2
The Model

We consider the following generalization of the linear ECM:

$$\Delta \boldsymbol{y}_t = \boldsymbol{f}(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}) + \sum_{i=1}^{p} \boldsymbol{\Gamma}_i \Delta \boldsymbol{y}_{t-i} + \boldsymbol{\epsilon}_t, \tag{2.1}$$

where $\boldsymbol{y}_t \in \mathbb{R}^n$ is a $I(1)$ vector of cointegrated series, the scalar $z_t = \boldsymbol{\beta}'\boldsymbol{y}_{t-1} \sim I(0)$ is a unique linear cointegration relationship, and $\boldsymbol{f}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ is a possibly nonlinear function. The model has a linear cointegration relation but a nonlinear dynamics towards the long-run equilibrium.

The first issue is establishing the existence of such a model. It must be shown that a cointegrated vector $\boldsymbol{y}_t$ may have an error correction representation as in Model (2.1). In the linear case we have the Granger representation theorem, Engle and Granger (1987) [6]. In the nonlinear framework a similar result have been established under three different set of assumptions. In Saikkonen (2005) [19] $\boldsymbol{\epsilon}_t$ may have a GARCH structure but $\boldsymbol{f}(x) - (\boldsymbol{\alpha}_0 + \boldsymbol{\alpha}_1 x) \longrightarrow 0$ when $|x| \longrightarrow \infty$. Kristensen and Rahbek (2010) [14] make the same hypothesis on the limits of $\boldsymbol{f}$, but $\boldsymbol{\epsilon}_t$ must be an independent and identically distributed (IID) innovation. Saikkonen (2008) [20] can be seen as a generalization of Bec and Rahbek (2004) [3], where the function $\boldsymbol{f}$ must be a linear combination of linear functions, but not necessarily the same in the extremes while $\boldsymbol{\epsilon}_t$ may have a GARCH structure.

The most important assumption is the linearity of $\boldsymbol{f}$ in the limit. Since it is present in every existence proof, we will use it throughout the paper.

Assumption 1 $\boldsymbol{f}(\cdot) : \mathbb{R} \rightarrow \mathbb{R}^n$ *is such that, for some* $K_1, K_2 \in \mathcal{R}^n$ *and* $\alpha_1, \alpha_2 \in \mathcal{R}^n$, $\lim_{x\rightarrow\infty} \boldsymbol{f}(x) - (K_1 + \alpha_1 x) = 0$ *and* $\lim_{x\rightarrow-\infty} \boldsymbol{f}(x) - (K_2 + \alpha_2 x) = 0$

This restriction lead to the wide usage of smooth transition models, where a weighting function (called transition function) is used to combine two or more linear functions. The most common transition functions are the logistic and the exponential. If

$$\boldsymbol{f}(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}) = \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{y}_{t-1} + \{1 - \exp[-\lambda(\boldsymbol{\beta}'\boldsymbol{y}_{t-1} - c)^2]\}\boldsymbol{\delta}\boldsymbol{\beta}'\boldsymbol{y}_{t-1} \tag{2.2}$$

we have the exponential smooth transition model, where λ is the smoothness (velocity) of the transition and c is the location parameter. If

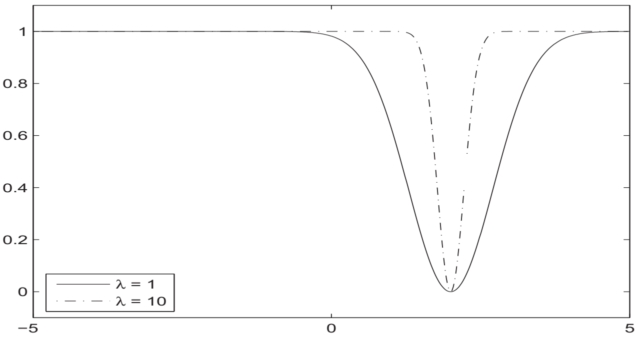
$$\boldsymbol{f}(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}) = \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{y}_{t-1} + \frac{1}{1+\exp[-\lambda(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}-c)]}\boldsymbol{\delta}\boldsymbol{\beta}'\boldsymbol{y}_{t-1} \tag{2.3}$$

we have the logistic smooth transition model, again with λ as the velocity of transition and c as the location parameter. Another example, less widespread, is a combination of logistic functions, from Suárez-Fariñas, Pedreira and Medeiros (2004) [24]

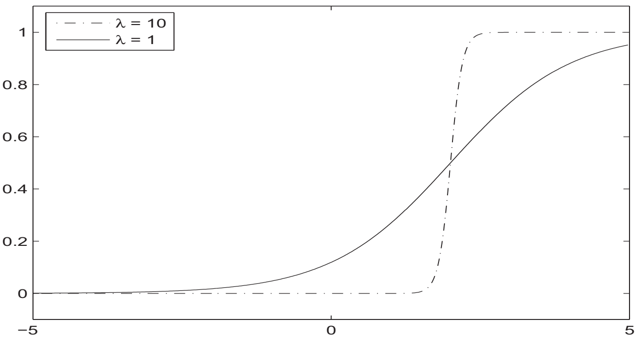
$$\begin{aligned}\boldsymbol{f}(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}) = \boldsymbol{\alpha}\boldsymbol{\beta}'\boldsymbol{y}_{t-1} + \Bigg\{ 1 &+ \frac{1}{1+\exp[-\lambda(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}-c)]} \\ &- \frac{1}{1+\exp[-\lambda(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}+c)]}\Bigg\}\boldsymbol{\delta}\boldsymbol{\beta}'\boldsymbol{y}_{t-1}.\end{aligned} \tag{2.4}$$

Figure 2.1 illustrates the shape of the transition functions discussed above. These transition functions generates adjustment functions $\boldsymbol{f}$ as shown in Figure 2.2. The exponential and double logistic models may be very similar depending on the value of the parameters. The exponential model has been used for no-arbitrage conditions in the presence of transaction costs, for example, addressing the PPP puzzle in Michael, Nobay and Peel (1997) [17]. However, the exponential model may account for little deviations from linearity, sometimes fitting its curve to better accommodate an outlier. We deem the double exponential model more adequate for these applications.

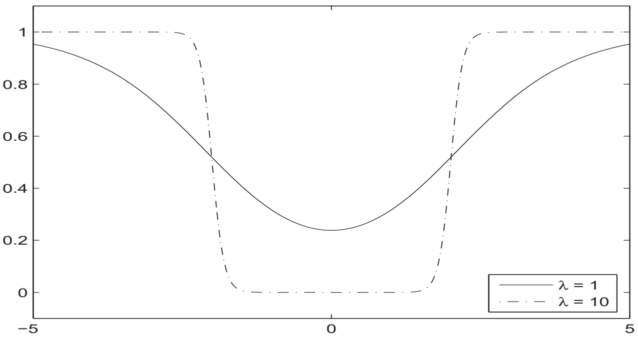
The logistic model is appropriate for cointegration of variables with different behavior when above or under a certain value, possibly zero. A very similar model was used in Hansen and Seo (2002) [10] for long and short bonds interest rates.



2.1(a): Exponential Function



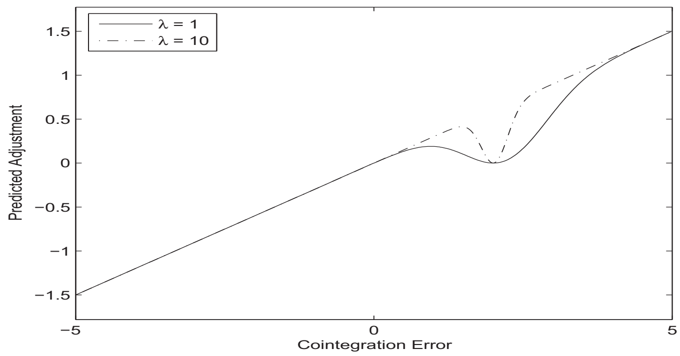
2.1(b): Logistic Function



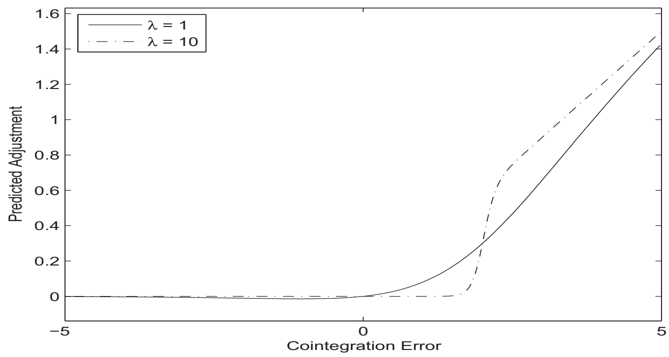
2.1(c): Double Logistic Function

Figure 2.1: Transition Functions

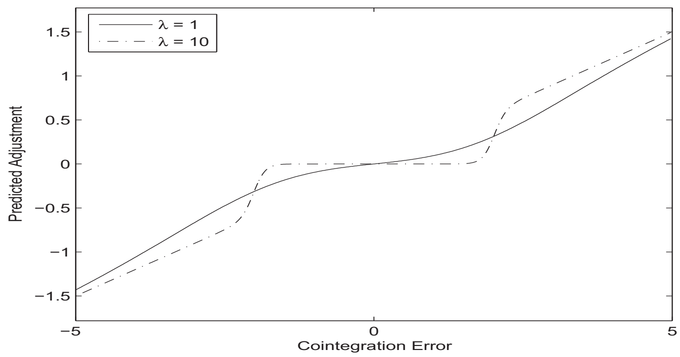
Each function is depicted with two different λ values and $c = 2$.



2.2(a): Exponential Model Adjustment



2.2(b): Logistic Model Adjustment



2.2(c): Double Logistic Model Adjustment

Figure 2.2: Adjustment Functions

Each function is depicted with two different λ values, $c = 2$, $\alpha = 0$ and $\delta = 0.3$.