7
Non-diffuse Curves

In this section we define a notion of rotation number for any non-diffuse curve in $\mathcal{L}_{\kappa_0}^{+\infty}$ and prove a bound on the total curvature of such a curve which depends only on its rotation number and κ_0 (prop. (7.8)).

(7.1) Lemma. *Suppose X is a connected, locally connected topological space and $C \neq \emptyset$ is a closed connected subspace. Let $\bigsqcup_{\alpha \in J} B_\alpha$ be the decomposition of $X \smallsetminus C$ into connected components. Then:*

(a) $\partial B_\alpha \subset C$ for all $\alpha \in J$.

(b) For any $J_0 \subset J$, the union $C \cup \bigcup_{\beta \in J_0} B_\beta$ is also connected.

Proof. Assume (a) is false, and let $p \in \partial B_\alpha \smallsetminus C$ for some α. Since C is closed and X locally connected, we can find a connected neighborhood $U \ni p$ which is disjoint from C. But $U \cap B_\alpha \neq \emptyset$ and B_α is a connected component of $X \smallsetminus C$, hence $U \subset B_\alpha$, contradicting the fact that $p \in \partial B_\alpha$. Therefore $\partial B_\alpha \subset C$ as claimed. Moreover, $\partial B_\alpha \neq \emptyset$, otherwise $X = B_\alpha \sqcup (X \smallsetminus B_\alpha)$ would be a decomposition of the connected space X into two open sets. Now, for $\beta \in J$, set $A_\beta = C \cup B_\beta = C \cup \overline{B}_\beta$. Each A_β is a union of two connected sets with non-empty intersection, hence is itself connected. Similarly, $C \cup \bigcup_{\beta \in J} B_\beta = \bigcup_{\beta \in J} A_\beta$ is connected as the union of a family of connected sets with a point in common. $\square$

We will also need the following well-known results.[1]

(7.2) Theorem. *Let $A \subset \mathbf{S}^2$ be a connected open set.*

(a) A is simply-connected if and only if $\mathbf{S}^2 \smallsetminus A$ is connected.

(b) If A is simply-connected and $\mathbf{S}^2 \smallsetminus A \neq \emptyset$, then A is homeomorphic to an open disk.

(c) Let $S_\pm \subset \mathbf{S}^2$ be disjoint and homeomorphic to $\mathbf{S}^1$. Then the closure of the region bounded by S_- and S_+ is homeomorphic to $\mathbf{S}^1 \times [-1, 1]$. $\square$

[1] Part (b) of (7.2) is an immediate corollary of the Riemann mapping theorem and part (c) is the 2-dimensional case of the annulus theorem.

(7.3) Lemma. *Let $U_\pm \subset \mathbf{S}^2$ be homeomorphic to open disks, $U_- \cup U_+ = \mathbf{S}^2$. Then*

$$U_- \cap U_+ \approx \mathbf{S}^1 \times (-1, 1).$$

Proof. We first make two claims:

(a) Suppose $C \approx \mathbf{S}^1 \times [-1, 1]$ and $h\colon \partial C_- \to \mathbf{S}^1 \times \{-1\}$ is a homeomorphism, where ∂C_- is one of the boundary circles of C. Then h may be extended to a homeomorphism $H\colon C \to \mathbf{S}^1 \times [-1, 1]$.

(b) Let M be a tower of cylinders, in the sense that:

 (i) $M_i \approx \mathbf{S}^1 \times [-1, 1]$ for each $i \in \mathbf{Z}$;

 (ii) $M = \bigcup_{i \in \mathbf{Z}} M_i$ and M has the weak topology determined by the M_i;

 (iii) $M_i \cap M_j = \emptyset$ for $j \neq i \pm 1$ and $M_i \cap M_{i+1} = S_i^+ = S_{i+1}^-$, where $S_i^\pm$ are the boundary circles of M_i.

 Then $M \approx \mathbf{S}^1 \times (-1, 1)$.

Claim (a) is obviously true if $C = \mathbf{S}^1 \times [-1, 1]$: Just set $H(z, t) = (h(z), t)$. In the general case let $F\colon C \to \mathbf{S}^1 \times [-1, 1]$ be a homeomorphism. Note that ∂C is well-defined as the inverse image of $\mathbf{S}^1 \times \{\pm 1\}$ ($p \in \partial C$ if and only if $U \smallsetminus \{p\}$ is contractible whenever U is a sufficiently small neighborhood of p). Hence ∂C consists of two topological circles, $\partial C_\pm = F^{-1}\big(\mathbf{S}^1 \times \{\pm 1\}\big)$. Let $f = F|_{\partial C_-}$ and let $g = h \circ f^{-1}\colon \mathbf{S}^1 \to \mathbf{S}^1$. As we have just seen, we can extend g to a self-homeomorphism G of $\mathbf{S}^1 \times [-1, 1]$. Now define $H\colon C \to \mathbf{S}^1 \times [-1, 1]$ by $H = G \circ F$. Then $H|_{\partial C_-} = g \circ f = h$, as desired.

To prove claim (b), let $H_0\colon M_0 \to \mathbf{S}^1 \times [-\frac{1}{2}, \frac{1}{2}]$ be any homeomorphism. By applying (a) to $M_{\pm 1}$ and $h_{\pm 1} = H_0|_{S_0^\pm}$, we can extend H_0 to a homeomorphism

$$H_1\colon M_0 \cup M_{\pm 1} \to \mathbf{S}^1 \times \left[-\tfrac{2}{3}, \tfrac{2}{3}\right],$$

and, inductively, to a homeomorphism

$$H_k\colon \bigcup_{|i| \leq k} M_i \to \mathbf{S}^1 \times \Big[-1 + \frac{1}{k+2}, 1 - \frac{1}{k+2}\Big] \qquad (k \in \mathbf{N}).$$

Finally, let $H\colon M \to \mathbf{S}^1 \times (-1, 1)$ be defined by $H(p) = H_i(p)$ if $p \in M_i$. Then H is bijective, continuous and proper, so it is the desired homeomorphism.

Returning to the statement of the lemma, note first that $\partial U_\pm \subset U_\mp$. Indeed, if $p \in \partial U_- \cap (\mathbf{S}^2 \smallsetminus U_+)$ then $p \notin U_- \cup U_+ = \mathbf{S}^2$, hence no such p exists. Let $h_\pm\colon B(0; 1) \to U_\pm$ be homeomorphisms, and define $f_\pm\colon [0, 1) \to \mathbf{R}$ by

$$f_\pm(r) = \sup\big\{d\big(p, \partial U_\pm\big) : p \in h_\pm(r\mathbf{S}^1)\big\},$$

where d denotes the distance on $\mathbf{S}^2$. We claim that $\lim_{r\to 1} f_\pm(r) = 0$. Observe first that $f_\pm$ is strictly decreasing, for if $q \in h_\pm(r_0\mathbf{S}^1)$, $r_0 < r$, then any geodesic joining q to $\partial U_\pm$ intersects $h(r\mathbf{S}^1)$. Hence the limit exists; if it were positive, then $U_\pm$ would be at a positive distance from $\partial U_\pm$, which is absurd.

Now choose $n \in \mathbf{N}$ such that

$$f_\pm(t) < \frac{1}{2}\min\big\{d\big(\partial U_-, \mathbf{S}^2 \smallsetminus U_+\big), d\big(\partial U_+, \mathbf{S}^2 \smallsetminus U_-\big)\big\}$$

for any $t > 1 - \frac{1}{n}$. Set

$$S_i = h_+\Big(\Big(1 - \frac{1}{n+i}\Big)\mathbf{S}^1\Big) \text{ for } i > 0 \text{ and } S_i = h_-\Big(\Big(1 - \frac{1}{n-i}\Big)\mathbf{S}^1\Big) \text{ for } i < 0.$$

Finally, let M_0 be the region of $U_- \cap U_+$ bounded by S_1 and S_{-1} and, for $i > 0$ (resp. < 0), let M_i the region bounded by S_i and S_{i+1} (resp. S_{i-1}). Using (7.2 (c)) we see that $U_- \cap U_+ = \bigcup M_i$ is a tower of cylinders as in claim (b), and we conclude that $U_- \cap U_+ \approx \mathbf{S}^1 \times (-1, 1)$. □

Remark. Another proof of the previous result can be obtained as follows: Since $U_\pm$ are each contractible, the Mayer-Vietoris sequence yields immediately that $U_- \cap U_+$ has the homology of $\mathbf{S}^1$. Together with a little more work it then follows from the classification of noncompact surfaces that $U_- \cap U_+ \approx \mathbf{S}^1 \times (-1, 1)$.

We now return to spaces of curves.

(7.4) Definitions. For fixed $\kappa_0 \in \mathbf{R}$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$, let C denote the image of C_γ and $D = -C$. Assuming γ non-diffuse (meaning that $C \cap D = \emptyset$), let $\hat{C}$ (resp. $\hat{D}$) be the connected component of $\mathbf{S}^2 \smallsetminus D$ containing C (resp. the component of $\mathbf{S}^2 \smallsetminus C$ containing D) and let $B = \hat{C} \cap \hat{D}$.

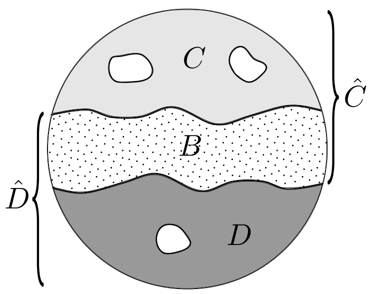

Figure 13: A sketch of the sets defined in (7.4) for a non-diffuse curve $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$. The lightly shaded region is C and the darkly shaded region is $D = -C$; both are closed. The dotted region represents B, which is homeomorphic to $\mathbf{S}^1 \times (-1, 1)$ by (7.5 (c)).

(7.5) Lemma. *Let the notation be as in (7.4).*

(a) *C and D are at a positive distance from each other.*

(b) *$B \subset \mathbf{S}^2 \smallsetminus (C \cup D)$ is open and consists of all $p \in \mathbf{S}^2$ such that: there exists a path $\eta\colon [-1,1] \to \mathbf{S}^2$ with*

$$\eta(-1) \in D, \quad \eta(1) \in C, \quad \eta(0) = p \quad \textit{and} \quad \eta(-1,1) \subset \mathbf{S}^2 \smallsetminus (C \cup D).$$

(c) *The set B is homeomorphic to $\mathbf{S}^1 \times (-1,1)$.*

Proof. The proof of each item will be given separately.

(a) This is clear, since C and D are compact sets which, by hypothesis, do not intersect.

(b) Being components of open sets, $\hat{C}$ and $\hat{D}$ are open, hence so is B.

Suppose $p \in B$. Since $p \in \hat{C}$, there exists $\eta_+\colon [0,1] \to \mathbf{S}^2$ such that

$$\eta_+(0) = p, \quad \eta_+(1) \in C \quad \text{and} \quad \eta_+[0,1] \subset \mathbf{S}^2 \smallsetminus D.$$

We can actually arrange that $\eta_+[0,1) \subset \mathbf{S}^2 \smallsetminus (C \cup D)$ by restricting the domain of η_+ to $[0,t_0]$, where $t_0 = \inf\big\{t \in [0,1] : \eta_+(t) \in C\big\}$ and reparametrizing; note that $t_0 > 0$ because B is open and disjoint from C. Similarly, there exists $\eta_-\colon [-1,0] \to \mathbf{S}^2$ such that

$$\eta_-(-1) \in D, \quad \eta_-(0) = p \quad \text{and} \quad \eta_-(-1,0] \subset \mathbf{S}^2 \smallsetminus (C \cup D).$$

Thus, $\eta = \eta_- * \eta_+$ satisfies all the requirements stated in (b).

Conversely, suppose that such a path η exists. Then $p \in \hat{C}$, for there is a path $\eta_+ = \eta|_{[0,1]}$ joining p to a point of C while staying outside of D at all times. Similarly, $p \in \hat{D}$, whence $p \in B$.

(c) The set $\hat{C}$ is open and connected by definition. Its complement is also connected by (7.1 (b)), as it consists of D and the components of $\mathbf{S}^2 \smallsetminus D$ distinct from $\hat{C}$. From (7.2 (a)) it follows that $\hat{C}$ is simply-connected. Further, $\hat{C} \cap D = \emptyset$, hence the complement of $\hat{C}$ is non-empty and (7.2 (b)) tells us that $\hat{C}$ is homeomorphic to an open disk. By symmetry, the same is true of $\hat{D}$.

We claim that $\hat{C} \cup \hat{D} = \mathbf{S}^2$. To see this suppose $p \notin C$, and let A be the component of $\mathbf{S}^2 \smallsetminus C$ containing p. If $A \cap D \neq \emptyset$ then $A = \hat{D}$ by definition. Otherwise $A \cap D = \emptyset$, hence there exists a path in $\mathbf{S}^2 \smallsetminus D$

joining p to ∂A. By (7.1 (a)), $\partial A \subset C$, consequently $A \subset \hat{C}$. In either case, $p \in \hat{C} \cup \hat{D}$.

We are thus in the setting of (7.3), and the conclusion is that

$$B = \hat{C} \cap \hat{D} \approx \mathbf{S}^1 \times (-1, 1). \qquad \square$$

In what follows let ∂B_γ be the restriction of B_γ to $[0, 1] \times \{0, \rho_0 - \pi\}$, let

$$\hat{B} = \mathrm{Im}(B_\gamma) \smallsetminus \mathrm{Im}(\partial B_\gamma),$$

and let

$$\bar{B}_\gamma \colon \mathbf{S}^1 \times [\rho_0 - \pi, 0] \to \mathbf{S}^2$$

be the unique map satisfying $\bar{B}_\gamma \circ (\mathrm{pr} \times \mathrm{id}) = B_\gamma$, $\mathrm{pr}(t) = \exp(2\pi i t)$.

(7.6) Lemma. *Let $\kappa_0 \in \mathbf{R}$ and suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is non-diffuse. Then:*

(a) For any $t \in [0, 1]$, $B_\gamma\big(\{t\} \times (\rho_0 - \pi, 0)\big)$ intersects B.

(b) $B \subset \hat{B}$.

(c) $\bar{B}_\gamma^{-1}(q)$ is a finite set for any $q \in \mathbf{S}^2$ and $\bar{B}_\gamma \colon \bar{B}_\gamma^{-1}(\hat{B}) \to \hat{B}$ is a covering map.

Proof. We split the proof into parts.

(a) Note first that $B_\gamma(t, 0) \in C$ and $B_\gamma(t, \rho_0 - \pi) \in D$ for any $t \in [0, 1]$ by definition. Let

$$\theta_1 = \inf\big\{\theta \in [\rho_0 - \pi, 0] : B_\gamma(t, \theta) \in C\big\},$$
$$\theta_0 = \sup\big\{\theta \in [\rho_0 - \pi, \theta_1] : B_\gamma(t, \theta) \in D\big\}.$$

Then $\theta_0 < \theta_1$ by (7.5 (a)). Let $\eta = B_\gamma|_{\{t\} \times [\theta_0, \theta_1]}$. Then

$$\eta(\theta_0) \in D, \quad \eta(\theta_1) \in C \quad \text{and} \quad \eta(\theta_0, \theta_1) \subset \mathbf{S}^2 \smallsetminus (C \cup D)$$

by construction. Therefore, any point $\eta(\theta)$ for $\theta \in (\theta_0, \theta_1)$ satisfies the characterization of B given in (7.5 (b)), and we conclude that

$$B_\gamma\big(\{t\} \times (\theta_0, \theta_1)\big) \subset B.$$

(b) Let $B_0 = B \cap \mathrm{Im}(B_\gamma)$. By part (a), $B_0 \neq \emptyset$. Since $\mathrm{Im}(\partial B_\gamma) \subset C \cup D$, while $B \cap (C \cup D) = \emptyset$ by definition, $B \cap \mathrm{Im}(\partial B_\gamma) = \emptyset$. Hence,

$$B_0 = B \cap \bar{B}_\gamma\big(\mathbf{S}^1 \times (\rho_0 - \pi, 0)\big),$$

which is an open set because $\bar{B}_\gamma$ is an immersion, by (4.7 (a)). Since $\mathrm{Im}(B_\gamma)$ is compact, B_0 is also closed in B. But B is connected by (7.5 (c)), consequently $B_0 = B$ and $B \subset \hat{B}$.

(c) Let $q \in \mathbf{S}^2$ be arbitrary. The set $\bar{B}_\gamma^{-1}(q)$ is discrete because $\bar{B}_\gamma$ is an immersion, and it is compact as a closed subset of $\mathbf{S}^2$. Hence, it must be finite. Now suppose $q \in \hat{B}$. Let $\bar{B}_\gamma^{-1}(q) = \{p_i\}_{i=1}^n$ and choose disjoint open sets $U_i \ni p_i$ restricted to which $\bar{B}_\gamma$ is a diffeomorphism. Let $U = \bigcup_{i=1}^n U_i$ and

$$W = \bar{B}_\gamma(U_1) \cap \cdots \cap \bar{B}_\gamma(U_n) \smallsetminus \bar{B}_\gamma\big(\mathbf{S}^1 \times [\rho_0 - \pi, 0] \smallsetminus U\big).$$

Then W is a distinguished neighborhood of q, in the sense that $\bar{B}_\gamma^{-1}(W) = \bigsqcup_{i=1}^n V_i$ and $\bar{B}_\gamma \colon V_i \to W$ is a diffeomorphism for each i, where

$$V_i = \bar{B}_\gamma^{-1}(W) \cap U_i. \qquad \square$$

Parts (b) and (c) of (7.6) allow us to introduce a useful notion which essentially counts how many times a non-diffuse curve winds around $\mathbf{S}^2$.

(7.7) Definition. Let $\kappa_0 \in \mathbf{R}$ and suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is non-diffuse. We define the *rotation number* $\nu(\gamma)$ of γ to be the number of sheets of the covering map $\bar{B}_\gamma \colon \bar{B}_\gamma^{-1}(B) \to B$.

Remark. Suppose now that $\kappa_0 > 0$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is not only non-diffuse but also condensed (meaning that C is contained in a closed hemisphere). In this case, a “more natural” notion of the rotation number of γ is available, as described on p. 55. Let us temporarily denote by $\bar{\nu}(\gamma)$ the latter rotation number. We claim that $\bar{\nu}(\gamma) = \nu(\gamma)$ for any condensed and non-diffuse curve γ. It is easy to check that this holds whenever γ is a circle traversed a number of times. If γ_s ($s \in [0, 1]$) is a continuous family of curves of this type then $\nu(\gamma_s) = \nu(\gamma_0)$ and $\bar{\nu}(\gamma_s) = \bar{\nu}(\gamma_0)$ for any s, since ν and $\bar{\nu}$ can only take on integral values and every element in their definitions depends continuously on s. Moreover, it follows from (6.2) that any condensed and non-diffuse curve is homotopic through curves of this type to a circle traversed a number of times.

(7.8) Proposition. *Let $\kappa_0 \in \mathbf{R}$ and suppose that $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is non-diffuse. Then there exists a constant K depending only on κ_0 such that*

$$\operatorname{tot}(\gamma) \leq K\nu(\gamma).$$

Proof. It is easy to check that being non-diffuse is an open condition. Using (2.8), we deduce that the closure of the subset of all C^2 non-diffuse curves in $\mathcal{L}_{\kappa_0}^{+\infty}$ contains the set of all (admissible) non-diffuse curves. Therefore, we lose no generality in restricting our attention to C^2 curves.

Let $b \in B$ be arbitrary; we have $B = -B$, hence $-b \in B$ also. Let $\hat{\gamma}$ be the other boundary curve of B_γ:

$$\hat{\gamma}(t) = B_\gamma(t, \rho_0 - \pi) = -\cos\rho_0\,\gamma(t) - \sin\rho_0\,\mathbf{n}(t) \quad (t \in [0,1]).$$

Then

$$\hat{\gamma}'(t) = \big(\kappa(t)\sin\rho_0 - \cos\rho_0\big)\gamma'(t) = \frac{\sin(\rho_0 - \rho(t))}{\sin\rho(t)}\gamma'(t) \quad (t \in [0,1]).^2 \tag{1}$$

(Here, as always, $\kappa = \cot\rho$ is the geodesic curvature of γ.) In particular, the unit tangent vector $\hat{\mathbf{t}}$ to $\hat{\gamma}$ satisfies $\hat{\mathbf{t}} = \mathbf{t}$. By (2.21), the geodesic curvature $\hat{\kappa}$ of $\hat{\gamma}$ is given by

$$\hat{\kappa}(t) = \cot(\rho(t) - (\rho_0 - \pi)) = \cot(\rho(t) - \rho_0) \quad (t \in [0,1]). \tag{2}$$

Define $h, \hat{h} \colon [0,1] \to (-1,1)$ by

$$h(t) = \langle \gamma(t), b \rangle \quad \text{and} \quad \hat{h}(t) = \langle \hat{\gamma}(t), b \rangle\,. \tag{3}$$

These functions measure the “height” of γ and $\hat{\gamma}$ with respect to $\pm b$. We cannot have $|h(t)| = 1$ nor $|\hat{h}(t)| = 1$ because the images of γ and $\hat{\gamma}$ are contained in C and D respectively, which are disjoint from B (by definition (7.4)). Also,

$$h'(t) = |\gamma'(t)|\,\langle b, \mathbf{t}(t)\rangle\,, \qquad \hat{h}'(t) = \frac{\sin(\rho_0 - \rho(t))}{\sin\rho(t)}h'(t). \tag{4}$$

Let Γ_t be the great circle whose center on $\mathbf{S}^2$ is $\mathbf{t}(t)$,

$$\Gamma_t = \big\{ \cos\theta\,\gamma(t) + \sin\theta\,\mathbf{n}(t) \,:\, \theta \in [-\pi, \pi) \big\}.$$

We have $\gamma(t)$, $\hat{\gamma}(t) \in \Gamma_t$ by definition. Moreover, the following conditions are equivalent:

(i) $b \in \Gamma_t$.

[2] In this proof, derivatives with respect to t are denoted using a $'$ to simplify the notation.

(ii) $h'(t) = 0$.

(iii) $\hat{h}'(t) = 0$.

(iv) The segment $B_\gamma\big(\{t\} \times (\rho_0 - \pi, 0)\big)$ contains either b or $-b$.

The equivalence of the first three conditions follows from (4). The equivalence (i)$\leftrightarrow$(iv) follows from the facts that $b \notin C \cap D$ and that Γ_t is the union of the segments $\pm B_\gamma\big(\{t\} \times (\rho_0 - \pi, 0)\big)$ and $\pm C_\gamma\big(\{t\} \times [0, \rho_0]\big)$ (see fig. 6, p. 37). The equivalence of the last three conditions tells us that h and $\hat{h}$ have exactly $2\nu(\gamma)$ critical points, for each of $B_\gamma^{-1}(b)$ and $B_\gamma^{-1}(-b)$ has cardinality $\nu(\gamma)$, by definition (7.7).

Suppose that τ is a critical point of h and $\hat{h}$. Because $b \in \Gamma_\tau \setminus (C \cup D)$, we can write

$$b = \cos\theta\,\gamma(\tau) + \sin\theta\,\mathbf{n}(\tau), \text{ for some } \theta \in (\rho_0 - \pi, 0) \cup (\rho_0, \pi). \tag{5}$$

A straightforward calculation shows that:

$$h''(\tau) = \langle \gamma''(\tau), b \rangle = \frac{|\gamma'(\tau)|^2}{\sin\rho(\tau)} \sin(\theta - \rho(\tau)).$$

Using (5) and $0 < \rho(\tau) < \rho_0$ we obtain that either

$$-\pi < \theta - \rho(\tau) < 0 \quad \text{or} \quad 0 < \theta - \rho(\tau) < \pi.$$

In any case, we deduce that $h''(\tau) \neq 0$. The proof that τ is a nondegenerate critical point of $\hat{h}$ is analogous: one obtains by another calculation that

$$\hat{h}''(\tau) = \frac{|\gamma'(\tau)|^2}{\sin^2(\rho(\tau))} \sin(\rho_0 - \rho(\tau)) \sin(\theta - \rho(\tau)),$$

and it follows from the above inequalities that $\hat{h}''(\tau) \neq 0$. In particular, two neighboring critical points $\tau_0 < \tau_1$ of h (and $\hat{h}$) cannot be both maxima or both minima for h (and $\hat{h}$). We will prove the proposition by obtaining an upper bound for $\operatorname{tot}\big(\gamma|_{[\tau_0,\tau_1]}\big)$.

We first claim that $B_\gamma|_{[\tau_0,\tau_1]\times[\rho_0-\pi,0]}$ is injective. Suppose for concreteness that $h' < 0$ throughout (τ_0, τ_1) and that $b = B_\gamma(\tau_0, \theta_0)$, $-b = B_\gamma(\tau_1, \theta_1)$, where $\theta_0, \theta_1 \in (\rho_0 - \pi, 0)$. Let $\alpha = \alpha_1 * \alpha_2 * \alpha_3$ be the concatenation of the curves $\alpha_i \colon [0,1] \to \mathbf{S}^2$ given by

$$\begin{aligned} \alpha_1(t) &= B_\gamma\big(\tau_0\,,\,(1-t)\theta_0\big), \quad \alpha_2(t) = \gamma\big((1-t)\tau_0 + t\tau_1\big), \\ \alpha_3(t) &= B_\gamma\big(\tau_1\,,\,t\theta_1\big), \end{aligned}$$

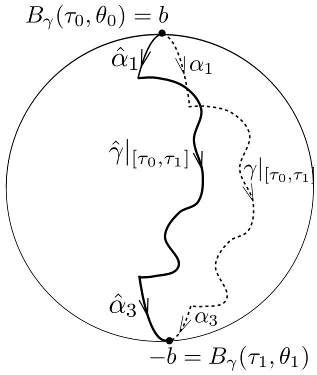

Figure 14: An illustration of the boundary of the rectangle $R = B_\gamma|_{[\tau_0,\tau_1]\times[\rho_0-\pi,0]}$ considered in the proof of (7.8).

as sketched in fig. 14. Similarly, let $\hat{\alpha}$ be the concatenation of the curves $\hat{\alpha}_i\colon [0,1] \to \mathbf{S}^2$,

$$\hat{\alpha}_1(t) = B_\gamma\big(\tau_0\,,\,(1-t)\theta_0 + t(\rho_0 - \pi)\big), \quad \hat{\alpha}_2(t) = \hat{\gamma}\big((1-t)\tau_0 + t\tau_1\big),$$
$$\hat{\alpha}_3(t) = B_\gamma\big(\tau_1\,,\,(1-t)(\rho_0 - \pi) + t\theta_1\big).$$

Define six functions $h_i, \hat{h}_i\colon [0,1] \to [-1,1]$ by the formulas

$$h_i(t) = \langle \alpha_i(t), b\rangle \quad \text{and} \quad \hat{h}_i(t) = \langle \hat{\alpha}_i(t), b\rangle \quad (i = 1,2,3).$$

Note that h_2 is essentially the restriction of h to $[\tau_0, \tau_1]$ and similarly for $\hat{h}_2$ (see (3)). Moreover, all of these functions are monotone decreasing. For $i = 2$ this is immediate from (4) and the hypothesis that $h' < 0$ on (τ_0, τ_1). For $i = 1, 3$ this follows from the fact that α_i, $\hat{\alpha}_i$ are geodesic arcs through $\pm b$, and our choice of orientations for these curves.

Because the map $B_\gamma|_{[\tau_0,\tau_1]\times[\rho_0-\pi,0]}$ is an immersion, if B_γ is not injective then either α and $\hat{\alpha}$ intersect each other, or one of them has a self-intersection. We can discard the possibility that either curve has a self-intersection from the fact that all functions h_i, $\hat{h}_i$ are monotone decreasing. Further, since $B \approx \mathbf{S}^1 \times (-1,1)$, we can find a Jordan curve $\beta\colon [0,1] \to B$ through $\pm b$ winding once around the $\mathbf{S}^1$ factor. If α and $\hat{\alpha}$ intersect (at some point other than $\alpha(0) = \hat{\alpha}(0)$ or $\alpha(1) = \hat{\alpha}(1)$), then this must be an intersection of γ and $\hat{\gamma}$. This is impossible because β, which has image in B, separates C and D, which contain the images of γ and $\hat{\gamma}$, respectively.

Thus, $R = B_\gamma|_{[\tau_0,\tau_1]\times[\rho_0-\pi,0]}$ is diffeomorphic to a rectangle, and its boundary consists of $\hat{\gamma}|_{[\tau_0,\tau_1]}$, $\gamma|_{[\tau_0,\tau_1]}$ (the latter with reversed orientation) and the two geodesic arcs $B_\gamma\big(\{\tau_0\} \times [\rho_0 - \pi, 0]\big)$ and $B_\gamma\big(\{\tau_1\} \times [\rho_0 - \pi, 0]\big)$. Recall from (4.7) that $\frac{\partial B_\gamma}{\partial t}$ is always orthogonal to $\frac{\partial B_\gamma}{\partial \theta}$. Using Gauss-Bonnet we deduce that

$$\Big(\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}+\frac{\pi}{2}\Big)+\int_{\tau_0}^{\tau_1}\hat{\kappa}(t)\,|\hat{\gamma}'(t)|\;dt-\int_{\tau_0}^{\tau_1}\kappa(t)\,|\gamma'(t)|\;dt+\text{Area}(R)=2\pi.$$

Using (1), (2) and the fact that $\text{Area}(R)<\text{Area}(\mathbf{S}^2)=4\pi$ we obtain:

$$\int_{\tau_0}^{\tau_1}\Big(\cot\rho(t)+\frac{\sin(\rho_0-\rho(t))}{\sin\rho(t)}\cot(\rho_0-\rho(t))\Big)\,|\gamma'(t)|\;dt<4\pi. \tag{6}$$

Let us see how this yields an upper bound for $\text{tot}\left(\gamma|_{[\tau_0,\tau_1]}\right)$. From $\cos(x)+\cos(y)=2\cos\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$ and $\left|\rho(t)-\frac{\rho_0}{2}\right|<\frac{\rho_0}{2}$ we deduce that

$$\begin{aligned}&\sin\rho(t)\Big(\cot\rho(t)+\frac{\sin(\rho_0-\rho(t))}{\sin\rho(t)}\cot(\rho_0-\rho(t))\Big)\\&=\cos\rho(t)+\cos(\rho_0-\rho(t))=2\cos\Big(\frac{\rho_0}{2}\Big)\cos\Big(\rho(t)-\frac{\rho_0}{2}\Big)\geq 2\cos^2\Big(\frac{\rho_0}{2}\Big).\end{aligned}$$

The Euclidean curvature K of γ thus satisfies

$$\begin{aligned}K(t)&=\sqrt{1+\kappa(t)^2}=\sqrt{1+\cot\rho(t)^2}=\csc\rho(t)\\&\leq\frac{1}{2\cos^2\left(\frac{\rho_0}{2}\right)}\Big(\cot\rho(t)+\frac{\sin(\rho_0-\rho(t))}{\sin\rho(t)}\cot(\rho_0-\rho(t))\Big).\end{aligned} \tag{7}$$

Combining (6) and (7) we obtain:

$$\text{tot}\left(\gamma|_{[\tau_0,\tau_1]}\right)=\int_{\tau_0}^{\tau_1}K(t)\,|\gamma'(t)|\;dt<\frac{2\pi}{\cos^2\left(\frac{\rho_0}{2}\right)}.$$

Extending γ to all of $\mathbf{R}$ by declaring it to be 1-periodic and choosing consecutive critical points $\tau_0<\tau_1<\cdots<\tau_{2\nu(\gamma)-1}<\tau_{2\nu(\gamma)}$, so that $\tau_{2\nu(\gamma)}=\tau_0+1$, we finally conclude from the previous estimate (with $[\tau_{i-1},\tau_i]$ in place of $[\tau_0,\tau_1]$) that

$$\text{tot}(\gamma)=\sum_{i=1}^{2\nu(\gamma)}\text{tot}\left(\gamma|_{[\tau_{i-1},\tau_i]}\right)<\frac{4\pi}{\cos^2\left(\frac{\rho_0}{2}\right)}\,\nu(\gamma).\qquad\square$$