9
Proofs of the Main Theorems

We will now collect some of the results from the previous sections in order to prove the theorems stated in §4. We repeat their statements here for convenience.

(3.2) Theorem. *Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$. Every curve in $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) lies in the same component as a circle traversed k times, for some $k \in \mathbf{N}$ (depending on the curve).*

Proof. By the homeomorphism $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ of (2.15), it does not matter whether we prove the theorem for $\mathcal{L}_{\kappa_1}^{\kappa_2}$ or for $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$. Further, by (2.25), it suffices to consider spaces of type $\mathcal{L}_{\kappa_0}^{+\infty}$, for $\kappa_0 \in \mathbf{R}$. If $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ is diffuse, then it is homotopic to a circle by (5.11). If it is condensed, then the same conclusion holds by (6.1) (when $\kappa_0 > 0$), (6.7) (when $\kappa_0 < 0$) and Little's theorem (when $\kappa_0 = 0$).

Assume then that γ is neither homotopic to a condensed nor to a diffuse curve. Since γ itself is non-condensed by hypothesis, (5.12) guarantees that we may find $\varepsilon > 0$ and a chain of grafts (γ_s) with $\gamma_0 = \gamma$ and $\gamma_s \in \mathcal{L}_{\kappa_0}^{+\infty}$ for all $s \in [0, \varepsilon)$. Let (γ_s), $s \in J$, be a maximal chain of grafts starting at $\gamma = \gamma_0$, where J is an interval of type $[0, \sigma)$ or $[0, \sigma]$. That such a chain exists follows by a straightforward argument involving Zorn's lemma, since the grafting relation is an equivalence relation, as proved in (5.6).[1] By hypothesis, no curve γ_s is diffuse, hence $\nu(\gamma_s)$ is well-defined and independent of s, and (7.8) yields that $\sigma < +\infty$. If the interval is of the first type, then we obtain a contradiction from (5.7), and if the interval is closed, then we can apply (5.12) to γ_σ to extend the chain, again contradicting the choice of J. We conclude that γ must be homotopic either to a condensed or to a diffuse curve. In any case, γ is homotopic in $\mathcal{L}_{\kappa_0}^{+\infty}$ to a circle traversed a number of times, as claimed. □

(3.3) Theorem. *Let $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$ and let $\sigma_k \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) denote any circle traversed $k \geq 1$ times. Then σ_k, σ_{k+2} lie in the same*

[1]By reasoning more carefully it would be possible to avoid using Zorn's lemma.

component of $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ *(resp.* $\mathcal{L}_{\kappa_1}^{\kappa_2}$*) if and only if*

$$k \geq \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor \qquad (\rho_i = \operatorname{arccot} \kappa_i, \ i = 1, 2).$$

Proof. This follows from the combination of (4.4), (8.2) and (8.4), if we use the homeomorphisms in (2.15) and (2.25). □

(9.1) Proposition. *Let* $\kappa_0 = \cot \rho_0 \geq 0$,

$$n = \left\lfloor \frac{\pi}{\rho_0} \right\rfloor + 1.$$

Then the set $\mathcal{O}_\nu$ *of all condensed curves* $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}(I)$ *satisfying* $\nu(\gamma) = \nu$ *for some fixed* $\nu \leq n - 2$ *is a contractible connected component of* $\mathcal{L}_{\kappa_0}^{+\infty}(I)$.

Proof. When $\kappa_0 = 0$, this result is equivalent to the assertion that the component of $\mathcal{L}_0^{+\infty}(I)$ containing a circle traversed once is contractible; this result is not new, and a proof can be found in [15]. When $\kappa_0 > 0$, (6.1) guarantees that $\mathcal{O}_\nu$ is weakly contractible and, in particular, connected. Proposition (8.4) then implies that $\mathcal{O}_\nu$ must be a connected component of $\mathcal{L}_{\kappa_0}^{+\infty}(I)$. Using (2.7(a)) we deduce that $\mathcal{O}_\nu$ is an open subset of this space. Hence $\mathcal{O}_\nu$ is also a Hilbert manifold, and it must be contractible by (2.7(b)). □

Remark. Note that if $\kappa_0 < 0$ (that is, if $\rho_0 > \frac{\pi}{2}$), then it is a consequence of (4.2) and (4.3) that $\mathcal{L}_{\kappa_0}^{+\infty}(I)$ has only $n = 2$ components, and the conclusion of (9.1) does not make sense in this case (no curve γ satisfies $\nu(\gamma) \leq 0$). Moreover, these two components are far from being contractible: Even for $\kappa_0 = -\infty$, the (co)homology groups of $\mathcal{I} = \mathcal{L}_{-\infty}^{+\infty}(I) \simeq \Omega\mathbf{S}^3 \sqcup \Omega\mathbf{S}^3$ are non-trivial in infinitely many dimensions.

Our main theorem is a combination of the three previous results.

(3.1) Theorem. *Let* $-\infty \leq \kappa_1 < \kappa_2 \leq +\infty$, $\rho_i = \operatorname{arccot} \kappa_i$ $(i = 1, 2)$ *and* $\lfloor x \rfloor$ *denote the greatest integer smaller than or equal to* x. *Then* $\mathcal{L}_{\kappa_1}^{\kappa_2}$ *has exactly* n *connected components* $\mathcal{L}_1, \dots, \mathcal{L}_n$, *where*

$$n = \left\lfloor \frac{\pi}{\rho_1 - \rho_2} \right\rfloor + 1$$

and $\mathcal{L}_j$ *contains circles traversed* j *times* $(1 \leq j \leq n)$. *The component* $\mathcal{L}_{n-1}$ *also contains circles traversed* $(n-1) + 2k$ *times, and* $\mathcal{L}_n$ *contains circles traversed* $n + 2k$ *times, for* $k \in \mathbf{N}$. *Moreover, each of* $\mathcal{L}_1, \dots, \mathcal{L}_{n-2}$ *is homotopy equivalent to* $\mathbf{SO}_3$ $(n \geq 3)$.

Proof. All of the assertions of the theorem but the last one follow from (4.2), (4.3) and the homeomorphism $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ of (2.15).

Assume that $n \geq 3$ and let $\sigma_k \in \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ be a circle traversed $k \leq n-2$ times. In the notation of (9.1), the connected component $\mathcal{L}_k(I)$ of $\mathcal{L}_{\kappa_1}^{\kappa_2}(I)$ containing σ_k is mapped to the component $\mathcal{O}_k$ under the homeomorphism $\mathcal{L}_{\kappa_1}^{\kappa_2}(I) \approx \mathcal{L}_{\kappa_0}^{+\infty}(I)$ of (2.25), because σ_k is mapped to another circle traversed k times (cf. (2.23)). Therefore, $\mathcal{L}_k(I)$ is contractible by (9.1). The last assertion of the theorem is deduced from this and the homeomorphism $\mathcal{L}_{\kappa_1}^{\kappa_2} \approx \mathbf{SO}_3 \times \mathcal{L}_{\kappa_1}^{\kappa_2}(I)$. □

Theorem (4.1) characterizes the connected components of $\mathcal{L}_{\kappa_1}^{\kappa_2}$ in terms of the circles that they contain. However, recall that $\mathcal{L}_{\kappa_1}^{\kappa_2}$ is homeomorphic to $\mathcal{L}_{\kappa_0}^{+\infty}$ for some κ_0, and for spaces of the latter type a more direct characterization in terms of the properties of a curve is also available.

(9.2) Theorem. *Let $\kappa_0 \in \mathbf{R}$ and let $\mathcal{L}_1, \dots, \mathcal{L}_n$ be the connected components of $\mathcal{L}_{\kappa_0}^{+\infty}$, as described in (4.1). Then:*

(i) $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ lies in $\mathcal{L}_j$ $(1 \leq j \leq n-2)$ if and only if it is condensed and has rotation number j.

(ii) $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ lies in $\mathcal{L}_{n-1}$ if and only if $\tilde{\Phi}_\gamma(1) = (-1)^{n-1}\tilde{\Phi}_\gamma(0)$ and either it is non-condensed or condensed with rotation number $\nu(\gamma) \geq n-1$.

(iii) $\gamma \in \mathcal{L}_{\kappa_0}^{+\infty}$ lies in $\mathcal{L}_n$ if and only if $\tilde{\Phi}_\gamma(1) = (-1)^{n}\tilde{\Phi}_\gamma(0)$ and either it is non-condensed or condensed with rotation number $\nu(\gamma) \geq n-1$.

Proof. This follows from (4.1) and (9.1). □

Recall that $\tilde{\Phi}\colon [0,1] \to \mathbf{S}^3$ is the lift of the frame $\Phi_\gamma\colon [0,1] \to \mathbf{SO}_3$ of γ to $\mathbf{S}^3$ (cf. (2.12)). When $-\infty \leq \kappa_0 < 0$ (resp. $\rho_1 - \rho_2 > \frac{\pi}{2}$) we have $n = 2$, and this characterization of the two components $\mathcal{L}_1$, $\mathcal{L}_2$ of $\mathcal{L}_{\kappa_0}^{+\infty}$ (resp. $\mathcal{L}_{\kappa_1}^{\kappa_2}$) may be simplified to: γ lies in $\mathcal{L}_i$ if and only if $\tilde{\Phi}_\gamma(1) = (-1)^i\tilde{\Phi}_\gamma(0)$.