3
Testing Linearity

We want to test whether $\boldsymbol{f}$ is a linear function or not. If $\boldsymbol{f}$ is the exponential or logistic function, this could be done by testing $\mathcal{H}_0 : \lambda = 0$ or $\mathcal{H}_0 : \boldsymbol{\delta} = 0$ in Equations (2.2), (2.3), or (2.4). The issue is that in any case there are unidentified parameters under the null hypothesis. One possible solution is to follow Davies (1987) [5] and perform a supLM test, as done in Hansen and Seo (2002) [10] and Seo (2007) [22]. Another possibility is to follow the ideas in Luukkonen, Saikkonen and Terasvirta (1988) [16] and substitute $\boldsymbol{f}$ by its Taylor expansion around the null hypothesis and test the polynomial coefficients. The latter will be our approach.

To establish the asymptotic distribution of the test, we will make two more assumptions.

Assumption 2 *The function* $\boldsymbol{f} : \mathbb{R} \to \mathbb{R}^n$ *is three times continuously differentiable.*

Assumption 3 ϵ_t *is a difference martingale sequence with respect to the filtration generated by* y_t.

The differentiability is needed in order to guarantee the validity of the Taylor expansion, while the difference martingale errors is a common assumption in error correction models. A important restriction is threshold models. Since the threshold function is not differentiable everywhere, it does not satisfy Assumption 2, which is essential to the validity of the test. However, in finite samples, a threshold model can always be approximated through a logistic smooth function by taking a large enough λ.

To establish the consistency of the test, we need one more hypothesis. First, define the following partial derivatives:

$$\boldsymbol{f}_{\psi_i}(x, \boldsymbol{\psi}) = \frac{\partial \boldsymbol{f}(x, \boldsymbol{\psi})}{\partial \psi_i}, \ \boldsymbol{f}_{\psi_i x}(x, \boldsymbol{\psi}) = \frac{\partial^2 \boldsymbol{f}(x, \boldsymbol{\psi})}{\partial \psi_i \partial x},$$
$$\boldsymbol{f}_{\psi_i \psi_j}(x, \boldsymbol{\psi}) = \frac{\partial^2 \boldsymbol{f}(x, \boldsymbol{\psi})}{\partial \psi_i \partial \psi_j}, \text{ and so on.}$$

Assumption 4 *The derivatives* $\boldsymbol{f}_{\psi_i x}(x, \boldsymbol{\psi})$ *and* $\boldsymbol{f}_{\psi_i \psi_j x}(x, \boldsymbol{\psi})$ *are limited in* x $\forall i, j \in \{1, \ldots, m\}$.

Note, for example, that the Smooth Transition Models presented in the previous sections attend Assumption 4.

One often cited shortcoming of the Taylor expansion approach is the local power characteristic of the test; see Hansen (1996) [9] for a discussion. The expansion is usually made around the $\mathcal{H}_0$ values for the parameters. Since the approximation becomes worse as the true parameters get farther from the null hypothesis, it is difficult to establish consistency for every true parameter set. Here we avoid this problem by making the expansion around the variable instead of making it around some parameters. By doing so, as a result of the finite variance of the variable, it is possible to limit the approximation errors.

This approach has two advantages over the supLM one. First, it is extremely simple and much faster computationally. While supLM demands bootstrap calculations, grid searches and involves non-standard asymptotic distributions, the Taylor expansion approach demands only a simple F-test. The computing time difference is of the order of 10^4. Second, since the test does not have a specific alternative hypothesis, it is consistent against a large set of nonlinearities. The supLM approach will be consistent only against the specific alternative being tested, but will also have power against other alternatives as well. Thus, since it is possible that the process is nonlinear but is not the alternative hypothesis, accepting $\mathcal{H}_0$ does not mean the process is linear and rejecting $\mathcal{H}_0$ does not guarantees the nonlinearity has the form being tested. This is true even for large samples.

The Taylor Theorem version we will use is stated in the appendix as Lemma 1. Using it to expand $\boldsymbol{f}$ around $\boldsymbol{\beta}'\boldsymbol{y}_{t-1} = 0$, Equation (2.1) becomes

$$\Delta \boldsymbol{y}_t = \boldsymbol{\theta}_0 + \boldsymbol{\theta}_1(\boldsymbol{\beta}'\boldsymbol{y}_{t-1}) + \boldsymbol{\theta}_2(\boldsymbol{\beta}'\boldsymbol{y}_{t-1})^2 + \boldsymbol{\theta}_3(\boldsymbol{\beta}'\boldsymbol{y}_{t-1})^3 + \sum_{i=1}^{p} \boldsymbol{\Gamma}_i \Delta \boldsymbol{y}_{t-i} + \boldsymbol{\epsilon}_t^*, \quad (3.1)$$

where $\boldsymbol{\epsilon}_t^* = \boldsymbol{\epsilon}_t + \left(\frac{1}{6}\right) \boldsymbol{f}^{(4)}(k_t)(\boldsymbol{\beta}'\boldsymbol{y}_{t-1})^4$, for some $k_t \in \mathbb{R}$, $\boldsymbol{\theta}_0 = \boldsymbol{f}(0)$, $\boldsymbol{\theta}_1 = \boldsymbol{f}^{(1)}(0)$, $\boldsymbol{\theta}_2 = \left(\frac{1}{2}\right) \boldsymbol{f}^{(2)}(0)$, and $\boldsymbol{\theta}_3 = \left(\frac{1}{6}\right) \boldsymbol{f}^{(3)}(0)$. $\boldsymbol{f}^{(i)}(0)$ is the ith-order derivative of $\boldsymbol{f}$ evaluated at 0. When $\boldsymbol{f}$ is linear, we have $\boldsymbol{\theta}_2 = \boldsymbol{\theta}_3 = \mathbf{0}$. When $\boldsymbol{f}$ is nonlinear, $\boldsymbol{f}^{(2)}(x) \neq 0$ for almost every $x \in \mathbb{R}$. The inequality will not hold if x is a point of inflexion of the function. This will be true for $x = 0$ if, for example, $\boldsymbol{f}$ is an odd function. We include the third term of the expansion to deal with this situations. To keep it concrete, take the exponential model. The θ values are $\boldsymbol{\theta}_0 = \mathbf{0}$, $\boldsymbol{\theta}_1 = \boldsymbol{\alpha} + \boldsymbol{\delta} exp\{-\lambda c^2\}$, $\boldsymbol{\theta}_2 = -4c\delta\lambda exp\{-\lambda c^2\}$, $\boldsymbol{\theta}_3 = 3\delta(2\lambda exp\{-\lambda c^2\} - 4c^2\lambda^2 exp\{-\lambda c^2\})$. If the location parameter, c, is zero, the function is odd, and $\boldsymbol{\theta}_2 = \mathbf{0}$, but $\boldsymbol{\theta}_3 \neq \mathbf{0}$. Therefore, the test will be able to detect the nonlinearity. To test $\mathcal{H}_0 : \boldsymbol{f}$ *is linear* against $\mathcal{H}_A : \boldsymbol{f}$ *is nonlinear* we propose the following procedure:

(a) Estimate $\widehat{\boldsymbol{\beta}}$ super-consistently. In our framework, it is enough to run an OLS in the equation $y_{1t} = \beta_1 + \beta_2 y_{2t} + \cdots + \beta_n y_{nt} + u_t$;[1]

(b) Estimate Equation (3.1) by OLS using $\widehat{\boldsymbol{\beta}}$ in place of $\boldsymbol{\beta}$. Then, perform a F-test for the following null hypothesis $\mathcal{H}_0 : \boldsymbol{\theta}_2 = \boldsymbol{\theta}_3 = \mathbf{0}$.

Proposition 1 *Under Assumptions 1–3 and* $\mathcal{H}_0 : \boldsymbol{\theta}_2 = \boldsymbol{\theta}_3 = \mathbf{0}$*, the F-statistic on the second stage of the proposed test has a* $\chi^2(2n)$ *asymptotic distribution. Moreover, under* $\mathcal{H}_A$ *and 4, the test is consistent.*

Proof: See Appendix in Section 8. ■

[1] In case of endogenous regressors, Dynamic OLS (DOLS) may be used.