2
Vector fields concepts

This chapter is a brief overview of vector field topology and how it carries over to the 2D discrete setting. The notation is defined for both an unstructured and a structured (grid) discrete vector field. For the structured vector field, we present two different ways to detect and classify singularities.

2.1
Continuous vector fields

A *vector field* $\mathbf{v}$ on a planar domain $\mathcal{D} \subset \mathbb{R}^2$ is a function assigning to each point $(x, y) \in \mathcal{D}$ a 2D vector $\mathbf{v}(x, y) = (v^x(x, y), v^y(x, y))$. Assuming that v^x and v^y are differentiable bivariate functions, then the *Jacobian* matrix of $\mathbf{v}$ at point (x_0, y_0) is:

$$J_{\mathbf{v}}(x_0, y_0) = \left[\begin{array}{cc} \frac{\partial v^x}{\partial x}(x_0, y_0) & \frac{\partial v^x}{\partial y}(x_0, y_0) \\ \frac{\partial v^y}{\partial x}(x_0, y_0) & \frac{\partial v^y}{\partial y}(x_0, y_0) \end{array} \right].$$

A point $(x_0, y_0) \in \mathcal{D}$ is *singular* for $\mathbf{v}$ if $\mathbf{v}(x_0, y_0) = (0, 0)$. According to the Hartman-Grobman theorem (9), a singular point can be partly classified by looking at the eigenvalues of the Jacobian matrix at that point:

- If the real parts of both eigenvalues are strictly negative, then the singular point is a *sink*.

- If the real parts of both eigenvalues are strictly positive, then the singular point is a *source*.

- If the real parts of both eigenvalues are non-zero real number with different signs (one positive and one negative), then the singular point is a *saddle*.

- If the real part of one of the eigenvalues is zero, the singularity is of higher order.

2.2
Discrete vector fields

In this dissertation we deal with sampled data of a vector field $\mathbf{v}$. Such data can be either unstructured, dealt with in Chapter 3, or structured (grid), the focus in Chapter 4.

2.2.1
Unstructured vector fields

Given a set of L unstructured points $\mathcal{P} = \{\mathbf{p}_1, \mathbf{p}_2, \ldots, \mathbf{p}_L\}$, where each point $\mathbf{p}_i \in \Omega \subset \mathbb{R}^2$ and the associated vector $\mathbf{v}_i(\mathbf{p}_i) \in \mathbb{R}^2$, denote the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_L\}$ by $\mathcal{V}$ (see Figure 2.1). A *vector field* is a map $\mathcal{F} : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ that associates to each point $\mathbf{p} \in \Omega$ a vector $\mathcal{F}(\mathbf{p})$. Since the vector field does not have any structure, the first preprocessing step, before we can process it, is to define a graph G.
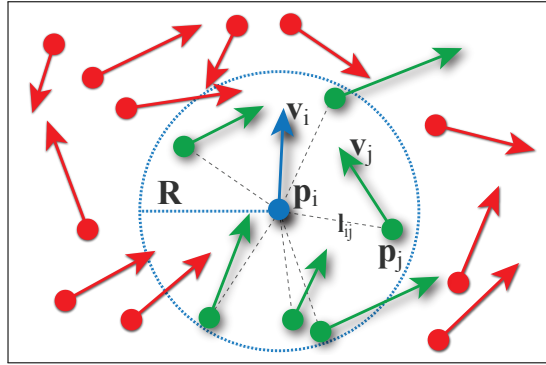

Figure 2.1: The support region and the induced graph

The nodes of G. There is a bijective mapping F from the nodes of graph G to the set of input points $\mathcal{P}$, which associates each node n_i on G the corresponding point $\mathbf{p}_i$ on $\mathcal{P}$, i.e., $F(n_i) = \mathbf{p_i}$ and $F^{-1}(\mathbf{p}_i) = n_i$.

The links of G. The connectivity between the nodes of G depends on a model parameter R, which represents the radii of balls centered at each point of $\mathcal{P}$, the nodes that correspond to points that are inside the ball centered at $\mathbf{p}_i$ are adjacent to n_i (see Figure 2.1). From now on, the graph defined for a given radius R will be denoted by G_R. More precisely, to define the links of G_R, the following rule is adopted:

"Node n_i is adjacent to node n_j through a link l_{ij} if and only if
$$||\mathbf{p}_i - \mathbf{p}_j|| < R."$$

The *first neighborhood* of a node n_i, denoted by $N(n_i)$, is the set of all adjacent nodes of n_i on G_R.

Interpolation. When needed, we can interpolate the values in-between the sample points using SPH like interpolation (7) such as:

$$\sum_{\|\mathbf{p}_i - \mathbf{p}\| \leq R} W_R(\|\mathbf{p}_i - \mathbf{p}\|) \cdot \mathbf{v}_i$$

2.2.2
Vector fields on a grid

Now we will suppose that we have the values of vector field $\mathbf{v}$ at the points (x_i, y_j) of a regular grid of size $M \times N$. We will denote $\mathbf{v}_{i,j} = (v_{i,j}^x, v_{i,j}^y) = \mathbf{v}(x_i, y_j)$, for $i = 1, \ldots, M$ and $j = 1, \ldots, N$.

Interpolation. In the grid we can also interpolate the values in-between the sample points. The simplest such interpolation is the bilinear interpolation, which can be written considering a grid cells as a unit square:

$$\mathbf{b}_{i,j} : [0, 1]^2 \rightarrow \mathbb{R}^2 \ ,$$
$$\mathbf{b}_{i,j}(x, y) = \mathbf{v}_{i,j} \cdot (1 - x)(1 - y) + \mathbf{v}_{i+1,j} \cdot x(1 - y) \qquad \text{(2-1)}$$
$$+ \mathbf{v}_{i,j+1} \cdot (1 - x)y + \mathbf{v}_{i+1,j+1} \cdot xy.$$

When the vector field is given on a grid we can easily detect and classify its singularities, as we will see in the next section.

2.3
Detection and classification of singularities

Two classical approaches for the detection of critical points on a regular 2D grid are described and explicitly showed. The first one searches where the interpolation of the vector field vanishes. The second one computes the winding numbers.

2.3.1
Singularities of the interpolation

In the bilinear interpolation case, the detection boils down to solving the system of quadratic equations $\mathbf{b}_{i,j}(x, y) = (0, 0)$, where $\mathbf{b}_{i,j}$ is defined in Equation (2-1). This can be explicitly solved by computing the roots of a

polynomial in y:

$$
\begin{aligned}
(\quad & -v_{01}^x\,v_{00}^y + v_{01}^x\,v_{10}^y + v_{11}^x\,v_{00}^y - v_{11}^x\,v_{10}^y + \\
& +v_{00}^x\,v_{01}^y - v_{00}^x\,v_{11}^y - v_{10}^x\,v_{01}^y + v_{10}^x\,v_{11}^y \quad) \quad \cdot \quad y^2 \\
+ \quad (\quad & 2\,v_{01}^x\,v_{00}^y - 2\,v_{00}^x\,v_{01}^y - v_{11}^x\,v_{00}^y - \\
& -v_{01}^x\,v_{10}^y + v_{10}^x\,v_{01}^y + v_{00}^x\,v_{11}^y \quad\quad\quad) \quad \cdot \quad y \\
+ \quad\quad & v_{00}^x\,v_{01}^y - v_{01}^x\,v_{00}^y
\end{aligned}
$$

To obtain the value of the x coordinate of the singular point we use the following expression:

$$
x = \frac{(v_{00}^y - v_{01}^y)\cdot y - v_{00}^y}{(v_{00}^y - v_{10}^y - v_{01}^y + v_{11}^y)\cdot y - v_{00}^y + v_{10}^y} \enspace .
$$

Sometimes this system degenerates to a lower degree polynomial. It can then have zero, one or two solutions. Each of them must be tested to lie in the quadrilateral. This method is the simplest way to detect first-order vector field singularities.

2.3.2
Winding numbers

The winding number counts the number of turns the vector fields achieves along a given closed curve Γ. It can be computed from the angular component of the vector field $\theta(\mathbf{v}(\mathbf{p}))$ at point $\mathbf{p} \in \Gamma$ by:

$$
w_\Gamma(\mathbf{v}(\mathbf{p})) = \frac{1}{2\pi} \oint_\Gamma d\theta(\mathbf{v}(\mathbf{p}))
$$

The winding number is zero if the region inside Γ does not contain critical points. If Γ contains a single saddle, then $w_\Gamma(\mathbf{v}) = -1$. If it contains a single sink or source, it will be $+1$.

The winding number is computed for each cell of the discrete grid using for Γ the square that bounds the cell. With the linear interpolation on the edges, we get the contribution of edge $(x_0, y_0) \to (x_1, y_0)$ to the above integral explicitly:

$$
\begin{aligned}
w_{00 \to 10} = \arctan &\left(\frac{v_{00}^{x\,2} - v_{00}^x\,v_{10}^x - v_{00}^y\,v_{10}^y + v_{00}^{y\,2}}{v_{10}^y\,v_{00}^x - v_{00}^y\,v_{10}^x} \right) \\
& - \arctan \left(\frac{v_{00}^x\,v_{10}^x - v_{10}^{x\,2} + v_{00}^y\,v_{10}^y - v_{10}^{y\,2}}{v_{10}^y\,v_{00}^x - v_{00}^y\,v_{10}^x} \right)
\end{aligned}
$$

Summing over the four edges gives the desired winding number.

2.3.3
Singularity classification

To classify the singularities, as we mention in Section 2.1, one must look at the Jacobian matrix. Here we provide the explicit Jacobian matrix of the bilinear interpolation $\mathbf{b}_{00}$:

$$
\begin{bmatrix}
v_{11}^x\, y - v_{00}^x\, \bar{y} + v_{10}^x\, \bar{y} - v_{01}^x\, y \;;\; v_{11}^x\, x - v_{00}^x\, \bar{x} - v_{10}^x\, x + v_{01}^x\, \bar{x} \\
v_{11}^y\, y - v_{00}^y\, \bar{y} + v_{10}^y\, \bar{y} - v_{01}^y\, y \;;\; v_{11}^y\, x - v_{00}^y\, \bar{x} - v_{10}^y\, x + v_{01}^y\, \bar{x}
\end{bmatrix} ,
$$

where $\bar{x} = 1 - x$ and $\bar{y} = 1 - y$. The eigenvalues are directly computed using the trace and determinant of the matrix.

These tools are used in Chapter 4 to display to the user the topological changes such as the creation, destruction, or change of type of a singularity at a fixed grid point.