),\, (1-u)\theta_-(t) + u\theta_+(t)\big) \quad (t,\, u \in [0,1]),$$

again by construction. We claim that $d(x, \partial B_\pm) = R$ for any $x \in \partial B_\mp$. Suppose for a contradiction that $d(p, \partial B_-) < R$ for some $p \in \partial B_+$, and let pq be a geodesic of length $d(p, \partial B_-)$, with $q \in \partial B_-$. Choose neighborhoods $U \ni p$ and $V \ni q$ such that $d(x, y) > R$ for any $x \in U$, $y \in V$. Since $p, q \in \partial B_\pm$, by choosing a sufficiently large $n \in \mathbf{N}$, we may find $x \in \partial A^n_+ \cap U$ and $y \in \partial A^n_- \cap V$ with $d(x, y) < R$, a contradiction. Similarly, if $d(p, \partial B_-) = R + \varepsilon$ for some $\varepsilon > 0$, choose neighborhoods $U \ni p$ and $V \ni q$ such that $d(x, y) \ge R + \frac{\varepsilon}{2}$ for any $x \in U$ and $V \ni q$. Let $N \in \mathbf{N}$ be so large that $2^{-N} < \frac{\varepsilon}{2}$. Since $p, q \in \partial B_\pm$, we may find some $n > N$ and $x \in \partial A^n_+ \cap U$, $y \in \partial A^n_- \cap V$. Then $d(x, y) \ge R + \frac{\varepsilon}{2} > R + 2^{-N}$, again a contradiction. The assumption that $d(q, \partial B_+) \neq R$ for some $q \in \partial B_-$ also yields a contradiction. We conclude that B is a good band of width R.

If $r \colon \mathcal{A} \to \mathcal{G}$ is the map which associates to an acceptable band A the good band B obtained by the process described above, then $r(A) = A$ whenever $A \in \mathcal{G}$. In addition, we see by induction that the map $A \mapsto A^n$ is continuous on $\mathcal{A}$ for every $n \in \mathbf{N}$. Given $\varepsilon > 0$, we can arrange that $\|A^n - A^m\|_{C^0} < \varepsilon$ for any $A \in \mathcal{A}$ by choosing $m,\, n \ge N$ and a sufficiently large $N \in \mathbf{N}$. Hence,

$r\colon \mathcal{A} \to \mathcal{G}$ is a retraction. □

(6.12) Corollary. *The space $\mathcal{G}$ is contractible.*

Proof. This is an immediate consequence of (6.10) and (6.11). □

(6.13) Definition. Let B be a good band of width R. A *track* of B is a curve on $\mathbf{S}^2_\nu$ of length R joining a point of ∂B_+ to a point of ∂B_-.

In other words, a track is a length-minimizing geodesic joining ∂B_+ to ∂B_-; in particular, it is a smooth curve. Also, if Γ_1, Γ_2 are two tracks through $p \in \partial B_+$ and $q \in \partial B_-$ then $\Gamma_1 = \Gamma_2$, since two geodesics on $\mathbf{S}^2$ intersect at a pair of antipodal points, and p and q do not map to the same point nor to a pair of antipodal points on $\mathbf{S}^2$ under the covering map.

(6.14) Lemma. *Let B be a good band. Then two tracks of B cannot intersect at a point lying in $\mathring{B}$.*

Proof. Suppose for the sake of obtaining a contradiction that two tracks p_1q_1 and p_2q_2, with $p_i \in \partial B_+$ and $q_i \in \partial B_-$, intersect at a point $x \in \mathring{B}$ (see fig. 10). Then one of the following must occur:[5]

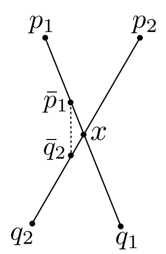

Figure 10:

(i) $xq_1 = xq_2$;

(ii) $xq_1 > xq_2$;

(iii) $xq_1 < xq_2$.

If (i) holds, let $\bar{p}_1, \bar{q}_2$ be points on p_1x and xq_2, respectively, which lie in a normal neighborhood of x. Then, by the triangle inequality,

$$R = p_1q_1 = p_1x + xq_2 > p_1\bar{p}_1 + \bar{p}_1\bar{q}_2 + \bar{q}_2q_2.$$

This contradicts the fact that B is a good band of width R.

[5]Here ab denotes the segment of the corresponding geodesic and also its length.

If (ii) holds then $R = p_1q_1 > p_1x + xq_2$. Again, this contradicts the fact that p_1q_1 is a path of minimal length joining p_1 to ∂B_-. Similarly, if (iii) holds then $R = p_2q_2 > p_2x+xq_1$, contradicting the fact that p_2q_2 is a path of minimal length joining p_2 to ∂B_-. □

Remark. Note that this result may be false for an acceptable band. In the proof, we have implicitly used the fact that if pq is a path of minimal length joining $p \in \partial B_+$ to ∂B_- then pq is also a path of minimal length joining q to ∂B_+, and this is not necessarily true for an acceptable band.

(6.15) Lemma. *Every point in the image of a good band B lies in a unique track of B.*

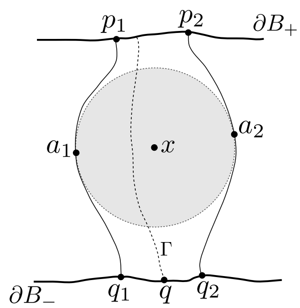

Figure 11:

Proof. Let R be the width of B and let $T \subset \mathrm{Im}(B)$ consist of all points which lie on some track of B. It is clear from the definitions that $\partial B_\pm \subset T$. We claim that $a \in T$ if and only if

$$d(a, \partial B_+) + d(a, \partial B_-) = R \tag{7}$$

The existence of a track through a implies that $d(a, \partial B_+) + d(a, \partial B_-) \le R$. If the inequality were strict, then there would exist a path of length less than R joining ∂B_+ to ∂B_-, which is impossible. Conversely, suppose (7) holds, and let $p \in \partial B_+$, $q \in \partial B_-$ be the points of ∂B_+ (resp. ∂B_-) which are closest to a. Then the concatenation of the geodesics pa and aq is a path of length R joining ∂B_+ to ∂B_-, i.e., a track. Hence, $a \in T$.

The characterization of T that we have established implies that the latter is a closed set. Now suppose that $x \notin T$, let V be the component of $\mathring{B} \smallsetminus T$ containing x (see fig. 11, where V is depicted as a gray open ball). Since T is closed, any point in ∂V lies in T. Choose points $a_1, a_2 \in \partial V \smallsetminus (\partial B_+ \cup \partial B_-)$ such that the (unique) tracks p_iq_i going through a_i do not coincide, where $p_i \in \partial B_+$ and $q_i \in \partial B_-$ $(i = 1, 2)$. Such points a_i exist because otherwise $V = \mathring{B}$, which is absurd since any point on a track lies in T. Because the tracks are distinct, at least one of $p_1 \neq p_2$ or $q_1 \neq q_2$ must hold. Assume without loss of generality that $q_1 \neq q_2$, and let $q \in \partial B_-$ be such that it is possible to join q to x in $\mathrm{Im}(B)$ without crossing p_1q_1 nor p_2q_2. Let Γ be a track through q. Then Γ joins q to ∂B_+, but it does not intersect p_1q_1 nor p_2q_2 by (6.14). It follows that Γ must contain points of V, a contradiction which shows that $T = \mathrm{Im}(B)$. In other words, every point of $\mathrm{Im}(B)$ lies in a track of B; uniqueness has already been established in (6.14). □

(6.16) Corollary. *Let B be a good band of width R. Then $d(a, \partial B_+) + d(a, \partial B_-) = R$ for any $a \in \mathrm{Im}(B)$.* □

(6.17) Lemma. *Let B be a good band of width R and let $0 < r < R$. Then the set γ_r consisting of all those points in $\mathring{B}$ at distance r from ∂B_+ is (the image of) a closed admissible curve whose radius of curvature ρ satisfies $r \leq \rho \leq \pi - R + r$ almost everywhere.*

Proof. For $p \in \mathring{B}$, let $\Gamma_p \colon [0, R] \to \mathbf{S}^2_\nu$ denote the unique track through p, parametrized by arc-length, with $\Gamma_p(0) \in \partial B_-$ and $\Gamma_p(1) \in \partial B_+$. Define vector fields $\mathbf{n}$ and $\mathbf{t}$ in $\mathring{B}$ by letting $\mathbf{n}(p)$ be the unit tangent vector to Γ_p at p and $\mathbf{t}(p) = \mathbf{n}(p) \times p$. We claim that the restriction of $\mathbf{n}$ (and consequently that of $\mathbf{t}$) to any compact subset K of $\mathring{B}$ satisfies a Lipschitz condition. Let $d_0 < \min\{d(K, \partial B_+), d(K, \partial B_-)\}$, let $a_0, a_1 \in K$, with a_1 close to a_0, and consider the (spherical) triangle having Γ_{a_0}, Γ_{a_1}, a_0a_1 as sides and a_0, a_1, a_2 as vertices (see fig. 12). The point a_2 must lie outside of $\mathring{B}$ by (6.14). Let p_0 be the point where the geodesic segment a_0a_2 intersects $\partial B_\pm$. Then

$$a_0a_2 \geq a_0p_0 \geq d_0.$$

Hence, by the law of sines (for spherical triangles) applied to $\triangle a_0a_1a_2$,

$$\frac{\sin a_2}{\sin(a_0a_1)} = \frac{\sin a_1}{\sin(a_0a_2)} \leq \frac{1}{\sin d_0},$$

Using parallel transport we may compare

$$\frac{\angle(\mathbf{n}(a_0), \mathbf{n}(a_1))}{a_0a_1} \quad \text{with} \quad \frac{\sphericalangle a_2}{a_0a_1} \approx \frac{\sin a_2}{\sin(a_0a_1)}$$

to obtain a Lipschitz condition satisfied by the former, but we omit the computations.

Now given $p \in \mathring{B}$ at distance r from ∂B_+, $0 < r < R$, let γ_r be the integral curve through p of the vector field $\mathbf{t}$. Then γ_r is parametrized by arc-length

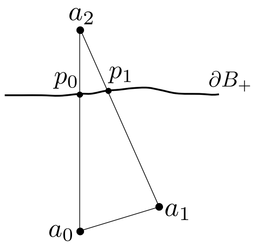

Figure 12:

and its frame is given by

$$\Phi_{\gamma_r}(t) = \begin{pmatrix} | & | & | \\ \gamma_r(t) & \mathbf{t}(\gamma_r(t)) & \mathbf{n}(\gamma_r(t)) \\ | & | & | \end{pmatrix}$$

by construction. If $d(t) = d(\gamma_r(t), \partial B_+)$ then $\dot{d} \equiv 0$, since $\mathbf{t}(\gamma_r(t))$ is orthogonal to the track through $\gamma_r(t)$ for every t. Hence d is constant, equal to r, and γ_r is a closed curve. Moreover, since $\mathbf{t}$ and $\mathbf{n}$ satisfy a Lipschitz condition when restricted to the image of γ_r, we see that the entries of Φ_{γ_r} are absolutely continuous with bounded derivative. In particular, these derivatives belong to L^2. We conclude that γ_r is admissible.

For $r - R < \theta < r$, the curve $\gamma_{r-\theta}$ is the translation of γ_r by θ (as defined on p. 24, eq. (8)) by construction. Since this curve is regular, we deduce from (6) in (4.7) that the radius of curvature ρ of γ_r satisfies

$$0 < \rho(t) - \theta < \pi$$

for all t at which ρ is defined and all θ in $(r - R, r)$. Therefore, $r \leq \rho \leq \pi - R + r$ a.e.. □

(6.18) Corollary. *Let B be a good band of width R and let $0 < r < R$. Then the central curve $\gamma_{\frac{R}{2}}$ is an admissible curve whose radius of curvature is restricted to $\left[\frac{R}{2}, \pi - \frac{R}{2}\right]$.* □

Before finally presenting a proof of (6.7), we extend the definition of the regular band of a curve to any space $\mathcal{L}_{\kappa_1}^{\kappa_2}$.

(6.19) Definition. Let $\gamma \in \mathcal{L}_{\kappa_1}^{\kappa_2}$. The (*regular*) *band* B_γ spanned by γ is the map:

$$B_\gamma \colon [0,1] \times [\rho_1 - \pi, \rho_2] \to \mathbf{S}^2, \quad B_\gamma(t,\theta) = \cos\theta\,\gamma(t) + \sin\theta\,\mathbf{n}(t).$$

The statement and proof of (4.7) still hold, except for obvious modifications.

Proof of (6.7). By (2.10), we may assume that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ $(\kappa_0 < 0)$ is of class C^2. Let ρ_γ denote its radius of curvature, $\rho_0 = \operatorname{arccot} \kappa_0$,

$$\rho_1 = \frac{\pi - \rho_0}{2}, \quad \kappa_1 = \cot \rho_1 \tag{8}$$

(compare (2.26)) and let η be the translation of γ by ρ_1. Then the radius of curvature ρ_η of η satisfies $\rho_1 < \rho_\eta < \pi - \rho_1$. Since ρ_η is continuous, there exists $\bar{\rho}_1$ with $\rho_1 < \bar{\rho}_1 < \frac{\pi}{2}$ such that

$$\bar{\rho}_1 < \rho_\eta < \pi - \bar{\rho}_1.$$

In particular, the regular band of η may be extended from $[0,1] \times [-\rho_1, \rho_1]$ to $[0,1] \times [-\bar{\rho}_1, \bar{\rho}_1]$. Consider the space $\mathcal{G}$ of good bands of width $R = 2\bar{\rho}_1$ and the corresponding space $\mathcal{A} \supset \mathcal{G}$ of acceptable bands. Let B_0 the regular band of η (whose image is the same that of the regular band of γ), and B_1 be the regular band of a condensed circle in $\mathcal{L}_{\kappa_0}^{+\infty}$ traversed ν times, where ν is the rotation number of γ. The combination of (6.9), (6.12) and (6.18) yields a homotopy $s \mapsto \eta_s$ from $\eta = \eta_0$ to a circle η_1, where η_s is the central curve of a good band B_s, $s \in [0,1]$. Moreover, (6.18) guarantees that the radius of curvature ρ_{η_s} of η_s satisfies $\bar{\rho}_1 \leq \rho_{\eta_s} \leq \pi - \bar{\rho}_1$ for each $s \in [0,1]$. Consequently,

$$\rho_1 < \rho_{\eta_s} < \pi - \rho_1 \quad \text{for each } s \in [0,1]$$

and it follows that $s \mapsto \eta_s$ is a path in $\mathcal{L}_{-\kappa_1}^{+\kappa_1}$ from η to a circle. If we let γ_s be the translation of η_s by $-\rho_1$, then γ_0 is the original curve γ, and $s \mapsto \gamma_s$ is a path in $\mathcal{L}_{\kappa_0}^{+\infty}$ from γ to a circle γ_1 traversed ν times.

We have proved that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ lies in the same component of $\mathcal{L}_{\kappa_0}^{+\infty}$ as a circle traversed a number of times. The latter space may be replaced by $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ without altering the conclusion by the usual trick of substituting γ_s by $\Phi_{\gamma_s}(0)^{-1}\gamma_s$ $(s \in [0,1])$. □