8
Homotopies of Circles

Let $k \geq 1$ be an integer. The *bending of the k-equator* is an explicit homotopy (to be defined below) from a great circle traversed k times to a great circle traversed $k+2$ times. It is an "optimal" homotopy of this type, in the following sense: It is possible to deform a circle traversed k times into a circle traversed $k+2$ times in $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ if and only if we may carry out the bending of the k-equator in this space (meaning that the absolute value of the geodesic curvature is bounded by κ_1 throughout the bending). A special case of this construction was considered by Saldanha in [12].

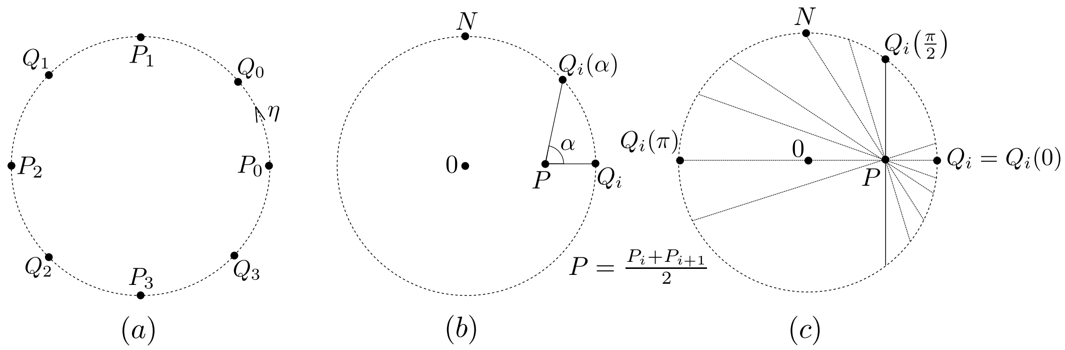

Figure 15:

Let $N = (0,0,1) \in \mathbf{S}^2$ be the north pole, let

$$\eta(t) = \big(\cos(2k\pi t), \sin(2k\pi t), 0\big) \quad (t \in [0,1])$$

be a parametrization of the equator traversed $k \geq 1$ times ($k \in \mathbf{N}$) and let

$$P_i = \eta\Big(\frac{i}{2k+2}\Big), \quad Q_i = \eta\Big(\frac{i+\frac{1}{2}}{2k+2}\Big) \quad (i = 0, 1, \dots, 2k+1),$$

as illustrated in fig. 15 (a) for $k = 1$. Define $Q_i(\alpha)$ (see fig. 15 (b)) to be the unique point in the geodesic through N and Q_i such that

$$\sphericalangle Q_i\Big(\frac{P_i + P_{i+1}}{2}\Big)Q_i(\alpha) = \alpha \quad (-\pi \leq \alpha \leq \pi,\ i = 0, 1, \dots, 2k+1).$$

Let $A_i(\alpha) \subset \mathbf{S}^2$ be the arc of circle through $P_iQ_i(\alpha)P_{i+1}$, with orientation determined by this ordering of the three points, and define

$$\sigma_{\alpha,i}\colon \Big[0, \frac{1}{2k+2}\Big] \to \mathbf{S}^2 \quad (0 \leq \alpha \leq \pi,\ i = 0, \dots, 2k+1)$$

to be a parametrization of $A_i((-1)^i\alpha)$ by a multiple of arc-length, as illustrated in fig. 16 below for $k = 1$. Note that $A_i(0)$ is just $\frac{k}{2k+2}$ of the equator, while $A_i(\pi)$ is the "complement" of $A_i(0)$, which is $\frac{k+2}{2k+2}$ of the equator.

Let $\sigma_\alpha\colon [0,1] \to \mathbf{S}^2$ be the concatenation of all the $\sigma_{\alpha,i}$, for i increasing from 0 to $2k+1$ (as in fig. 16). Then σ_0 is the equator traversed k times, while σ_π is the equator traversed $k+2$ times, in the opposite direction. The curve σ_α is closed and regular for all $\alpha \in [0, \pi]$. However, its geodesic curvature is a step function, taking the value $(-1)^i\kappa(\alpha)$ for $t \in (\frac{i}{2k+2}, \frac{i+1}{2k+2})$, where $\kappa(\alpha)$ depends only on α. At the points $t = \frac{i}{2k+2}$ the curvature is not defined, except for $\alpha = 0, \pi$, when the curvature vanishes identically.

We are only interested in the maximum value of $\kappa(\alpha)$ for $0 \leq \alpha \leq \pi$, which can be easily determined. For any α, the center of the circle C of which $A_i(\alpha)$ is an arc is contained in the plane Π_1 through 0, Q_i and N, since this plane is the locus of points equidistant from P_i and P_{i+1} (Π_1 is the plane of figures 15 (b) and 15 (c)). By definition, C is contained in the plane Π_2 through P_i, $Q_i(\alpha)$ and P_{i+1}. Thus, the center of C lies in the line $\Pi_1 \cap \Pi_2 = PQ_k(\alpha)$, and the segment of this line bounded by $\mathbf{S}^2$ is a diameter of C. Clearly, this diameter is shortest when $\alpha = \frac{\pi}{2}$ (see fig. 15 (c)). (More precisely, the shortest chord through a point lying in the interior of a circle is the one which is perpendicular to the diameter through this point; the proof is an exercise in elementary geometry.) The corresponding spherical radius is $\rho = \frac{k\pi}{2k+2}$, hence *the maximum value attained by $\kappa(\alpha)$ for $0 \leq \alpha \leq \pi$ is*

$$\kappa(\tfrac{\pi}{2}) = \cot\Big(\frac{k\pi}{2k+2}\Big) = \tan\Big(\frac{\pi}{2k+2}\Big),$$

and the minimum value is $-\kappa(\frac{\pi}{2})$.

(8.1) Definition. Let σ_α be as in the discussion above ($0 \leq \alpha \leq \pi$) and assume that

$$\kappa_1 > \tan\Big(\frac{\pi}{2k+2}\Big). \tag{1}$$

The *bending of the k-equator* is the family of curves $\eta_s \in \mathcal{L}_{-\kappa_1}^{+\kappa_1}(I)$ given by:

$$\eta_s(t) = \big(\Phi_{\sigma_{s\pi}}(0)\big)^{-1}\sigma_{s\pi}(t) \quad (s, t \in [0,1]).$$

Note that η_0 is the equator of $\mathbf{S}^2$ traversed k times and η_1 is the equator traversed $k+2$ times, in the same direction. The following result is an

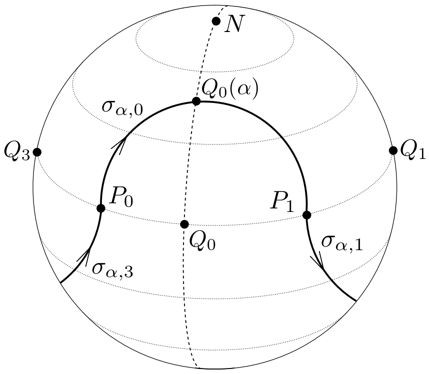

Figure 16: An illustration of the bending of the 1-equator. The curve σ_α is the concatenation of $\sigma_{\alpha,0}, \dots, \sigma_{\alpha,3}$.

immediate consequence of the discussion above.

(8.2) Proposition. *Let $\kappa_0 = \cot\rho_0 \in \mathbf{R}$ and let $\sigma_k, \sigma_{k+2} \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ be circles traversed k and $k+2$ times, respectively. Then σ_k lies in the same component of $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ as σ_{k+2} if*

$$k \geq \left\lfloor \frac{\pi}{\rho_0} \right\rfloor. \tag{2}$$

Proof. Let $\rho_1 = \frac{\pi - \rho_0}{2}$, so that $\kappa_1 = \cot\rho_1$ satisfies (1). Let γ_s ($s \in [0,1]$) be the image of the bending η_s of the k-equator under the homeomorphism $\mathcal{L}_{-\kappa_1}^{+\kappa_1}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$ of (2.26). Then γ_0 is some circle traversed k times, while γ_1 is a circle traversed $k+2$ times. Using (4.4) we deduce that $\sigma_k \simeq \gamma_0 \simeq \gamma_1 \simeq \sigma_{k+2}$, hence σ_k and σ_{k+2} lie in the same component of $\mathcal{L}_{\kappa_0}^{+\infty}(I)$. □

(8.3) Corollary. *Let $\rho_i = \operatorname{arccot}(\kappa_i)$, $i = 1, 2$, and suppose that $\rho_1 - \rho_2 > \frac{\pi}{2}$. Let $\sigma_{k_0}, \sigma_{k_1} \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) be two parametrized circles traversed k_0 and k_1 times, respectively. Then σ_{k_0} and σ_{k_1} lie in the same connected component if and only if $k_0 \equiv k_1 \pmod 2$.*

Proof. Under the homeomorphism $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$ of (2.25), the condition $\rho_1 - \rho_2 > \frac{\pi}{2}$ translates into $\rho_0 > \frac{\pi}{2}$. The result is an immediate consequence of (2.15), (4.4) and (8.2). □

Homotopies of condensed curves

The previous corollary settles the question of when two circles in $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ lie in the same component for $\kappa_0 < 0$. Because of this, we will assume for the rest of the section that $\kappa_0 \geq 0$; the following proposition implies the converse to (8.2), and together with it, settles the same question in this case.

(8.4) Proposition. *Let* $\kappa_0 = \cot \rho_0 \geq 0$ *and let*

$$n = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor + 1.$$

Suppose that $s \mapsto \gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ *is a homotopy, with* γ_0 *condensed and* $\nu(\gamma_0) \leq n-2$ $(s \in [0,1])$. *Then* γ_s *is condensed and* $\nu(\gamma_s) = \nu(\gamma_0)$ *for all* $s \in [0,1]$.

In particular, taking γ_0 to be a circle σ_k traversed k times for $k \leq n-2$, we conclude that it is not possible to deform σ_k into a circle traversed $k+2$ times in $\mathcal{L}_{\kappa_0}^{+\infty}$. The proof of (8.4) will be broken into several parts. We start with the definition of an equatorial curve, which is just a borderline case of a condensed curve.

(8.5) Definition. Let $\kappa_0 \geq 0$. We shall say that a curve $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is *equatorial* if the image C of its caustic band is contained in a closed hemisphere, but not in any open hemisphere. Let

$$H_\gamma = \left\{ p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle \geq 0 \right\}$$

be a closed hemisphere containing γ, and let

$$E_\gamma = \left\{ p \in \mathbf{S}^2 : \langle p, h_\gamma \rangle = 0 \right\}$$

denote the corresponding *equator*. Also, let $\check{\gamma}\colon [0,1] \to \mathbf{S}^2$ be the curve given by

$$\check{\gamma}(t) = C_\gamma(t, \rho_0).$$

(8.6) Lemma. *Let* $\kappa_0 \geq 0$, *let* $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ *be an equatorial curve of class* C^2. *Then:*

(a) The hemisphere H_γ *and the equator* E_γ *defined above are uniquely determined by* γ.

(b) The geodesic curvature $\check{\kappa}$ *of* $\check{\gamma}$ *is given by:*

$$\check{\kappa} = \cot(\rho_0 - \rho) > 0.$$

Proof. Suppose that $C = \mathrm{Im}(C_\gamma)$ is contained in distinct closed hemispheres H_1 and H_2. Then it is contained in the closed lune $H_1 \cap H_2$. Since the curves $\gamma, \check{\gamma}$, whose images form the boundary of C, have a unit tangent vector at all points, they cannot pass through either of the points in $E_1 \cap E_2$ (where E_i is the equator corresponding to H_i). It follows that C is contained in an open hemisphere, a contradiction which establishes (a).

For part (b) we calculate:[1]

$$\check{\gamma}'(t) = |\gamma'(t)|\big(\cos\rho_0 - \kappa(t)\sin\rho_0\big)\mathbf{t}(t) \tag{3}$$
$$\check{\gamma}''(t) = |\gamma'(t)|^2\big(\cos\rho_0 - \kappa(t)\sin\rho_0\big)\big(-\gamma(t) + \kappa(t)\mathbf{n}(t)\big) + \lambda(t)\mathbf{t}(t), \tag{4}$$

where κ, $\mathbf{t}$ and $\mathbf{n}$ denote the geodesic curvature of and unit and normal vectors to γ, respectively, and the value of $\lambda(t)$ is irrelevant to us. Hence,

$$\check{\kappa} = \frac{\langle \check{\gamma}\,,\, \check{\gamma}' \times \check{\gamma}''\rangle}{|\check{\gamma}'|^3} = \frac{\kappa\cos\rho_0 + \sin\rho_0}{|\cos\rho_0 - \kappa\sin\rho_0|} = \frac{\cos(\rho_0 - \rho)}{|\sin(\rho - \rho_0)|} = \cot(\rho_0 - \rho). \qquad \square$$

(8.7) Lemma. *Let $\kappa_0 \geq 0$ and $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ be an equatorial curve of class C^2. Take $N \in E_\gamma$ and define $h, \check{h}\colon [0,1] \to \mathbf{R}$ by*

$$h(t) = \langle \gamma(t), N\rangle\,, \quad \check{h}(t) = \langle \check{\gamma}(t), N\rangle\,. \tag{5}$$

(a) The following conditions are equivalent:

(i) $\pm N \in \Gamma_\tau$ for some $\tau \in [0,1]$.

(ii) $\tau \in [0,1]$ is a critical point of h.

(iii) $\tau \in [0,1]$ is a critical point of $\check{h}$.

(b) If τ is a common critical point of h, $\check{h}$, then $h''(\tau)\check{h}''(\tau) < 0$.

(c) If $\tau < \bar{\tau}$ are neighboring critical points then $h''(\tau)h''(\bar{\tau}) < 0$ and $\check{h}''(\tau)\check{h}''(\bar{\tau}) < 0$.

Recall that Γ_t is the great circle

$$\Gamma_t = \big\{\cos\theta\,\gamma(t) + \sin\theta\,\mathbf{n}(t) \,:\, \theta \in [-\pi, \pi)\big\}.$$

Part (b) implies in particular that all critical points of h, $\check{h}$ are nondegenerate.

Proof. A straightforward calculation using (3) shows that:

$$h'(t) = |\gamma'(t)|\,\langle N, \mathbf{t}(t)\rangle\,, \quad \check{h}'(t) = \frac{\sin(\rho(t) - \rho_0)}{\sin\rho(t)}h'(t) \qquad (t \in [0,1]). \tag{6}$$

The equivalence of the conditions in (a) is immediate from this and the definition of Γ_t.

[1]For the rest of the section we denote derivatives with respect to t by a $'$ to unclutter the notation.

From $\pm N \in E_\gamma$ and $C = \operatorname{Im}(C_\gamma) \subset H_\gamma$, it follows that $\pm N \notin C\big([0,1]\times(0,\rho_0)\big)$. Thus, if τ is a critical point of h, $\check{h}$, i.e., if $N \in \Gamma_\tau$ then we can write

$$N = \cos\theta\,\gamma(\tau) + \sin\theta\,\mathbf{n}(\tau) \quad \text{for some } \theta \in [\rho_0 - \pi, 0] \cup [\rho_0, \pi]. \tag{7}$$

Another calculation, with the help of (4), yields:

$$h''(\tau) = \frac{|\gamma'(\tau)|^2}{\sin\rho(\tau)}\sin\big(\theta - \rho(\tau)\big), \quad \check{h}''(\tau) = \frac{|\gamma'(\tau)|^2}{\sin^2\rho(\tau)}\sin\big(\theta - \rho(\tau)\big)\sin\big(\rho(\tau) - \rho_0\big)$$

Taking the possible values for θ in (7) and $0 < \rho(\tau) < \rho_0$ into account, we deduce that

$$h''(\tau)\check{h}''(\tau) = \frac{|\gamma'(\tau)|^4}{\sin^3\rho(\tau)}\sin^2\big(\theta - \rho(\tau)\big)\sin\big(\rho(\tau) - \rho_0\big) < 0,$$

since all terms here are positive except for $\sin\big(\rho(\tau) - \rho_0\big)$. This proves (b).

For part (c), suppose that $\tau < \bar{\tau}$ are neighboring critical points, but $h''(\tau)h''(\bar{\tau}) > 0$. This means that h' vanishes at $\tau, \bar{\tau}$ and takes opposite signs on the intervals $(\tau, \tau+\varepsilon)$ and $(\bar{\tau}-\varepsilon, \bar{\tau})$ for small $\varepsilon > 0$. Hence, it must vanish somewhere in $(\tau, \bar{\tau})$, a contradiction. The proof for $\check{h}$ is the same. □

Let $\kappa_0 \geq 0$, $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ be an equatorial curve and $\operatorname{pr}\colon \mathbf{S}^2 \to \mathbf{R}^2$ denote the stereographic projection from $-h_\gamma$, where $H_\gamma = \big\{p \in \mathbf{S}^2 : \langle p, h_\gamma\rangle \geq 0\big\}$. As for any condensed curve, we may define a (non-unique) continuous angle function θ by the formula:

$$\exp(i\theta(t)) = \mathbf{t}_\eta(t), \quad \eta(t) = \operatorname{pr}\circ\gamma(t) \quad (t \in [0,1]);$$

here $\mathbf{t}_\eta$ is the unit tangent vector, taking values in $\mathbf{S}^1$, of the plane curve η. The function θ is strictly decreasing since $\kappa_0 \geq 0$, and

$$2\pi\nu(\gamma) = \theta(0) - \theta(1).$$

(8.8) Lemma. *Let $\kappa_0 \geq 0$, $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ be an equatorial curve of class C^2 and*

$$n = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor + 1.$$

Then $\nu(\gamma) \geq n - 1$.

Proof. Let $C = \operatorname{Im}(C_\gamma)$, $H = H_\gamma$ be the closed hemisphere containing γ and $E = E_\gamma$ be the corresponding equator, oriented so that H lies to its left. It

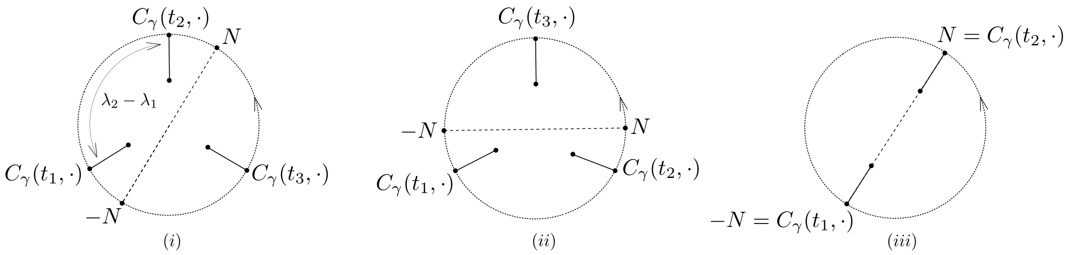

Figure 17: Three possibilities for an equatorial curve γ. The circle represents E_γ and its interior represents H_γ, seen from above.

follows from the combination of (11.1), (11.5) and (11.2) that either we can find two antipodal points in $C \cap E$ or we can choose $t_1 < t_2 < t_3$ and $\theta_i \in \{0, \rho_0\}$ such that 0 is a convex combination of the points $C_\gamma(t_i, \theta_i) \in C \cap E$. There are three possibilities, as depicted in fig. 17; the only difference between the first two is the order of the points in the orientation of E.

In cases (i) and (ii), choose N in E so that

$$\langle C_\gamma(t_2, \theta_2), N\rangle = -\langle C_\gamma(t_1, \theta_1), N\rangle > 0.$$

Let h and $\check{h}$ be as in (5) and define latitude functions λ, $\check{\lambda}$ by

$$\lambda(t) = \arcsin(h(t)), \quad \check{\lambda}(t) = \arcsin(\check{h}(t)) \qquad (t \in [0,1]).$$

Let $\tau_1 < \cdots < \tau_{k_1}$ be all the common critical points of these functions in the interval $[t_1, t_2)$, and let

$$m_j = \min\{\lambda(\tau_j), \check{\lambda}(\tau_j)\}, \quad M_j = \max\{\lambda(\tau_j), \check{\lambda}(\tau_j)\}.$$

From (8.7 (a)), we deduce that

$$M_j - m_j = \rho_0 \quad \text{for all } j = 1, \dots, k_1, \tag{8}$$

while from (8.7 (b)) and (8.7 (c)), we deduce that the τ_j are alternatingly maxima and minima of λ (resp. minima and maxima of $\check{\lambda}$) as j goes from 1 to k_1, whence

$$M_j > m_{j+1} \quad \text{for all } j = 1, \dots, k_1 - 1. \tag{9}$$

Let

$$\lambda_2 = \max\left\{\lambda(t_2), \check{\lambda}(t_2)\right\} \quad \text{and} \quad \lambda_1 = \min\{\lambda(t_1), \check{\lambda}(t_1)\} = -\lambda_2.$$

Then $\lambda_2 - \lambda_1$ is just the angle between $C_\gamma(t_1, \cdot) \cap E$ and $C_\gamma(t_2, \cdot) \cap E$ measured along E, as depicted in fig. 17(i). For the rest of the proof we consider each case separately.

In case (i),

$$m_1 \leq \lambda_1 \quad \text{and} \quad \lambda_2 \leq M_{k_1}. \tag{10}$$

Combining (8), (9) and (10), we find that

$$k_1\rho_0 = \sum_{j=1}^{k_1}(M_j - m_j) > \sum_{j=1}^{k_1-1}(m_{j+1} - m_j) + M_{k_1} - m_{k_1} = M_{k_1} - m_1 \geq \lambda_2 - \lambda_1. \tag{11}$$

Let there be k_2 (resp. k_3) critical points of h, $\check{h}$ in the interval $[t_2, t_3)$ (resp. $[t_3, t_1 + 1)$), where for the latter we are considering γ as a 1-periodic curve. Then an analogous result to (11) holds for k_2 and k_3, and summing all three inequalities we conclude that

$$k_1 + k_2 + k_3 > \frac{2\pi}{\rho_0} \geq 2(n-1).$$

In case (i), the number of half-turns of the tangent vector to the image of γ under stereographic projection through $-h_\gamma$ in $[0,1]$ is given by $k_1+k_2+k_3-2$. Hence,

$$\nu(\gamma) = \frac{k_1 + k_2 + k_3 - 2}{2} > n - 2,$$

as claimed.

In case (ii), a direct calculation using basic trigonometry shows that

$$\begin{aligned} &m_1 < \arcsin(\cos\rho_0 \sin\lambda_1) = -\arcsin(\cos\rho_0\sin\lambda_2) \\ \text{and} \quad &M_{k_1} > \arcsin(\cos\rho_0\sin\lambda_2). \end{aligned}$$

Combining this with (8) and (9), we obtain that

$$\begin{aligned} k_1\rho_0 = \sum_{j=1}^{k_1}(M_j - m_j) &> \sum_{j=1}^{k_1-1}(m_{j+1} - m_j) + M_{k_1} - m_{k_1} \\ &= M_{k_1} - m_1 > 2\arcsin(\cos\rho_0\sin\lambda_2), \end{aligned}$$

and similarly for k_2 and k_3, where the latter denote the number of critical points of h, $\check{h}$ in the intervals $[t_2, t_3)$ and $[t_3, t_1 + 1)$, respectively. More precisely, we have

$$k_1 + k_2 + k_3 > \frac{2}{\rho_0}\sum_{i=1}^{3}\arcsin(\cos\rho_0\sin\lambda_{2i}), \tag{12}$$

where $\lambda_4 = \max\{\lambda(t_3), \check{\lambda}(t_3)\}$, $\lambda_6 = \max\{\lambda(t_1), \check{\lambda}(t_1)\}$ and these latitudes are measured with respect to the chosen points $\pm N$ corresponding to each of the intervals $[t_2, t_3]$ and $[t_3, t_3 + 1]$. In case (ii), the number of half-turns of the tangent vector to the image of γ under stereographic projection through $-h_\gamma$ in $[0,1]$ is given by $k_1+k_2+k_3-2$. Hence, it follows from (12) and lemma (8.9) below that

$$\nu(\gamma)=\frac{k_1+k_2+k_3+2}{2}>\Big(\frac{\pi}{\rho_0}-2\Big)+1\geq n-2,$$

as we wished to prove.

Finally, in case (iii), we may choose $\pm N\in E\cap C$, that is, we may find $t_1<t_2$ and $\theta_i\in\{0,\rho_0\}$ such that

$$N=C_\gamma(t_2,\theta_2)=-C_\gamma(t_1,\theta_1).$$

In this case $\lambda_2-\lambda_1=\pi$ and

$$\nu(\gamma)=\frac{k_1+k_2-2}{2},$$

where k_1 (resp. k_2) is the number of critical points of h, $\check{h}$ in $[t_1,t_2]$ (resp. $[t_2,t_1+1]$). Note that t_1, t_2 are critical points of h which are counted twice in the sum k_1+k_2 (under the identification of t_1 with t_1+1); this is the reason why we need to subtract 2 from k_1+k_2 to calculate the number of half-turns of the tangent vector. Using (9) one more time, we deduce that

$$k_1\rho_0=\sum_{j=1}^{k_1}(M_j-m_j)>\sum_{j=1}^{k_1-1}(m_{j+1}-m_j)+M_{k_1}-m_{k_1}=M_{k_1}-m_1=\lambda_2-\lambda_1=\pi;$$

similarly, $k_2\rho_0>\pi$. Therefore,

$$\nu(\gamma)=\frac{k_1+k_2-2}{2}>\frac{\pi}{\rho_0}-1\geq n-2.$$ □

Here is the technical lemma that was invoked in the proof of (8.8).

(8.9) Lemma. *Let* $\lambda_2+\lambda_4+\lambda_6=\pi$, $0\leq\lambda_i\leq\frac{\pi}{2}$ *and* $0<\rho_0\leq\frac{\pi}{2}$. *Then*

$$\arcsin(\cos\rho_0\sin\lambda_2)+\arcsin(\cos\rho_0\sin\lambda_4)+\arcsin(\cos\rho_0\sin\lambda_6)\geq\pi-2\rho_0$$

Proof. Let $f\colon[0,\pi]\to\mathbf{R}$ be the function given by $f(t)=\arcsin(\cos\rho_0\sin t)$. Then

$$f''(t)=-\frac{\sin^2\rho_0\cos\rho_0\sin t}{\big(1-\cos^2\rho_0\sin^2 t\big)^{\frac{3}{2}}},$$

so that $f''(t)\leq 0$ for all $t\in(0,\pi)$ and f is a concave function. Consequently,

$$f(s_1a+s_2b+s_3c)\geq s_1f(a)+s_2f(b)+s_3f(c) \qquad (13)$$

for any $a, b, c \in [0, \pi]$, $s_i \in [0, 1]$, $s_1 + s_2 + s_3 = 1$. Define $g\colon T \to \mathbf{R}$ by $g(x, y, z) = f(x) + f(y) + f(z)$, where

$$T = \left\{(x, y, z) \in \mathbf{R}^3 : x + y + z = \pi,\ \ x,\ y,\ z \in \left[0, \tfrac{\pi}{2}\right]\right\}.$$

In other words, T is the triangle with vertices $A = (0, \frac{\pi}{2}, \frac{\pi}{2})$, $B = (\frac{\pi}{2}, 0, \frac{\pi}{2})$ and $C = (\frac{\pi}{2}, \frac{\pi}{2}, 0)$. It follows from (13) (applied three times) that

$$g(s_1 u + s_2 v + s_3 w) \geq s_1 g(u) + s_2 g(v) + s_3 g(w) \tag{14}$$

for any $u, v, w \in T,\ s_i \in [0, 1]$, $s_1 + s_2 + s_3 = 1$. Moreover, a direct verification shows that

$$g(A) = g(B) = g(C) = 2 \arcsin(\cos \rho_0) = \pi - 2\rho_0.$$

If $p \in T$ then we can write

$$p = s_1 A + s_2 B + s_3 C \text{ for some } s_1,\ s_2,\ s_3 \in [0, 1] \text{ with } s_1 + s_2 + s_3 = 1.$$

Therefore, (14) guarantees that

$$g(p) \geq s_1 g(A) + s_2 g(B) + s_3 g(C) = \pi - 2\rho_0. \qquad \square$$

Proof of (8.4). If γ_s is condensed for all $s \in [0, 1]$, then $s \mapsto \nu(\gamma_s)$ is defined and constant, since it can only take on integral values. Thus, if the assertion is false, there must exist $s \in [0, 1]$, say $s = 1$, such that γ_s is not condensed. By (6.1), γ_0 is homotopic to a circle traversed $\nu(\gamma_0)$ times. Moreover, the set of non-condensed curves is open. Together with (2.10), this shows that there exist C^2 curves γ_{-1}, γ_2 such that:

(i) There exist a path joining γ_{-1} to γ_0 and a path joining γ_1 to γ_2 in $\mathcal{L}_{\kappa_0}^{+\infty}(I)$;

(ii) γ_{-1} is condensed and has rotation number $\nu(\gamma_0)$;

(iii) γ_2 is not condensed.

Consider the map $f\colon \mathbf{S}^0 \to \mathcal{L}_{\kappa_0}^{+\infty}(I)$ given by $f(-1) = \gamma_{-1}$, $f(1) = \gamma_2$. The existence of the homotopy γ_s ($s \in [0, 1]$) tells us that f is nullhomotopic in $\mathcal{L}_{\kappa_0}^{+\infty}(I)$. By (2.10), f must be nullhomotopic in $\mathcal{C}_{\kappa_0}^{+\infty}(I)$. In other words, we may assume at the outset that each γ_s is of class C^2 ($s \in [0, 1]$).

With this assumption in force, let s_0 be the infimum of all $s \in [0, 1]$ such that γ_s is not condensed, and let $\gamma = \gamma_{s_0}$. Then γ must be condensed by (11.2), and it must be equatorial by our choice of s_0. In addition, $\nu(\gamma_s)$ must be constant ($s \in [0, s_0]$), since it can only take on integral values. This contradicts (8.8). $\square$