Pontifícia Universidade Católica do Rio de Janeiro

Pablo Hector Seuanez Salgado

Three Essays in Finance

Tese de Doutorado

Thesis presented to the Postgraduate Program in Economics of the Departamento de Economia, PUC–Rio as partial fulfillment of the requirements for the degree of Doutor em Economia

Advisor : Prof. Vinicius do Nascimento Carrasco
Co–Advisor: Prof. João Manoel Pinho de Mello

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Abstract

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This thesis is composed of three theoretical essays in Finance. In Chapter 1, I explore the effect of product market structure on the liquidation value of industry assets. I show that the contest for the gains from market concentration among firms with financial constraints leads to their expending of insufficient efforts to redeploy assets across industries and to significant liquidation discounts when compared to an efficient benchmark. Equilibrium distress costs and private costs of leverage should increase with the rents associated to market concentration in the product market. In Chapter 2, I revisit the canonical costly state verification model. I show that financial intermediation in a costly state verification model has a cost not yet analyzed: it allows for the existence of multiple equilibria, some of which are characterized by borrowers defaulting on their loans because they expect other borrowers to do the same (i.e. bad equilibria arise due to strategic complementarities in entrepreneurs' actions). I propose two mechanisms that fully implement the desired equilibrium allocation. Finally in Chapter 3, I analyze a continuous time principal-agent model where a risk-neutral agent protected by limited liability is hired to perform multiple tasks. In this setting, I show that economies of scope naturally emerge as combining multiple tasks into a unique job relaxes the agent's limited-liability constraint. The analysis has several implications for job design i.e. the optimal grouping of several tasks into a unique job.

Keywords

Fire-Sales; Stochastic Games; Product Market Competition; Costly State Verification; Multiple Equilibria; Principal Agent Problem.

Resumo

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Esta tese é composta por três artigos teóricos em finanças. No capítulo 1, eu analiso o efeito da estrutura do mercado de produto de uma indústria no valor de liquidação dos seus ativos. Eu mostro que a competição pelos ganhos que resultam da concentração da estrutura de mercado entre firmas com restrições financeiras resulta na realização de esforços insuficientes para reempregar ativos em outras indústrias e a descontos de liquidação quando comparado com um benchmark socialmente efficiente. Em equilíbrio, custos de estresse financeiro e custos privados de alavancagem devem crescer com as rendas associadas à concentração de mercado. No capítulo 2, eu analiso o modelo canônico de verificação custos de estado e mostro que a intermediação financeira nessa classe de modelos tem um custo ignorado até o momento: permite a existência de múltiplos equilíbrios, alguns dos quais são caracterizados por devedores deixando de pagar suas dívidas por anteciparem que outros devedores farão o mesmo (equilíbrios indesejáveis surgem como consequência da complementariedade estratégica das decisões de repagamento dos empreendedores). Eu proponho dois mecanismos que implementam completamente a alocação de equilíbrio desejada. Finalmente no Capítulo 3, eu analiso um modelo de agente principal em tempo contínuo onde um agente neutro ao risco é contratado para realizar múltiplas tarefas. Nesse ambiente, eu mostro que economias de escala naturalmente surgem uma vez que a combinação de múltiplas tarefas num mesmo trabalho relaxa a restrição de participação do agente. A análise tem inúmeras implicações para o desenho de um trabalho, isto é, o agrupamento ótimo de várias tarefas num único emprego.

Palavras–chave

Vendas a Preços de Fire-Sales; Jogos Estocásticos; Competição no Mercado de Produto; Verificação Custosa de Estado; Equilíbrios Múltiplos; Problema de Agente Principal;.

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1 Product Market Competition and the Severity of Distressed Asset Sales

When in financial distress firms often transfer specialized assets to low valuation users. For instance, debtors provide creditors with the right to foreclose on the debtor's assets in the event of default (Aghion and Bolton (1992), Hart and Moore (1994), Hart Moore (1998), Hart (1995)). In addition, when a common adverse shock sidelines industry specialists and firms rely on the proceeds from divestitures to alleviate financial constraints, firms sell assets to liquidity endowed non-specialists (Shleifer and Vishny (1992)).[1]

In many instances it is particularly costly to transfer specialized assets to creditors, as opposed to redeploying them even if at dislocated prices. Creditors frequently engage in expensive and lengthy court procedures to foreclose on collateral (Djankov et al. (2008), Campbell et al. (2011)) and are unlikely to be the second best users of specialized industry assets.[2]

But, if they anticipate costly foreclosures, why do firms not take further actions to minimize the costs of financial distress by reallocating capital more efficiently? What are the drivers behind excessive continuation, despite the risk of having assets turned over to financiers, their lowest value users?

In this paper, I address these issues focusing on firms that operate in specialized industries. I argue that, if financial distress clusters at the industry level, firms' contest for rents arising from market concentration leads them to make insufficient efforts to deploy assets across industries when compared to a constrained efficient benchmark.[3] The result relies on the observation that the transfer of assets to non-specialists necessarily leads to changes in the structure of the market in which specialists operate. Firms that weather the storm of an industry-wide shock without severely compromising productive capacity benefit from the increase in profits resulting from the easing of competition within the product market. Incentives to be among the last survivors might

[1]Empirical papers studying fire-sales include, among others, Asquith et. al. (1994), Pulvino (1998), Schlingerman et. al. (2002), and Campbell et. al. (2012).

[2]I extensively discuss this claim later.

[3]More specifically, I consider in the constrained efficient benchmark the outcome determined by a social planner who maximizes aggregate firm value while facing the same institutional and informational constraints as private agents.

be so great that firms engage in a war of attrition as industry conditions deteriorate.

The shipping industry is an example of how competition in the product market affects industry participants' decision to redeploy capital, potentially leading to excessively high distress costs. The sudden drop in global activity that followed the financial crisis of 2008 was a devastating blow to the shipping industry with the volume of container shipping in 2009 experiencing its first yearly decline in history. Prominent industry figures urged for capacity reduction among shipowners to reduce oversupply and restore profitability. However, according to an industry report, *few are willing to take actions. This is so since shipowners are so selfish that they wish to benefit from others' scrapping.*[4]

Capacity reductions were insufficient and assets ended up in the hands of financiers. In a report on the state of the shipping industry during that period Ernst & Young claimed: *ship values fell, leaving many owners with debt that outweighed their asset's values, and technically breaching loan-to-value covenants - an industry standard banking covenant.*[5]

I formalize the above ideas in a dynamic stochastic model cast in continuous time. In the model, two identical firms that have made industry-specific investments compete in a duopolistic market until the time they are unable to meet their financial obligations. When this occurs, they lose control of their productive assets to creditors through a costly foreclosure process. Firms can attempt to mitigate distress costs before financial constraints bind by searching for potential buyers for their assets. These buyers, who are hard to find, have a higher valuation than the creditor but are nonetheless not as specialized as industry insiders.

In this environment, firms face a simple trade-off in the competitive equilibrium. They can adopt a precautionary strategy characterized by the early initiation of search to increase the likelihood that their assets are redeployed before financial constraints bind. On the other hand, firms can refrain from search to outlast rivals and reap the gains from market concentration that results from another firm's exit. I prove that the search game faced by firms is characterized by strategic substitutability in players' actions - when a firm increases its search efforts, its rival optimally responds in opposite manner - and derive its Markov Perfect Equilibrium using tools from the theory of supermodular games (Topkis (1979), Milgrom and Roberts (1990), Vives (1990)).

[4]The Economist, Sea of Troubles July 30th, 2009 and Gao Yanming, speech delivered at the World Shipping Summit of 2009

[5]Ernst & Young Restructuring thought - leadership series, All at sea - How can shipping stay afloat?

In the competitive equilibrium, search and asset redeployability are below the constrained efficient benchmark. There are two reasons for this inefficiency. First, a firm that exits the market does not internalize the benefits in the form of less competition that its departure bestows on its surviving rival. Second, a social planner has preferences over a state contingent market structure that disregards firms' identities. However, in the competitive equilibrium each firm competes to be the one that becomes a monopolist. Firms' insufficient search efforts contribute to higher costs of financial distress in the form of lower liquidation values and higher allocative inefficiencies.

An increase in monopoly rents exacerbates the departure of the competitive equilibrium from the constrained efficient benchmark. After all, as a monopolistic market structure becomes more attractive when compared to a duopolistic one, the social planner chooses to revert earlier to a monopoly. By contrast, firms make less effort to redeploy assets once the prize from continuing operations in the product market increases.

These results depend crucially on the characteristics of the environment I study; firms operating in a specialized industry suffering an aggregate shock. Because the adverse shock is industry-wide, assets are displaced to liquidity endowed outsiders; and because assets are specialized, their new owners put them to alternative uses. Together, these assumptions imply that actions to mitigate financial distress lead to changes in product market structure.

One interesting feature of the model is its tractability. I derive all results considering general diffusions and obtain closed form solutions for the special case where evolution of uncertainty is described by a geometric brownian motion.

1.1 Literature Review

This study is related to a diverse array of papers. First and foremost, it contributes to the literature that studies the determinants of assets' liquidation value. In a seminal contribution, Shleifer and Vishny (1992) introduce the idea of fire-sales; a forced sale of a real asset at a dislocated price. They explore the determinants of assets' liquidation value focusing on potential buyers of assets. In their paper, firms suffer shocks that are imperfectly correlated so they partially insure each other through a resale market for specialized assets. I on the other hand focus on potential sellers of assets and assume the shock is perfectly correlated across industry participants so partial insurance is impossible.

Previous papers have focused on two main channels through which product market competition interacts with the firm's financial structure.

Bolton and Scharfstein (1990) and Poitevin (1989) explore the long-purse explanation of predatory behavior. In these papers, a firm's competitive actions in the product market become more aggressive when rivals are in financial distress, as healthy firms try to drive suffering competitors into bankruptcy. In the second class of models (Brander and Lewis (1986), Maksimovic (1988)), the financial structure of a firm affects how it competes in the product market because limited liability creates a conflict of interest between bondholders and equityholders (Jensen and Meckling(1976)). For example, in Brander Lewis (1986) more leveraged firms have incentives to pursue output strategies that raise returns in good states and lower returns in bad states. In this paper, I rule out predatory behavior on the part of firms. Furthermore, I deliberately abstract from output decisions so as to isolate the linkage between product market competition, a firm's financial structure, and its efforts to redeploy assets across industries.

In this sense, my study is closely related to the extensive literature that studies exit in duopoly (Ghemawat and Nalebuff (1985 ,1990), Fudenberg and Tirole (1986)). However, there are important differences as well. These papers focus on economic distress, while I deal with financial distress and the interaction between past financing decisions and costly present actions. In particular, to highlight the importance of assuming that firms have a fragile financial structure, I analyze a situation where unlevered firms would never divest corporate assets.

Several important papers study how capital is allocated across industries over time (Dixit (1989), Abel and Easterly (1994), Ramey and Shapiro (2001), Eisfeldt and Rampini (2006)). They are mainly concerned with the costs that firms incur in reversing their past investment decisions and how these costs affect firms' incentives to invest in the first place. However, they ignore the effect of product market competition on firms' decisions, which is the focus of this paper.

In terms of methodology, I follow the capital structure models first presented in Fischer et. al. (1989), Leland (1994), and Leland and Toft (1996) in that divestiture or bankruptcy decisions are endogenous and claims on the firm are contingent on a state variable. In these models liquidation decisions can be chosen continuously, whereas I assume that shareholders can only sell assets at discrete points in time that represent the stochastic meeting of an interested buyer. This turns out to be a significant departure in terms of both methodology and results. Dupuis and Wang (2002) and Hugonnier et. al. (2011) also study dynamic stochastic optimization problems when agents have liquidity constraints. In particular, Hugonnier et. al. (2011) also add financial frictions to the firm's financing decisions by relaxing the assumption in Leland (1994) that equityholders can instantaneously inject capital into the firm whenever they choose.

This paper also relates to the literature that studies stochastic games (Dutta and Rustichini (1993), Grenadier (1996), Weeds (2002), Manso(2011)). Like Manso (2011), I use powerful results from the theory of supermodular games to analyze equilibrium behavior.

1.2 Model and Assumptions

Two firms compete in the product market for widgets. Each firm continuously produces one unit of output until it loses the control rights over its division. From this point on, each firm continues with its other activities, which for simplicity are assumed to be riskless. In the widget market firms face a downward-sloping inverse demand curve of the following form

$$P = xD(Q), \tag{1-1}$$

where $D(\cdot)$ is the deterministic component of the inverse demand curve with $D'(\cdot) < 0$ and x is an industry-wide shock. At each moment in time, Q takes value in $\mathcal{Q} = \{1, 2\}$ depending on the number of active firms in the widget market.[6]

The aggregate industry shock x is the fundamental source of uncertainty in the model. I assume that the shock is a diffusion process governed by the equation

$$dx = \mu_x(x)dt + \sigma_x(x)dB, \tag{1-2}$$

where dB is the increment of a standard Wiener process. The initial value of the shock is $x = x_0$, which is strictly positive. The general diffusion process of equation (1-2) embeds several special cases that are often used in the finance literature. For example, if $\mu_x(x) = \mu_x x$ and $\sigma_x(x) = \sigma_x x$, then x follows a geometric brownian motion and has a log-normal distribution. If $\mu_x(x) = \lambda(\mu_x - \delta_t)dt$ and $\sigma_x(x) = \sigma_x x$, then x follows a mean-reverting process with proportional volatility.

The value of the widget division depends on the cash-flow it generates. I normalize production costs to zero and assume that it is costless to shut down or restart production whenever x changes sign. If positive, the price given by (1-1) equals each active division's instantaneous cash-flow, which can be rewritten in differential form using (1-1) and (1-2) as

$$d\delta = \mu_\delta(\delta)dt + \sigma_\delta(\delta)dB, \tag{1-3}$$

[6]The divestiture of both firms yields $Q = 0$, but I ignore this case since it is uninteresting.

where $\mu_\delta(\delta) = \mu_x(\delta/D(Q))D(Q)$ and $\sigma_\delta(\delta) = \sigma_x(\delta/D(Q))D(Q)$.[7] Agents are risk-neutral and discount future cash flows at the risk-free interest rate r.

The *fundamental value* of a division when Q firms are active is a measure of its economic value to insiders. It is defined by

$$\Pi(x_t, Q) \equiv \mathbb{E}_t \left[\int_t^\infty e^{-r(s-t)} \delta_s^+(Q) ds \right], \tag{1-4}$$

where $\delta^+ = \max\{\delta, 0\}$. The above expression is finite if the discount rate r is greater than the expected growth rate of cash flows μ_δ. The division's fundamental value with Q firms equals the present value of all positive cash-flows earned from operations under an invariant market structure (Q is constant in equation (1-4)), with time running on forever. The above definition captures the fact that a firm never finds it optimal to divest its widget division if it is unlevered.[8]

Nonetheless, firms *are* levered and face financial constraints. More specifically, when the state variable first falls to $x_C < x_0$ each firm must return an amount of M in cash to its debtholders.[9]

I assume that to raise the necessary cash to honor their financial obligations, firms must sell corporate assets, more specifically the widget division. This assumption relies on two more primitive forces, high costs of rescheduling debt and high costs of raising outside capital and contrasts Leland (1994) who assumes that shareholders have deep pockets and continue to inject resources into the firm as long as they choose.[10]

Asset sales occur as follows. *When* a firm meets a potential buyer and the state variable is x, the buyer makes a take-it or leave-it offer to buy the widget division for an amount of $\phi x + M$.[11] After paying off its debt M the

[7] If μ_x and σ_x are homogeneous of degree one, as is the case with geometric brownian motion, then $\mu_x = \mu_\delta$ and $\sigma_x = \sigma_\delta$

[8] This allows me to focus on the interaction between product market competition, a firm's financial structure, and its efforts to redeploy assets across industries.

[9] The presence of this financial constraint may be justified on several grounds. When industry conditions are sufficiently poor, debtholders might refuse to roll over short-term debt, either because they fear that current management lacks the knowledge to run the business properly, or to prevent being expropriated through asset substitution when risk-shifting becomes highly attractive. In any case, the firm will have agreed to this financial constraint when signing the debt contract to increase the income that can be pledged to financiers.

[10] High costs of rescheduling debt and high costs of raising outside capital are central features of fire-sales models, otherwise the firm would never sell specialized assets to low valuation users. They can be justified on several grounds. Creditors might be dispersed and difficult to coordinate. Furthermore the issue of new securities might be precluded given the existence of debt overhang on the part of firms or asymmetry of information between inside management and outside investors. Finally inefficiencies in the bankruptcy process might be sufficiently large so as to dissuade the firm from seeking Chapter 11 protection to continue with all its operations.

[11] All qualitative results of this paper are robust to the assumption regarding the distri-

selling firm keeps ϕx.[12] If a firm has not divested its division before x reaches x_C, creditors foreclose on the debtors assets.

In the model, firms earn more by redeploying assets than by having them foreclosed. This will be true if, apart from the direct costs of the foreclosure process, financiers are particularly poor matches for specialized industrial assets. Pulvino (1998) finds that, in times where the airline industry is depressed, the discount associated to selling aircraft to financial institutions is 30%, compared to an average overall discount of 14%. Ramey and Shapiro (2001) analyze the gradual unwinding of an industry suffering from a negative aggregate shock. They show that when firms have some financial slack and time to search for potential buyers, specialized assets are redeployed in all sorts of industries, but not financial institutions.[13]

There is a caveat nevertheless: there is no centralized market for corporate assets. Selling industrial plants and oil tankers is not as easy as selling blue-chip stocks or government bonds. There is *market illiquidity* in that firms must search for buyers for their divisions.[14] When searching over an infinitesimal period of time $[t, t+dt]$, a firm incurs a cost of cdt and meets a buyer with probability λdt. Therefore, before the financial constraint binds, firm i can only redeploy its asset at random times $T_1^i < T_2^i < \cdots < T_n^i < \cdots$, such that $\{T_1^i, \cdots, T_n^i - T_{n-1}^i, \cdots\}$ are i.i.d. random variables with an exponential distribution with intensity λ. Firms cannot recall a previously rejected offer.

Many alternative interpretations can justify the assumption that meetings between buyers and sellers occur at discrete random times, as opposed to continuously. A buyer may always be accessible, but might demand a due diligence to protect himself from buying damaged or low quality assets. The due diligence takes time and is concluded at the random times defined above. Alternatively, random times might represent the moments at which slow moving outside money becomes available to wealthy investors (Duffie (2010), Acharya et. al. (2012)). If this pool of outside money is not immediately tapped, it is used to buy assets of other distressed industries. Because a firm never knows precisely when a buyer will be found, the existence of market illiquidity severely diminishes the option value of waiting to divest. It further implies that there

bution of the bargaining power between buyers and sellers.

[12]I assume shareholders cannot immediately tunnel (Johnson et. al. 2000)) the resources obtained in a divestiture. For example, the entrepreneur may have credibly promised to use the proceeds of asset sales toward debt repayment (Smith and Warner (1979)).

[13]Shleifer and Vishny (2011) claim financiers quickly sell the collateral they repossess even if at severe discounts, which suggests financiers' low disposition to own real assets.

[14]The market for used commercial aircraft analyzed in Pulvino (1998) is an example of an extremely thin market were transactions are mainly privately negotiated. He claims that on average there are only one or two monthly transactions of any particular aircraft model.

is always a positive probability that assets end up with creditors, irrespective of the search efforts that firms adopt.

Asset sales affect the competitive environment in which firms operate. After the divestiture of one firm, its surviving rival earns monopoly profits from that moment until the time it too chooses or is forced to depart with its division. According to this assumption, assets are redeployed elsewhere by the outsider and not on the widget industry and is starker than necessary. The results of the paper are unaltered if the exit of one firm exerts a positive externality on the surviving firm, even if only for a short period.[15]

I finally make three assumptions that impose some restrictions on model parameters. The first two incorporate important features concerning the outsiders' valuation of the widget division. First, firms face a weakly negative sloping demand curve for their assets. This assumption reflects the empirical evidence concerning the pricing of real assets in illiquid markets (Benmelech and Bergman (2011),Campbell et. al. (2011)), that shows that an asset sale by one firm places a downward pressure on the value of similar assets. Formally, when the state variable is x, the first firm to divest earns $\phi_H x$ after debt repayment while the second firm only earns $\phi_L x$, where $\phi_H \geq \phi_L$. This assumption actually attenuates the incentives to continue operations when industry conditions are poor. I restrict attention to the following parameter configurations:

Assumption 1 $D(1) - D(2) > \lambda(\phi_H - \phi_L)$.

The second feature that concerns outsiders' valuation is the presence of *asset illiquidity* as in Shleifer and Vishny (1992). Because assets used in the widget production process are industry-specific and cannot be fully redeployed elsewhere, prices fetched at divestitures are below their value in best use. Asset illiquidity is reflected in the following assumption:

Assumption 2 $r(M + \phi_H x) < xD(2) \;\; \forall x \geq x_C$

According to Assumption 2, an insider earns more from operating a division than selling it and investing the proceeds at the risk free rate. Assumption 2 also implies that $(M + \phi_L x) < (M + \phi_H x) < \Pi(x, 2) < \Pi(x, 1)$ so a firm only fetches a price that is below the division's fundamental value, irrespective of industry structure or the value of x.

[15]The surviving firm might only earn monopoly profits until a new competitor enters the widget industry, reestablishing a duopolistic market structure. This would be the case if the financier mothballed assets until economic conditions improved and a specialized buyer with enough financial muscle became available. Alternatively, the buyer might take some time to learn to use the asset to produce widgets, so once again monopoly would be only temporary.

The presence of asset illiquidity is an important feature of the model. Based on economic considerations alone, an industry insider would never sells its division to a less productive user, but might consider doing so due to the implications of having a fragile financial structure.[16]

Due to asset illiquidity, a firm should naturally seek a buyer among fellow industry participants which are their high valuation users. Nonetheless, I assume that when a division is liquidated, irrespective of the liquidation price, no firm is able to raise capital to buy its competitor.[17]

Finally, the third assumption concerns the market structure of the widget industry. It is used to derive a normative benchmark against which the competitive equilibrium is compared. More specifically, I restrict attention to the more interesting situation where aggregate profits in the product market are increasing in the quantity produced, so, absent financial constraints, a duopolistic market structure maximizes aggregate firm value.

Assumption 3 *Aggregate industry revenues given by* $R(Q) = QD(Q)$ *are increasing in* Q*. In particular* $D(1) < 2D(2)$*.*

The parameters in the model have straightforward interpretations. The parameter ϕ measures how second best users of the asset value it in excess of financial institutions. It depends not only on the asset's physical attributes but also on the expediency of the legal process involving the repossession of collateral by creditors.[18] The intensity of arrivals λ measures the "thickness" of the market for corporate assets and depends on the number of potential buyers that can redeploy the asset and are financially unconstrained.

1.3 A Benchmark: (Constrained) Efficient Divestiture Decisions

In this section, I analyze the behavior of a central planner who maximizes aggregate firm value and has the power to determine (i) firms' search efforts and (ii) whether firms accept a bid once a buyer is found. Because the value of short-term debt is always M, the maximization of aggregate firm value and aggregate shareholder wealth amounts to the same thing. The solution to the

[16]This is in stark contrast to Leland (1994), where, at the time bankruptcy is declared, the expected profits from continuing operations are strictly lower than outstanding debt due to the option value of liquidating the firm, which is always positive.

[17]It might be the case that initial industry conditions are not sufficiently healthy so as to provide one firm with enough financial slack to buy its rival, or the model might only be applicable to cases where firms are unable to hoard sufficient cash to buy competitors. Alternatively, acquiring other firms might be prohibited by regulation.

[18]The parameter ϕ also captures the correlation of the shock across industries, because asset liquidity depends on the cash-flow of potential buyers. For example, if industries where the widget division could potentially be redeployed are facing a downturn precisely when the widget industry is suffering, then ϕ will be low.

social planner's problem will provide a useful benchmark to the case where firms compete in an oligopolistic industry.

Not even the social planner obtains a fully efficient outcome.[19] Because firms have a non-renegotiable liability and markets are illiquid, the allocative inefficiency that results from the transfer of assets to financiers is present even in the benchmark case. More specifically, irrespective of the search strategy adopted, with strictly positive probability a firm becomes unable to honor its loan before it meets a buyer. The central planner's optimal search effort balances the benefits of having assets at the hand of their highest valuation users for the longest period of time with the expected costs of transferring assets to financiers, their lowest valuation users.

1.3.1 Benchmark Search Strategies

One important feature of the optimal search strategy follows immediately from the model's assumptions. Since search is costly, completely reversible, and there are no gains from receiving a bid and turning it down, a firm searches for a buyer only when it intends to sell its division to the first buyer it meets. A firm never turns down an offer for its division because it could do better by not searching in the first place.

To derive the central planner's divestiture policy, I work backwards. I initially characterize optimal search when only one division remains and then turn to the case were both divisions are active. Furthermore, recall that the aggregate shock is a time-homogeneous Markov process. Taking the number of operating firms Q as fixed, I thus focus on strategies that are a function of the current value x of the aggregate shock.

With one division remaining, a search strategy is a measurable markovian stochastic process given by u and taking values in $\mathcal{U} = \{0, 1\}$, where search is realized if and only if $u = 1$.[20] The social planner's expected payoff from adopting strategy u, is

$$J_M(x; u) \equiv \mathbb{E}^{x,u}\left[\int_0^{\tau \wedge \tau_C} e^{-rt}(x_t D(1) - cu_t)dt + e^{-r\tau}\phi_L x_\tau \mathbf{1}_{\{\tau < \tau_{\mathbf{C}}\}}\right]. \tag{1-5}$$

Expression (1-5) has the following interpretation. The first term inside brackets represents the discounted sum of the monopolist's net profits over the time period in which it operates the widget division. Instantaneous net profits

[19] The solution to the social planner's problem is only constrained efficient. The social planner is limited to act based on the same information as individual agents.

[20] As will soon become clear, nothing would change if the search process were allowed to take values in the interval $[0, 1]$. Because the Hamilton Jacobi-Bellman Equation is linear in the search intensity, the optimal search effort is of the bang-bang type, so u would only take values in $\{0, 1\}$ in equilibrium.

are given by operating cash-flows $xD(1)dt$ minus search costs $cudt$. A firm operates the division from time 0 until the minimum of two stopping times. The first one is $\tau = \tau(u)$, a stopping time that depends on the search process u and represents the first time that the firm meets a buyer. The second is $\tau_C = \inf\{t : x_t \leq x_C \ , \ x_0 > x_C\}$, the hitting time of the process x at x_C. The second term inside brackets is the present value of the benefits of redeploying assets in alternative uses, as opposed to having them foreclosed. Because one division has already been divested, prices fetched when redeploying the monopolist's assets are depressed so $\phi = \phi_L$. Finally note that the expectation depends not only on the current value of the aggregate industry shock x but also on the search strategy u adopted.

Let $m(x)$ be the total expected discounted profits from operating one division when the state variable is x and search is optimal. Then

$$m(x) \equiv \sup_u J_M(x, u). \tag{1-6}$$

I assume that $m(x)$ and all other value functions in this section are finite and differentiable, which I will later prove to be true. First, $m(x_C) = 0$. Furthermore, by the Strong Markov property, $m(x)$ satisfies the following equation in the search region:

$$m(x) = (xD(1) - c)dt + e^{-rdt}\lambda dt \phi_L \mathbb{E}^x[(x + dx)] + e^{-rdt}(1 - \lambda dt)\mathbb{E}^x\left[m(x + dx)\right]. \tag{1-7}$$

Equation (1-7) is an inter-temporal consistency condition. At any given time the value of a monopolist, given by $m(x)$, equals the sum of net profits accrued during an instant of time dt and the expected value of being a monopolist after dt has elapsed, discounted back in time. In the search region, during the interval of time dt the division yields profits of $xD(1)dt$ from operations minus search costs of cdt, hence the first term in equation (1-7). With the passing of time of dt, with probability λdt the firm meets a buyer and sells the division earning $\phi_L(x + dx)$ after honoring its financial obligations, hence the second term in equation (1-7). Alternatively, with probability $(1 - \lambda dt)$, the firm does not meet a buyer and remains as a monopolist at the new value of the state variable given by $x + dx$. This is captured by the third term in equation (1-7).

Using Itô's Lemma to expand $m(x + dx)$ yields, after cancellation of all terms of order higher than dt, the following Hamilton-Jacobi-Bellman (HJB) equation for $m(x)$ in the search region:

$$rm(x) = xD(1) - c + \mathcal{L}m(x) + \lambda[\phi_L x - m(x)], \tag{1-8}$$

where $\mathcal{L}$ is given by

[21]
$$\mathcal{L}g \equiv \frac{1}{2}g''\sigma^2 + g'\mu. \tag{1-9}$$

Similar reasoning leads to the following HJB equation for $m(x)$ in the absence of search:

$$rm(x) = xD(1) + \mathcal{L}m(x) \tag{1-10}$$

The social planner determines the search region optimally. In the search region, the instantaneous expected gain from searching must exceed the instantaneous cost. The instantaneous expected gain from searching, given by $\lambda[\phi_L x - m(x)]$, is a product of two terms. The first term, λdt, is the probability of finding a buyer over the interval of time dt. The second term, given by $[\phi_L x - m(x)]$, is the net gain (or loss) from selling the division when industry conditions are summarized by x. The instantaneous cost of searching over the interval of time dt is cdt. Therefore $\lambda[\phi_L x - m(x)] > c$ in the search region, whereas the opposite inequality holds when there is absence of search. Continuity of the value function across the boundary that divides the two regions yields

$$\lambda[\phi_L x_M - m(x_M)] = c. \tag{1-11}$$

A cutoff rule summarizes optimal search; there exists a unique x_M such that the firm searches if and only if $x \leq x_M$. The threshold x_M is implicitly defined by the above equality.

The following HJB equation summarizes the above discussion:

$$rm(x) = xD(1) + \mathcal{L}m(x) + \{\lambda[\phi_L x - m(x)] - c\}\mathbf{1}_{\{x \leq x_M\}}\ \forall x \geq x_C. \tag{1-12}$$

Now that I have completely characterized the value of operating one division, I turn to the case where both firms are still active. Let $d(x)$ be the value function of operating both firms when market structure is still duopolistic. A search strategy is a measurable markovian stochastic processes w taking values in $\mathcal{W} = \{0, 1, 2\}$, where $w = 1$ denotes search by one firm and $w = 2$ by both.[22] Then

$$d(x) \equiv \sup_w \mathbb{E}^{x,w}\left[\int_0^{\tau \wedge \tau_C} e^{-rs}(2xD(2) - cw_t)dt + e^{-r\tau}(\phi_H x_\tau + m(x_\tau))\mathbf{1}_{\{\tau < \tau_\mathbf{C}\}}\right]. \tag{1-13}$$

The interpretation of expression (1-13) is similar to that of expression (1-5). In particular $\tau = \tau(w)$ is once again a stopping time that represents the meeting of a buyer and τ_C is as previously defined a standard hitting time. However, there are some differences. The first term inside brackets is now a sum of the net profits of both firms. More importantly, the second term inside brackets

[21] $\mathcal{L}$ is the the infinitesimal generator of the Itô diffusion. The infinitesimal generator of an Itô Diffusion x is the operator $\mathcal{L}$ that associates to a suitable function f of x its instantaneous expected change. It is defined by $\mathcal{L}f(x) = \lim_{t \downarrow 0} \frac{\mathbb{E}^x[f(x_t)] - f(x)}{t}$.

[22] Since both divisions are identical, there is no need to discriminate among them.

is the present value of the sum of two components. The first $\phi_H x_\tau$ is the benefit that accrues to the firm when redeploying the first widget division. The second $m(x_\tau)$ is the value to the central planner of continuing operations under a monopolistic market structure from the moment that one firm exits the industry. The problem of the monopolist is thus embedded in the optimization problem of the social planner when two divisions are operational.

The social planner's optimal search strategy when two divisions are active is intuitive. When industry conditions are sufficiently healthy, it is optimal to operate both divisions and no search takes place. This follows from Assumption 3, which guarantees that a duopoly generates more aggregate profits than a monopoly. On the other hand, as industry conditions deteriorate sufficiently and the binding of the financial constraint approaches, it is more attractive to redeploy corporate assets even if prices fetched are below their fundamental value.

One might initially think the social planner randomly picks a firm to initiate search, but this is not true. The social planner has a state contingent preference over market structures - duopoly versus monopoly - but is indifferent between which firm actually exits first. Therefore he determines that both firms initiate search simultaneously to maximize contact intensity with buyers.[23] Mathematically, optimality of simultaneous search results from the linearity of the HJB equation in search intensity. Once the first division is sold, the social planner proceeds according to the rule previously derived when operating only one division.

Let x_D be the threshold that determines the social planner's search strategy when two firms remain. Then

$$rd(x) = 2xD(2) + \mathcal{L}d(x) + 2\{\lambda[m(x) + \phi_H x - d(x)] - c\}\mathbf{1}_{\{x \leq x_D\}} \; \forall x \geq x_C \qquad (1\text{-}14)$$

Once again, the threshold is determined optimally which implies that

$$\lambda[m(x_D) + \phi_H x_D - d(x_D)] = c. \qquad (1\text{-}15)$$

According to Equation (1-15), when $x = x_D$ the instantaneous expected gain from searching equals its costs. After the first firm exits, profits to the surviving firm increase. The social planner fully internalizes the benefits of less competition in the product market. This is why, when contemplating the divestiture of the first division, the social planner adds the value of a monopoly to the liquidation value fetched and compares this to the value of a duopoly.

[23] Since each firm meets a buyer with probability λdt during the infinitesimal instant of time dt, the probability that they both meet a buyer simultaneously is $(\lambda dt)^2 \approx 0$.

The following proposition gives conditions under which the solutions to the HJB equations are the value functions of interest.

Proposition 4 *Let $\hat{m}(\cdot)$ be a differentiable function satisfying (1-12), with bounded derivative and $\hat{m}(x_C) = 0$. Then $\hat{m}(\cdot)$ is the value function of the optimization problem (1-6). Similarly let $\hat{d}(\cdot)$ be a differentiable function satisfying (1-14), with bounded derivative and $\hat{d}(x_C) = 0$. Then $\hat{d}(\cdot)$ is the value function of the optimization problem (1-13). Search strategies are of the cutoff type and characterized by (1-11) and (1-15), where $x_D > x_M$.*

The proof of Proposition 4 is given in the Appendix and relies on a standard verification argument.[24] Cutoff strategies attain the supremum in the HJB equation and are therefore optimal policies. Furthermore, they are uniquely defined.

Some features of the social planner's solution are worth highlighting. The pair of thresholds $\{x_M, x_D\}$ fully determines the efficient state contingent market structure. For $x > x_D$, expected financial distress costs are negligible and thus overpowered by economic considerations. From Assumption 3 a duopoly is optimal. Furthermore, for values of the industry-wide shock in this range, one active division is better than none. Therefore, irrespective of the prevailing market structure, there is no search whatsoever for $x > x_D$, which in turn implies that the divestment hazard rate - the probability of a divestiture occurring in the interval $[t, t + dt]$ - is 0.

Alternatively, expected financial distress costs are too high to be disregarded when $x \in (x_M, x_D)$ and a monopoly is optimal. To see why it is better to sell only one division instead of selling both of them altogether one must understand how divisions are socially valued under a duopoly. One division is valued as a monopolist, while the remaining division is valued by its marginal profits above the monopoly level. When x is in this intermediate range, the marginal division's profit is too low to risk losing assets to creditors if the financial constraint is eventually binding. Nonetheless, monopoly profits are still high enough to take this risk. Therefore, if two divisions are still active when x enters (x_M, x_D) then maximal efforts are made to revert to a monopoly and the hazard rate is 2λ. On the other hand, if one division has already been divested then the desired market structure prevails and the hazard rate is 0.

For x below x_M search is maximal irrespective of the number of active firms since even a monopolist does not generate enough economic value at the hands of a specialist. The following figure shows divestiture hazard rates and optimal search strategies conditional on prevailing market structure.

[24]In optimal control, verification theorems show that a smooth candidate solution to the HJB equation is equal to the value function of the problem at hand.

The following proposition shows how efficient market structure and search efforts vary with the parameters of the model.

Proposition 5 *The following comparative statics results hold:*

1. The higher the monopoly profits measured by $D(1)$ the earlier the first division is sold and the later the second division is sold (in expectation). That is, x_D is increasing and x_M is decreasing in $D(1)$;

2. $x_M(\phi_L)$ and $x_D(\phi_H)$ are increasing functions;

3. x_M and x_D vary non-monotonically with respect to λ;

4. The average price of the first (second) division sold is increasing (decreasing) in $D(1)$;

5. The probability of the first (second) division ending up at the hand of financiers is decreasing (increasing) in $D(1)$.

The intuition behind the first part of the previous proposition is the following. As monopoly profits increases while duopoly profits remain constant, the value of the marginal division decreases. The social planner therefore becomes less willing to take risks to keep it operational at the hands of a specialist under an industry downturn. Greater efforts are made to switch to a monopolistic market structure, which implies earlier initiation of search and an earlier divestiture of the first division in expectation. By contrast, the division that survives after the first one is sold becomes more valuable with an increase in $D(1)$ and is divested later based on similar reasoning.

According to the second part of the proposition, as the asset becomes more redeployable firms initiate search earlier to reduce the expected social loss from the seizure of assets by creditors, their lowest valuation users. The non-monotonicity of search strategies with respect to the meeting intensity λ might appear awkward at first but, apart from being intuitive, also strikes me as a particularly interesting feature of the model. If $\lambda < c\phi^{-1}$, then the optimality conditions given by (1-11) and (1-15) are never satisfied before x reaches x_C. Effective search costs given by c/λ are too high (when compared to ϕ) to make search worthwhile, regardless of the likelihood of the violation of financial constraints. The model thus predicts that severe market illiquidity causes severe liquidation discounts. In the extreme case when there is no search, the divestiture threshold is exogenously given as in Merton (1974) and Longstaff and Schwartz (1995).

As the meeting intensity rises, search becomes more attractive and threshold levels continuously increase at first. Ultimately, this behavior is reversed. To see this note that as $\lambda \to \infty$, it is optimal to stop searching altogether again since the social value of the division at the hands of a specialist is above what the outsider is willing to pay. Divesting when the market is perfectly liquid resembles trading a ten dollar bill for a five dollar bill. It is always better to postpone divestment as much as possible in an attempt to avoid the destruction of value that accompanies the separation of the asset from its high valuation user.[25]

Shleifer and Vishny (1992) show that firms fetch low prices if they are forced to immediately sell assets when few potential buyers are available. In this paper I allow some flexibility in the firms' divestiture decision and endogenize search efforts, obtaining different results for some parameter configurations. Because a decrease in market liquidity destroys the option value of waiting to divest, firms may exert more efforts to redeploy assets precisely when they are hard to find, which leads to higher liquidation prices.The last two statements of the proposition follow mechanically from the first one. As $D(1)$ increases and search is initiated earlier, higher liquidation prices are obtained and the likelihood of foreclosures decrease.

1.4 Duopoly: Markov Perfect Equilibrium

In this section I analyze firms' strategic behavior when they compete in a duopoly. In the competitive equilibrium, optimal search decisions must be part of a Nash equilibrium solution in search strategies. In a Nash equilibrium firms choose strategies that maximize expected lifetime discounted profits, taking as given their rival's behavior. I assume that firms adopt stationary Markov strategies, with current actions depending on the history of the game only through its influence on the current environment and look for a pair of strategies that constitute a Markov Perfect Equilibrium (MPE).[26]

When choosing their strategies, firms face a simple tradeoff. Firms can adopt a precautionary strategy with early initiation of search to increase the average liquidation price fetched for their divisions. On the other hand, firms can postpone searching in an attempt to outlast rivals and win the contest for a monopoly. I am particularly interested in how competition for a monopoly impacts the liquidation value of the first division divested. After all, the problem faced by a monopolist is exactly the same as the one faced by a social planner operating only one division, so no inefficiencies are present after the second division is sold.

[25] This option value of waiting always increases with λ and is ultimately so high that it becomes optimal never to sell. Mathematically, the problem becomes ill defined when $\lambda = \infty$ since the optimal time to divest is $inf\{t : x_t > x_C\}$.

[26] I formally define the game played by firms in the Appendix. A MPE remains a Nash equilibrium when the markovian restriction is dropped, which is quite reassuring. However, there may also exist other equilibria in which player's strategies are path-dependent.

The following lemma is important in characterizing equilibrium behavior in a simple way.

Lemma 6 *The best response of a firm to any of its rival's strategy is a search strategy of the threshold type.*

According to Lemma 6, firm i adopts in response to its rival's behavior a strategy that can be completely summarized by a unique number $x_i \in \mathbb{R}$. According to this strategy, firm i searches if and only if $x \in (x_C, x_i]$.

1.4.1 Equilibrium Characterization

In this section I characterize firms' equilibrium behavior. I first analyze the case where firms adopt asymmetric strategies and then turn to the symmetric case. In the asymmetric equilibrium, I call the first firm to initiate search the Leader and its rival the Follower. These are merely roles played by firms, irrespective of their identities.

Follower

Taking the Leader's strategy at x_L as fixed, the Follower faces a standard one-firm stochastic optimization problem. The Follower's best response is a threshold strategy, so his value function given by $f(x)$ is easy to characterize. The following ODEs describe the evolution of the value of the Follower at the different regions of the state variable:

$$rf(x) = D(2)x + \mathcal{L}f(x) \;\; \forall x \in [x_L, \infty) \tag{1-16}$$

$$rf(x) = D(2)x + \mathcal{L}f(x) + \lambda[m(x) - f(x)] \;\; \forall x \in [x_F, x_L) \tag{1-17}$$

$$rf(x) = D(2)x - c + \mathcal{L}f(x) + \lambda[m(x) - f(x)] + \lambda[\phi_H x - f(x)] \;\; \forall x \in [x_C, x_F) \tag{1-18}$$

Equation (1-16) is straightforward and is valid if $x \in [x_L, \infty)$, when neither firm searches. Equation (1-17) determines the evolution of the value of the Follower when $x \in [x_F, x_L)$ and only the Leader searches for a buyer. The Leader's effort to find a buyer exerts an externality of $\lambda[m(x) - f(x)]dt$ on the Follower, who, with probability λdt, becomes a monopolist over each instant of time dt. This externality is always positive since $m(x) > f(x)$ for any value of x and is responsible for making the role of the Follower greater than that of the Leader. Finally, equation (1-18) represents the evolution of the value of the Follower when $x \in [x_C, x_F)$ and both firms search for buyers.

The best response to the Leader's strategy is for the Follower to initiate search at the threshold $x_F = BR(x_L)$, where $BR(\cdot)$ is the best reply function. x_F is implicitly characterized by

$$\lambda[\phi_H x_F - f(x_F)] = c. \tag{1-19}$$

Equation (1-19) is an optimality condition similar to the ones derived in the previous section. It implicitly determines x_F by equalizing the instantaneous expected benefit of searching for a buyer to its instantaneous costs. Finally, it is also required that $f(x_C) = 0$ and that $f(\cdot)$ have bounded derivative.

Leader

Now, assume the Follower's strategy fixed at x_F and let $l(x)$ be the value function of the Leader when the industry-wide shock is x. The following set of ODEs determine the evolution of the value of the Leader in the relevant regions:

$$rl(x) = D(2)x + \mathcal{L}l(x) \ \ \forall x \in [x_L, \infty) \tag{1-20}$$

$$rl(x) = D(2)x - c + \mathcal{L}l(x) + \lambda[\phi_H x - l(x)] \ \ \forall x \in [x_F, x_L) \tag{1-21}$$

$$rl(x) = D(2)x - c + \mathcal{L}l(x) + \lambda[\phi_H x - l(x)] + \lambda[m(x) - l(x)] \ \ \forall x \in [x_C, x_F) \tag{1-22}$$

The ODEs of Leader and Follower only differ in the region $[x_F, x_L)$. When the aggregate shock x lies in this region, the Leader finds a buyer and liquidates its division with probability λdt over the interval dt while the Follower becomes the monopolist when this happens. The best response of the Leader, given by $x_L = BR(x_F)$, is characterized by

$$\lambda[\phi_H x_L - l(x_L)] = c. \tag{1-23}$$

Once again, Equation (1-23) defines the Leader's optimal response to the Follower's strategy.

I now characterize the firms' value functions when they adopt the same strategies.

Symmetric Case

In the symmetric case, both firms initiate search simultaneously, so:

$$rs(x) = D(2)x + \mathcal{L}s(x) \;\; \forall x \in [x_S, \infty) \tag{1-24}$$

$$rs(x) = D(2)x - c + \mathcal{L}s(x) + \lambda[\phi_H - s(x)] + \lambda[m(x) - s(x)] \;\; \forall x \in [x_C, x_S) \tag{1-25}$$

Once again the optimal threshold is defined implicitly by

$$\lambda[\phi_H x_S - s(x_S)] = c. \tag{1-26}$$

1.4.2 Game of Strategic Substitutes

In this section, I prove that the search game played by firms is one of strategic substitutes. In games of strategic substitutes, an increase in one player's actions leads to a decrease in the other player's best reply to that action, i.e. the best reply function of a given player is decreasing. Once established, this ordinal property in players' best-reply functions takes us a long way. In particular it can be used to prove existence of equilibrium by an application of Tarski's Fixed Point Theorem.[27]

Proposition 7 *The best reply function $BR(\cdot)$ is decreasing in rival's strategies. The search game played by firms is one of strategic substitutes.*

The intuition behind the result of Proposition 7 is straightforward. Assumption 1 guarantees that the value of operating as a monopolist, given by $m(x)$, is greater than the value of operating as a duopolist, irrespective of the type of equilibrium that ensues or the role each firm adopts. Therefore, as one player exerts more efforts to redeploy assets, its rival, enticed by the opportunity to become the monopolist, reacts in opposite manner by reducing search.

[27]By reversing the natural order of one of the player's strategy set, a game of strategic substitutability becomes a game of strategic complementarity. One can then apply the results of supermodular games, which are based on lattice theory, monotonicity results in lattice programming, and Tarski's Fixed Point Theorem. Tarski's Fixed Point Theorem shows the existence of a fixed point for increasing functions defined on a complete lattice. Supermodular games are developed in Topkis (1979), Milgrom and Roberts (1990), and Vives (1990). Vives (2001) provides a survey of the literature.

Characterizing the Equilibrium Set

Using the result of Proposition 7, it is possible to characterize the set of MPE of the game. Let the set of MPE of the search game be given by $\mathcal{E}$, where

$$\mathcal{E} = \{(x, y) \in \mathbb{R}^2 : \ x = BR(y) \text{ and } y = BR(x)\} \tag{1-27}$$

Proposition 8 *The set $\mathcal{E}$ is non-empty and contains a unique symmetric equilibrium. Furthermore, after reversing the natural order of one of the player's strategies, the set $\mathcal{E}$ has a largest and smallest element.*

Proposition 8 follows from standard results in supermodular games. In particular, existence of equilibrium follows from Tarski's Fixed Point Theorem. The largest and smallest elements of $\mathcal{E}$ can be obtained by an iterative procedure of eliminating strictly dominated strategies. Furthermore, starting from an arbitrarily high x_0, repeated iteration of best-reply functions until convergence is reached yields the largest and smallest elements of $\mathcal{E}$.[28] The game has a unique Markov equilibrium if and only if the algorithm just described yields the same threshold strategy for both firms, in which case the game is dominance solvable.

The symmetric equilibrium is the only one characterized by equalization of payoffs among firms. In all asymmetric equilibria, the firm adopting the role of the Follower earns a higher payoff than that of its rival, the Leader. Proposition 8 also highlights another notion of symmetry. The equilibrium set $\mathcal{E}$ is symmetric in the sense that reversing the roles adopted by firms in equilibrium also yields an equilibrium of the search game. Mathematically,

$$(x, y) \in \mathcal{E} \Longleftrightarrow (y, x) \in \mathcal{E}. \tag{1-28}$$

The following result also holds.

Proposition 9 *The MPE equilibrium set is completely unordered (an anti-chain).*

In words, Proposition 9 states the following. Fix two equilibria of the search game. Then it cannot be the case that both firms search strictly more in one equilibrium than in the other.

Proposition 9 rules out the type of coordination failures that are pervasive in games of strategic complementarities. If firms could meet at the start of the game and choose a MPE in $\mathcal{E}$, they would not reach an agreement unless

[28]More specifically, initially calculate $x_F^1 = BR(\infty)$ and $x_L^1 = BR(x_F^1)$. Then, proceed inductively, computing $x_F^n = BR(x_L^{n-1})$ and $x_L^n = BR(x_F^{n-1})$ until convergence has been achieved i.e. $|x_j^n - x_j^{n-1}| < \varepsilon$, for $j \in \{L, F\}$.

monetary transfers between them were allowed.[29] Among all equilibria in $\mathcal{E}$, a given firm prefers the one where its rival initiates search earliest. Since there is an inherent conflict between the two firms, the most preferred equilibrium by one firm is naturally the least preferred by the other. Proposition 9 highlights why coordination efforts to reduce oversupply in the shipping industry ultimately failed, as each firm waited for its rivals to bear the brunt of capacity reductions.

Inefficiency of Competitive Equilibrium

I now compare the competitive equilibrium with the efficient benchmark derived in Section 1.3 and show how liquidation discounts vary with model parameters.

Proposition 10 *Let (x_F, x_L) and (x_S, x_S) belong to $\mathcal{E}$, where $x_F < x_L$. Then $x_M < x_F < x_S < x_L < x_D$. Furthermore, compared to the efficient benchmark, (i) the average price fetched for the first division that is liquidated is too low and (ii) the division ends up at the hand of financiers, their lowest valuation users, too often.*

The ordering of the thresholds portrays how the contest for a monopolistic position leads to insufficient efforts to redeploy assets across industries. Irrespective of the type of equilibrium that ensues, firms initiate search efforts too late (thresholds are below x_D) when compared to the efficient benchmark. This occurs because firms do not internalize the social benefit of their exit, bestowed upon their rival in the form of rents arising from market concentration. Consequently, contact intensity with potential buyers is below first best and market structure takes too long to revert to a monopoly. This in turn has two major implications, both of which impact distress costs. First, on average, prices fetched for the first division are too low since they are proportional to x and firms are only willing to divest when conditions have deteriorated significantly. Second, since there is less time to find a buyer before the financial constraint binds, assets end up at the hand of financiers, their lowest valuation users, too often.

Both types of equilibrium are qualitatively different. In the asymmetric equilibrium, the Follower only searches when x has fallen significantly in its quest to become the monopolist. By doing this, the Follower forces an early search strategy on the part of its rival, the Leader. This is in stark contrast to the efficient benchmark where the irrelevance of the identity of the firm to be awarded the most lucrative role made simultaneous initiation of search

[29] In particular, it is useless to allow firms to communicate with each other.

by both firms optimal. In the symmetric equilibrium, there is equalization of payoffs among firms, but this is not enough to restore efficiency and search is once again insufficient.

Comparative Statics

In performing comparative statics, I restrict attention to the symmetric equilibrium.

Proposition 11 *The following comparative static results hold:*

1. x_S is decreasing in the rents associated to a monopoly given by $D(1)$;

2. x_S is increasing in the outsider valuation ϕ_H;

3. threshold level varies non-monotonically with respect to λ;

4. average liquidation prices of the first division sold are decreasing in $D(1)$;

5. the probability that the first division ends up at the hand of financiers is increasing in $D(1)$.

The main message of Proposition 11 is that equilibrium distress costs increase with the rents linked to concentration in the product market. As $D(1)$ increases while $D(2)$ remains fixed, firms postpone their search efforts to try to outlast their rivals. This behavior leads in equilibrium to lower liquidation prices and more instances of covenant violations with the resulting transfer of assets to financiers.

It is instructive to compare the results of Proposition 11 with the efficient benchmark outcome.[30] The first part of Proposition 11 states that, as monopoly rents increase, the longer it takes in expectation for the market to revert to a monopoly *ceteris paribus*. This is exactly the opposite of what is prescribed by efficiency considerations, where an increase in monopoly profits results in a quicker exit of the first firm. While larger monopoly rents lead a social planner to revert quicker to a more concentrated market structure, it increases the competing firms' desire to outlast each other. Furthermore, as $D(1)$ increases, so does the liquidation discount. The model thus predicts that industries where monopoly rents are higher are characterized by larger departures from the socially optimal benchmark.

[30]The results of Proposition 11 do not depend on Assumption 3, which states that aggregate profits are increasing in Q

1.5 Relation to the Empirical Literature

There is empirical evidence supporting many of the model's assumptions. For example, the transfer of specialized assets to financial institutions appears to be particularly costly. Using hedonic regressions, Pulvino (1998) estimates price discounts associated with the sale of aircraft. He concludes that during recessions, when finding a lessee is difficult, financial institutions pay a discount of 30% to the average market price, whereas other outsiders pay a discount of only 14%. Djankov et. al. (2008) show that debt enforcement procedures are highly inefficient; in only 36% of the time the firm is kept as a going concern.

There is also evidence that financial constraints play an important role in the transfer of specialized assets to third parties. Lang et. al. (1995) show that selling firms have poor performance and high leverage. Pulvino (1998) estimates that the likelihood of asset transfers to financiers increases by approximately 35% when the seller is financially constrained, but only during market recessions.

Benmelech and Bergman (2011) and Campbell et. al. (2011) identify the existence of fire-sale externalities. Fire-sales externalities occur when, by selling into an illiquid market, one firm negatively effects the collateral value of similar assets. In the model, fire-sales externalities occur through two distinct channels. The first is direct and is captured by allowing that $\phi_H > \phi_L$, so that for the same level of the aggregate shock x the first firm to divest fetches a higher price than the surviving firm. The second channel operates through the surviving firm's equilibrium divestiture decision. Because the second firm to divest fetches very depressed prices, it chooses to stay operating in the product market longer and, conditional on eventually divesting its division, will do so after industry conditions have deteriorated significantly.

I now turn to the model's predictions. Takahashi (2012) estimates the impact of an exogenous demand decline on the exit decisions of movie theaters in the U.S. from 1949 to 1955. His study seems particularly appropriate to test the model I present. The growth of television forecasting in the U.S. during the 1950's was a negative aggregate demand shock to the movie industry. Because most movie theaters were single-screen, they could not adjust capacity, so their only decision was whether to exit the market. Lastly, most movie theaters had a fragile financial structure since they were heavily mortgaged. Takahashi (2012) explores the fact that competition among movie theaters was mainly at a local level thus varied from city to city. He shows that, beginning in 1949, the exit rate of movie theaters from the industry increase with the number of competitors. This is precisely the content of the first part of Proposition 11 that argues that as monopoly profits increase, firms postpone exit at the risk of higher distress costs. However, Takahashi (2012) remains silent on the prices fetched for redeployed assets or on the frequency of covenant violations.

Other papers have a greater focus on the interaction between the degree of competition in the product market and the severity of distressed asset sales. Opler and Titman (1994) estimate that, during industry-wide distress, firms with higher leverage experience a more significant drop in their equity value and this effect is more pronounced in concentrated industries. Acharya et. al. (2007) use data of defaulted firms and also find that costs due to financial distress are greater in industries that have specific assets and fewer firms.[31] These findings might be due to the mechanism presented in this paper, namely that excessive continuation leads to higher financial distress costs. However, industry concentration can proxy for a number of other things, the first coming to mind being the lack of competition among asset buyers (Shleifer and Vishny (1992)).

According to the model, the redeployability of assets by firms in financial distress is a positive outcome, all things considered. Alexander et. al. (1984) and Hite et. al. (1987) provide empirical evidence that positive stock-price reactions follow firms' asset sales announcements.

A considerable empirical literature analyzes how a firm's capital structure influences both the firm's and its competitors' decisions in the product market. Chevalier (1995) finds that an increase in debt on the part of one firms generates a positive impact on the shareholder wealth of its rivals and leads them to realize greater investments in the industry. She interprets this result as evidence that higher leverage softens competition in the product market. Phillips (1995) and Kovenock and Phillips (1997) analyze how a sharp debt increase on the part of a firm interacts with market structure by affecting its plant closing decisions and that of its rivals. They find that rivals are less likely to close plants and more likely to invest after a firm with high market share increases leverage. This finding is also consistent with the model's predictions.

1.6 Conclusion

Fire sales occur when financial distress is common to firms operating in a specialized industry (Shleifer and Vishny (1992)). Under these conditions, I argue that firms' attempts to mitigate financial distress through asset divestitures lead to changes in the firms' competitive environment in the product market. Because the adverse shock is industry-wide, assets are displaced to liquidity

[31] Asquithet et. al. (1994), Andrade and Kaplan (1998) and Pulvino (1998) find empirical evidence supporting the existence of distressed costs associated with fire-sales but remain silent on how these costs vary with market concentration.

endowed outsiders; and because assets are specialized, their new owners employ them in alternative uses. But then the redeployment of assets must reduce competition within the industry. In a setting where firms have some flexibility in their divestiture decision, I show that the costs of financial distress associated to the liquidation of specialized assets should be greater in industries where the gains arising from market concentration are larger.

This paper has implications for the study of a firm's optimal capital structure. According to the existing literature on the subject, the optimal debt-equity mix often involves trading off some advantage of debt against expected financial distress costs.[32] Because this paper shows that financial distress costs are larger in industries with significant rents arising from market concentration, firms operating in these industries should have ex-ante a lower leverage. By similar reasoning, they should also hoard more cash and avoid financing through hard securities that prevent ex-post renegotiation.

The model may also help clarify the cross-sectional determinants of aggregate industry leverage. Shleifer Vishny (1992) introduce the notion of *industry debt capacity* and argue that, in equilibrium, an increase in leverage by one firm crowds out potential leverage by rivals. Industries where monopoly rents and equilibrium financial distress costs are higher should have a smaller aggregate debt capacity than those where the prize from a monopoly position is less attractive.

The model has many other empirical predictions. For example, industries where rents arising from market concentration are lower should be characterized by a higher rate of asset divestitures, fewer instances of covenant violations, and a market structure more sensitive to economic conditions. As shown in the previous section, there is empirical evidence supporting some of my results. However, additional empirical work seems warranted since some of the predictions of the model remain untested. It might be particularly interesting to empirically disentangle the effect of the lack of buyers from the effect of the lack of sellers on discounts of liquidated assets in concentrated industries.

This paper also derives new results on how market illiquidity can impact liquidation prices in equilibrium. Shleifer and Vishny (1992) show that, when firms are forced to divest immediately, liquidation discounts relative to fundamental value decrease as potential buyers become more available. I in turn allow some flexibility in the firms' divestiture decision and endogenize redeployment efforts, obtaining different results for some parameter configur-

[32]The traditional static trade-off theory explores the tax benefits of debt. Alternative benefits have been highlighted among many others by Jensen and Meckling (1976), Ross (1977), Jensen (1986), and Harris and Raviv (1990). Harris and Raviv (1991) survey the theoretical literature on capital structure.

ations. Because a decrease in market liquidity destroys the option value of waiting to divest, firms may exert more effort to find a buyer precisely when buyers are hard to find, leading to lower price discounts in equilibrium.

Finally, the analysis also adds to the discussion regarding the interruption of operations of distressed firms (Baird (1986), Aghion et. al. (1992), Shleifer and Vishny (1992)). Previous literature has acknowledged that lack of competition among potential buyers of assets leads to severe liquidation discounts, which in turn should favor continuation of operations under bankruptcy protection as opposed to a forced liquidation followed by immediate auction of assets. This article shows the flip side of the coin, by highlighting the benefits of rules and procedures that promote early exit.

2 Coordinated Strategic Defaults and Financial Fragility in a Costly State Verification Model

A run on a bank takes place when a large number of its clients simultaneously renege on its services, promoting its disintermediation and occasionally its demise. The most common form of bank run occurs when depositors rush to withdraw their money because they fear the bank will be unable to honor all its liabilities at par. In this paper, we explore a different form of bank run; that which originates on the bank's asset side when a borrower defaults on his loan because he expects other borrowers to do the same. We refer to such a situation as a *coordinated strategic default.*[1]

There is evidence that coordinated strategic defaults occur across a variety of institutional arrangements. Krueger and Tornell (1999) document how the lack of transparent and effective bankruptcy procedures in Mexico during the 1995 crises led many borrowers to default, despite their full capacity to service their debt. Another case is Childreach, a microfinance program in Ecuador. According to Goering and Marx (1998) the program collapsed when "*the number of residents defaulting on their loans multiplied as the word spread that few people were paying*". Even the US, arguably the world's most financially developed country, has not gone unscathed. Guiso, Sapienza and Zingales (2012) document how households with underwater mortgages are more likely to strategically default on their loans if they are acquainted with someone who is also defaulting strategically.[2]

We analyze the issue of coordinated strategic defaults in a canonical model of entrepreneurial finance characterized by costly state verification (Townsend (1979), Gale and Hellwig (1985)). [3] In the model, a financial intermediary lends to a continuum of entrepreneurs at contractual terms endogenously chosen. To derive the optimal contract, we initially adopt the tradi-

[1] A strategic default occurs when the debtor has the financial means to pay off his debt, but chooses not to. It is thus an issue of the debtor's willingness to pay, not of his capability to do so.

[2] Bond and Rai (2009) present more evidence on coordinated strategic defaults in microfinance programs, while Vlahu (2008) focuses on corporate credit in Eastern Europe and Asia.

[3] The main feature of costly state verification models is that the entrepreneur observes his project's return free of charge, while the financial intermediary must perform a costly audit if it wishes to become informed.

tional mechanism design approach according to which the designer proposes a bayesian game that has among its possibly many equilibria one that maximizes a predefined value criterion.[4]

In this setting, we show that in the *good* equilibrium a standard debt contract provides entrepreneurs with incentives to repay their loan whenever they can while minimizing monitoring costs.

However, while repayment is *one* equilibrium of the optimal financial arrangement through which the bank finances projects (standard debt contract), it is not the only one. In the model the default by a group of debtors weakens the bank's financial position and hurts its monitoring capabilities, which ultimately makes the decision to default by any other entrepreneur more attractive. Such strategic complementarities in entrepreneurs' actions lead to multiplicity of equilibria. In some of them, a debtor declares default because he expects other debtors to do the same.

We establish that, apart from the good equilibrium, there is always an equilibrium in which *all* entrepreneurs default strategically. We refer to such an outcome as a *fully coordinated default*. One may argue that, due to communication and coordination costs, joint deviations by the whole set of entrepreneurs are not particularly worrisome. Nevertheless, we show that *partially coordinated default* equilibria always exist as well. In these equilibria, although some entrepreneurs repay their debt, a non-negligible subset of entrepreneurs default strategically.

In addition to establishing that banks may fall victim to coordinated defaults in a canonical model of financial contracting, the second goal of this paper is to consider alternatives banks may have to rule out these *bad* equilibria. We propose two main solutions, both of them sharing the following features: (i) to break the strategic complementarities among borrowers the bank must use what we call a *sequential audit strategy* and (ii) to be able to audit a given group that plays a special role in the sequential audit strategy the bank must secure a given amount of resources. The solutions differ mainly in the way the bank secures such resources.

The sequential audit strategy is implemented as follows. The bank first divides entrepreneurs into groups, which are then randomly ordered. Once the bank starts auditing, it does so sequentially, auditing entrepreneurs in group $n+1$ only after it has audited all defaulted projects in group n. *If* the bank can fully commit to audit entrepreneurs in the first group, such entrepreneurs find it optimal to report truthfully regardless of the announcements made

[4]The mechanism design approach is also used in Townsend (1979) and Gale and Hellwig (1985). It implicitly assumes that, when multiple equilibria are present, the desired one is chosen.

by other entrepreneurs. With the payments collected from entrepreneurs belonging to the first group the bank's monitoring resources increase and it can credibly commit to audit entrepreneurs in the second group as well. Proceeding inductively, we show that coordinated strategic defaults unravel and the good equilibrium is restored.

Together, sequentiality and asymmetric treatment of ex-ante identical individuals are central features of some theoretical models of bank runs. In Diamond and Dybvig (1983) for example, multiplicity of equilibria arise because ex-ante identical depositors are treated asymmetrically, according to a first-come first-serve basis (Jacklin (1987), Wallace (1996)). In this paper, however, multiplicity of equilibria is eliminated precisely when the bank starts to treat ex-ante identical borrowers asymmetrically and sequentially.

Finally, we propose two ways the bank can kick-start the sequential audit strategy, that is, secure the needed resources to audit entrepreneurs in the first group with certainty. First, the bank can set aside a small amount of capital ex-ante. While we assume that the bank incurs in an opportunity cost of hoarding cash, we show that the bank's capital buffer can be arbitrarily small. Furthermore, capital hoarding guarantees truth-telling on the part of entrepreneurs through a process that resembles the iterative deletion of strictly dominated strategies proposed by Bergemann and Morris (2009). As a consequence, truth-telling is the only rationalizable strategy for the entrepreneurs and, as Bergemann and Morris (2009) call it, implementation is robust.

The second way for the bank to secure the initial necessary resources is by contracting with the entrepreneur through a debt contract coupled with a properly designed forgiveness clause. We show that no capital needs to be put aside to implement this solution, so it is less costly than the first one (in fact, it involves no cost whatsoever). The adding of a forgiveness clause has a drawback, however, in that the strategy adopted by each individual entrepreneur now depends on his correct beliefs about others' default intentions, so robustness in entrepreneurs' decision-making process is compromised.

The rest of this paper is structured as follows: We review the related literature in Section 2, lay down the model and establish the existence of coordinated strategic defaults in costly state verification models in Section 3. Section 4 introduces the sequential audit strategy and presents two possible solutions to the problem of bad equilibria. Section 5 discusses the validity of our results under alternative modeling assumptions and Section 6 s concludes.

2.1 Related Literature

This work draws on a diverse array of papers. We extend the costly state verification environment developed in Townsend (1979) and Gale and Hellwig (1985) to the case in which a single financial intermediary lending to a continuum of entrepreneurs has a limited monitoring capacity due to budgetary issues. Like Diamond (1984), we show that delegation reduces the costs of monitoring a set of fully diversified loans, but in our paper it also exposes the bank to the possibility of coordinated strategic defaults.

Other papers also consider the existence of runs on the asset side of a financial intermediary (Vlahu (2008), Bond and Rai (2009)). However, they differ with ours on many accounts. For example, Vlahu (2008) and Bond and Rai (2009) adopt a global games framework and prove multiplicity of equilibria in repayment behavior, but do not derive optimal financial contracts. In Bond and Rai (2009), repayment incentives stem from the prospect of receiving future credit. By contrast, our results do not rely on intertemporal incentives, but on whether bankruptcy procedures are such that a bank must have a minimum of resources to collect its loans.[5]

An extensive theoretical literature focuses on strategic complementarity as a source of multiplicity of equilibria. In Farhi and Tirole (2012), the inability of authorities to commit not to bail out financial institutions after the realization of a negative shock creates, ex-ante, complementarities in their choice of leverage. On the other hand, in our paper, entrepreneurs' actions are complements because of the bank's potential inability to monitor all the projects it finances. We share with Silva and Kahn (1993), Basseto and Phelan (2008), and Bond and Hagerty (2010) the idea that bad equilibria may result from the principal's limited resources to discipline agents. Silva and Kahn (1993) examine the optimal provision of a public good for which exclusion is possible, but imperfect. They show that, if a sufficient number of agents in the economy free-ride the public good, then it is desirable to free-ride as well, because the probability of being caught and punished is low. In a crime prevention setting, Bond and Hagerty (2010) analyze how the optimal punishment intensity varies along the various existing equilibria, without addressing implementation issues. Basseto and Phelan (2008) study optimal taxation and show that, when the tax authority can only audit a fixed proportion of households due to a budget constraint, the optimal mechanism also has equilibria in which households misreport. In our paper,

[5]It must be said that we have learned a great deal about the functioning of microfinance institutions from Bond and Rai (2009) and their article served as a useful guide to many episodes of coordinated strategic defaults.

the amount of resources the bank has for auditing purposes is endogenous, as proceeds collected from entrepreneurs who pay up can be used to audit other entrepreneurs. This feature, which is only present in our work, plays a crucial role in all solutions we propose to eliminate bad equilibria.[6]

Many other papers propose modifications to the standard costly state verification model of Townsend (1985) and Gale and Hellwig (1985). Border and Sobel (1987) and Mookherjee and Png (1989) formally consider the possibility of random audits; Krasa and Vilamil (2000) analyze the case in which audits must be sequentially rational; Lacker and Weinberg (1989) propose a model in which the agent can fabricate cash-flows; and Winton (1995) considers a single entrepreneur contracting with many investors that have different degrees of seniority. None of these papers address the implications of the principal's limited resources on its ability to audit, and the resulting incentives to default.

Finally, our paper also relates to the mechanism design literature regarding full and robust implementation. The mechanism designer obtains full implementation when agents' behaviors lead to the desired outcome in *every* equilibria of the game induced by the mechanism. Our analysis shows that, with a continuum of entrepreneurs (and a symmetric audit strategy), a standard debt contract only *partially* implements the desired allocation. Another relevant issue is robustness, a measure of the complexity of the agents' decision process. Our sequential audit strategy induces truth-telling from entrepreneurs in a way that is related to the ideas of robust implementation put forth in Bergemann and Morris (2009).

2.2 The Model

We extend the model of Gale and Hellwig (1985) in two directions. First we assume that a unique investor lends to a continuum of entrepreneurs. Subsequently, we introduce a limit to the investor's monitoring capacity and analyze how entrepreneurs' repayment incentives are affected.

2.2.1 A Continuum of Entrepreneurs

There are two periods. At date 0, each of a continuum of identical entrepreneurs is endowed with a production technology which requires an initial investment of $I > 0$. Entrepreneurs have no wealth of their own, so they must borrow from a wealthy investor to undertake the project. Projects

[6]There is also empirical evidence supporting the existence of strategic complementarities in borrowers' default decisions. Using survey data on US households, Guiso et al. (2012) document that an agent who is acquainted with someone who has defaulted strategically is more likely to declare his intention to default strategically as well.

are risky and returns are i.i.d. across entrepreneurs, so the possibility of diversification makes it optimal for a unique agent to assume the role of a financial intermediary, as in Diamond and (1984). This agent, who is delegated the task of monitoring the credit it extends, will be called the *bank* throughout.

At date 1 production is realized yielding a total output of $f(s)$ when the state s is realized. We assume that $f(0) = 0$ and $\partial f(s)/\partial s > 0$, so that states are ordered with higher states implying higher returns. The probability distribution of the states is given by an absolutely continuous cumulative distribution function H, with density h and support in a compact interval $[0, \overline{s}]$. Capital markets are perfectly competitive so the bank's expected profit is zero. Without loss of generality, the interest rate is normalized to zero and the mass of entrepreneurs normalized to 1.

We adopt the standard assumptions of the costly state verification model regarding the asymmetry of information between entrepreneurs and the bank. More specifically, at date 1 each entrepreneur observes the return of his own project free of charge, however, an audit cost of $c(s)$ must be borne for the bank to become informed, where $\partial c(s)/\partial s \geq 0$.[7] The audit is usually interpreted as the process of determining an inventory of the debtor's assets and liabilities, such as a bankruptcy procedure.

We follow Gale and Hellwig (1985) and assume that, if the entrepreneur defaults on his loan, the bank can impose on him a constant non pecuniary cost of c_0. We do not require that the bank audit a given entrepreneur's project for the imposition of the non-pecuniary cost to be possible. This penalty can be interpreted in a number of ways, one being as the bank's decision to inform a credit bureau of the entrepreneur's failure to comply with the agreed upon financial contract.

We now address the problem of establishing the optimal contractual arrangement between the parties. Initially, we follow the standard mechanism design approach and search for a Bayesian game that, while providing entrepreneurs with incentives for truthful reporting, has among its possibly many equilibria one that minimizes total expected audit costs. We also require that the bank break-even in expectation. This approach implicitly assumes that, when multiple equilibria are present, the desired equilibrium is chosen.

When signing a contract, a given entrepreneur and the bank must agree upon several issues. The first is on the audit region $B \subseteq [0, \overline{s}]$, which determines when the bank pays the observation costs. We initially restrict attention to

[7]The existence of strategic defaults does not depend on the assumption that costs are weakly increasing in s. In particular, in Appendix B we argue that the existence of coordinated defaults continue to exist even under more general cost structures, as long as $c(s) > 0$ for all s.

deterministic audits on the part of the principal and analyze the case of random audits in Section 5. With a slight abuse of notation, let $B(s)$ be an indicator function defined in $[0, \overline{s}]$ and taking value 1 at states where audits occur and 0 otherwise. The second issue that must be agreed upon is on how parties share at each state the project's return net of observation costs, namely $f(s) - B(s)c(s)$. Let $R_b(s)$ and $R_e(s)$ be the return to the bank and the entrepreneur respectively when the state is s.

A contract can be represented by an array (R_b, R_e, B). An optimal contract provides the entrepreneur with incentives for truthful reporting, while minimizing expected audit costs. One important feature of the optimal contract directly follows from the stated assumptions. In an optimal contract, the entrepreneur is fully expropriated when found out to have misreported after an audit. As opposed to partial expropriation, full expropriation is optimal because it loosens the bank's budget constraint, thus allowing for a reduction in the audit region.

An optimal contract is incentive compatible if only if:

(i) there exists a constant D such that $R_b(s) = D$ whenever $B(s) = 0$;

(ii) for any states $s, \widehat{s}$ such that $B(s) = 1$ and $B(\widehat{s}) = 0,$ we have $D \geq R_b(s) + c(s)$.

Condition (i) specifies a constant repayment schedule for the entrepreneur in the no-audit region, while condition (ii) guarantees that it is never in the entrepreneur's interest to report a non-audit state, when the true state specifies that an audit be realized.[8] Using the above characterization of the set of Incentive Compatible contracts, Gale and Hellwig (1985) show that in the case of a single entrepreneur-investor pairing, the optimal contract takes the form of what they call a *standard debt contract*, which is characterized by:

$$B = \begin{cases} 0 \text{ if } f \geq D \\ 1 \text{ if } f < D \end{cases} \qquad R_b = \begin{cases} D \text{ if } B = 0 \\ f - c \text{ if } B = 1 \end{cases} \tag{2-1}$$

In the above characterization, D is the face value of debt. According to the standard debt contract, when the entrepreneur fails to repay D he is instantly audited and fully expropriated. The face value of debt is chosen so as to guarantee that the bank breaks even in expectation.

[9] Let s^D be such that $f(s^D) = D$, then D is implicitly defined by

[8]Irrespective of who pays for the audit costs, the entrepreneur bears these costs in equilibrium because the bank must break-even.

[9]We restrict the analysis to the interesting situation in which $c_0 \leq D$, so that costly audits must be realized to provide incentives for the entrepreneur to pay off his debt.

$$D(1 - H(s^D)) + \int_0^{s^D} [f(s) - c(s)]h(s)ds = I. \qquad (2\text{-}2)$$

In our setting, there is a continuum of entrepreneurs with i.i.d. projects and by the law of large numbers the bank knows the realized aggregate return. Hence, conditional on *all* entrepreneurs being truthful, it is as if the bank were dealing with a single representative borrower. Therefore, the following proposition holds:

Proposition 12 *When a single bank lends to a continuum of entrepreneurs with i.i.d. projects, the standard debt contract of Gale and Hellwig (1985) is an optimal contract.*

Proposition 1 shows that the results of Gale and Hellwig (1985) remain unaltered if we assume that a single agent lends to a continuum of entrepreneurs. The standard debt contract still provides each entrepreneur with incentives for truthful reporting while minimizing the bank's expected aggregate audit costs.

2.2.2 The Bank's Budget Constraint

We have been purposely silent as to whom - the bank or the entrepreneur - actually pays for the audit costs, but in the analysis that follows we assume explicitly that the bank must pay for these costs entirely.[10] More specifically, we make the following assumption:

Assumption 13 *Before an audit is realized, the bank must pay entirely for its costs. The bank can either set aside capital at* $t = 0$ *or use the proceeds collected from creditworthy entrepreneurs at* $t = 1$ *to pay for audits. Furthermore, conditional on having the necessary resources, the bank can credibly commit to audit entrepreneurs in default.*

In theory, the first part of Assumption 13 regarding who pays the audit costs is irrelevant, provided that the necessary resources are secured at the time the audit is to take place; since the bank only breaks even, audit costs are ultimately borne by the entrepreneur in equilibrium. However, the assumption of an unlimited budget to cover audit costs does not seem realistic and we wish to explore the implications of parting with it.[11]

[10]This assumption is starker than necessary and made for expositional convenience. All that is needed is that the bank incurs in any positive fraction of total costs of the audit.

[11]The case we study - where the creditor bears the costs of collecting loans - might be specially applicable to countries with less developed bankruptcy laws, as was Mexico in the early nineties (Tornell and Krueger (1999), Luna-Martinez (2000)) or some developing countries today. For example, Murdoch (1999) documents instances in which microfinance programs rely on the posting of collateral to grant credit, despite the absence of institutions that guarantee repossession by judicial means.

We will do so in the following section.

The second part of Assumption 13 regards the principal's commitment capabilities conditional on the availability of resources. It is instructive to be explicit about how our model compares to other papers in this regard. For example, Townsend (1979) and Gale and Hellwig (1985) assume the principal can fully commit to audit entrepreneurs in default, even if it is not optimal to spend resources on audits after the agent has revealed the true return from his project. By contrast, Krasa and Vilamil (2000), in a variant of the costly state verification environment, impose the restriction that audits be sequentially rational (i.e. the principal has zero commitment capacity). Our paper is thus an intermediate case; the principal can fully commit to use his resources, as long as they are available at $t = 1$.[12]

From now on, we incorporate Assumption 13 to the baseline model and analyze how debtors' repayment behavior changes. If entrepreneurs anticipate that audit resources are insufficient, the bank will be unable to provide them with repayment incentives. This situation can be amplified by opportunism on the part of debtors, since they benefit from actions that hurt the lender's financial capability and make the collection of loans less likely.

With the introduction of the bank's budget constraint, we must be explicit as to how the bank's resources evolve over time. Timing is as follows. At date 0, the bank chooses an initial capital level E, which is common knowledge among every agent in the economy. We assume that equity capital is costly, so the bank chooses the lowest level of capital in accordance with equilibrium behavior. The bank then signs a standard debt contract with all entrepreneurs, with face value D. Conditional on having the necessary resources, the bank fully commits to audit entrepreneurs in default. We will complete our description of the bank's audit strategy in a moment.

At date 1 each entrepreneur instantly observes his project's return and chooses whether to repay the loan. Let Λ denote the set of entrepreneurs who repay D to the bank and Λ^c the set of entrepreneurs who do not pay and declare default.[13] Entrepreneurs in default are either unable to repay their debt if project returns are lower than D, or are unwilling to do so. In the latter case, they default strategically.

[12]We assume the bank cannot raises resources at the time audits begin. This assumption may be justified on several grounds. For example, according to Diamond Rajan (2001) relationship lenders might be unable to raise resources when facing a liquidity shock, because they often lack the capacity to commit to use their specific ability to collect loans. Alternatively, the issue of new securities might be precluded given the existence of debt overhang on the part of the bank (Myers (1977)) or asymmetry of information between inside management and outside investors (Myers and Majluf (1984)).

[13]More specifically, in a direct mechanism entrepreneurs in Λ report a state $\hat{s}$ that prescribes payment of D, while entrepreneurs in Λ^c report state $\hat{s}$ in the audit region.

After observing aggregate repayment behavior, the bank decides which projects to audit, subject to its budget constraint. The bank can only use its capital E and the proceeds from creditworthy entrepreneurs, given by $D \int_{\Lambda} dH$, to pay for audit costs. Because the bank signs with each individual entrepreneur a standard debt contract, it only audits entrepreneurs who do not pay D. Remember that we have also assumed that the bank always audits a project in default if it has the necessary resources to do so. The only remaining question then is how the bank conducts audits when its budget constraint is binding. When this occurs, we assume the bank randomly chooses among projects in default until its resources are exhausted. This strategy can be argued to be rather arbitrary, but it seems to us as the most natural extension of the standard audit strategy of the costly state verification model to the case where there is a continuum of entrepreneurs.

We also adopt the following assumption:

Assumption 14 *The equilibrium notional value of debt D is such that the following inequality holds:*

$$D\left(1 - H\left(s^D\right)\right) > \int_0^{s^D} c\,(s)\,h\,(s)\,ds. \tag{2-3}$$

Assumption 1 guarantees that the amount the bank collects when all entrepreneurs report truthfully is more than enough to cover its audit costs. This appears to be a sensible assumption provided that the bank is willing to extend the standard debt contract to all entrepreneurs. In fact, using the bank's budget constraint given by equation (2-2), Assumption 1 is equivalent to

$$\int_0^{s^D} f(s)h(s)ds < I, \tag{2-4}$$

which states that the mean return in default states is insufficient to cover the initial investment.

2.2.3 Equilibrium

We now analyze the set of equilibria of the game induced by the standard debt contract and the bank's audit strategy. First, we characterize behavior on the part of entrepreneurs.

Proposition 15 *When confronted with a probability p of audit and a debt level D, there is a cutoff state s^* such that entrepreneur i declares default if and only if $s_i \leq s^*$.*

The intuition for the above result is clear. As the realized return of the project increases, so does the cost to the entrepreneur of being fully expropriated if found out to have reported untruthfully.

We now analyze how the probability p of each entrepreneur being audited is affected by the entrepreneurs' reports, given that the bank has fully committed to the audit strategy previously described. Suppose there's a cutoff state s^* such that every entrepreneur with $s_i \leq s^*$ declares default, while the remaining entrepreneurs pay off their debt. The bank collects $E+D(1-H(s^*))$ in resources to audit a total of $H(s^*)$ projects. Resources used to audit non-performing loans can either come from capital hoarded at the initial period, or repayments from performing loans.

If $E + D(1 - H(s^*))$ is greater than total audit costs, given by $\int_0^{s^*} c(s)h(s)ds$, then all entrepreneurs in default are audited. On the other hand, if the bank does not collect enough resources from the creditworthy entrepreneurs to audit all projects in default, then it randomly chooses which projects to audit until it runs out of cash. The audit probability faced by each entrepreneur is given by

$$p(E, s^*) = \min \left\{ \frac{E + D(1 - H(s^*))}{\int_0^{s^*} c(s)h(s)ds}, 1 \right\}. \tag{2-5}$$

We are now ready to define an equilibrium of the entrepreneur game induced by the mechanism.

Definition 16 *For a fixed capital buffer E, a repayment equilibrium is given by an ordered pair $\mathcal{E} = (s^*, p)$ such that:*

(i) $p = p(E, s^*)$ as in Equation (2-5);

(ii) entrepreneur i defaults if and only if $s_i \leq s^*$.

The definition of a repayment equilibrium implies that entrepreneurs form beliefs about the cutoff state s^* and then choose actions that maximize profits given these beliefs. In addition, beliefs are correct in equilibrium.

We also have the following proposition:

Proposition 17 *Under Assumptions 13 and 14, for any given level of capital E chosen by the bank ex-ante (in particular, $E = 0$), the standard debt contract derived in Proposition 12 coupled with the symmetric audit strategy has a truth-telling equilibrium given by $\mathcal{E} = (s^D, 1)$.*

Initially, it seems that nothing is changed by the introduction of the bank's budget constraint, since in a truth-telling equilibrium the bank can set $E = 0$ at $t = 0$. When entrepreneurs tell the truth, the bank secures sufficient resources from creditworthy entrepreneurs to audit those who declare default and the budget constraint implicit in Assumption 13 is slack. The standard debt contract coupled with a symmetric audit strategy is a mechanism that at least partially implements the desired allocation.

However, while truth-telling is *one* possible equilibrium of the mechanism induced by the standard debt contract and the bank's audit strategy, it is not unique when the bank's capital level is below a certain threshold. Entrepreneurs can default strategically, impairing the bank's financial position and auditing capability. This is formally stated in the following result:

Proposition 18 *Consider the standard debt contract coupled with the symmetric audit strategy. Under Assumptions 13 and 14 and if* $E < E_1 \equiv \frac{(D-c_0)\int_0^{\overline{s}} c(s)h(s)ds}{f(\overline{s})}$*:*

(i) there is a fully coordinated default equilibrium, characterized by all entrepreneurs declaring default, that is $s^* = \overline{s}$*;*

(ii) there is a partially coordinated default equilibrium, characterized by a threshold $s^* \in (s^D, \overline{s})$*.*

The threshold capital level E_1 is decreasing in c_0, the non-pecuniary penalty that the bank can impose on entrepreneurs. The reason is simple: keeping fixed all other factors that affect a borrower's decision to repay, a larger non-pecuniary penalty reduces a borrower's incentives to default. Hence, as c_0 increases, the bank can lower the audit probability - and thus the capital set aside ex-ante - while still maintaining repayment incentives intact.

Jointly, Propositions 17 and 18 highlight how entrepreneurs' repayment incentives are affected by the bank's health. The results seem to be in line with the realities of microfinance programs in particular. For example, according to van Maanen (2004) *'If the (repayment) percentage sinks below - say 90% - a growing percentage of the clients is tempted to join the 10% that seems to get away with non-payment. Once the percentage sinks below 80% it is very difficult to reverse that trend, because the virus travels faster than any medicine: 'Why should I repay to a MFI (microfinance institution) that is likely to go down? Let's wait and see what happens!"*

Proposition 18 also sheds some light on the potential limits of delegated monitoring. Diamond (1984) shows that, when financial intermediaries are fully diversified, delegation costs - given by the costs of providing the proper incentives to intermediaries, as opposed to entrepreneurs - are zero. In Diamond (1984), because the financial intermediary is fully diversified, the Law of Large Number eliminates the informational advantage that it may have over its depositors as a result of directly observing project returns. In this paper, however, the bank can always expropriate its depositors by claiming that it has suffered a full coordinated default, even if entrepreneurs report truthfully. Therefore, any potential benefit of delegated monitoring must be weighted against the costs of providing the intermediary with the proper incentives.

2.3 A general solution to rule out bad equilibria

In this section we show how the bank can prevent entrepreneurs from coordinating on an undesirable equilibrium by adopting what we call a *sequential audit strategy*. This strategy can be implemented by the bank if, once auditing is to take place, it has any strictly positive level of resources. We first describe the sequential audit strategy assuming that an amount δ in capital can be secured. Subsequently, we show how the bank can secure this positive amount.[14]

2.3.1 The Sequential Audit Solution

Suppose that, at the moment of the signing of the contract, the bank divides entrepreneurs into $1/\varepsilon$ groups of mass ε, where

$$\varepsilon = \frac{\delta}{\int_0^{\bar{s}} c(s,k)h(s)ds}. \tag{2-6}$$

Groups are then randomly ordered as $(g_1, g_2, ..., g_{1/\varepsilon})$ and this order is common knowledge among bank and entrepreneurs. Note that $\varepsilon \int_0^{\bar{s}} c(s,k)h(s)ds$ is exactly the amount of resources that the bank must have to audit all entrepreneurs belonging to one given group, regardless of their reporting strategy.

Once the bank starts auditing, it does so sequentially, auditing entrepreneurs in group $n+1$ only after it has audited all defaulted projects in group n. Hence, even though entrepreneurs are ex-ante identical, the bank treats them asymmetrically when adopting a sequential audit strategy.

With δ in capital, the bank can credibly commit to audit all entrepreneurs in g_1. Therefore the first group never defaults strategically, since doing so would automatically trigger an audit and the seizure of the entire project's return. We now show that if entrepreneurs in group n report truthfully, the bank collects enough resources to commit to audit all entrepreneurs belonging to group $n+1$ that have declared default.

Note that truthful reporting by a given group increases the bank's audit resources by

[14]Note that we propose a modification to the symmetric audit strategy but keep the financial contract (standard debt contract) unaltered. What if the bank held, as opposed to debt, an alternative security issued against future proceeds from project returns? In the costly state verification environment that we study (Gale and Hellwig (1985), Townsend (1979)), this would be of no help. For example, by holding equity the bank would need to monitor the entrepreneur in all states of the world, rather than just in a subset of states as with debt. This would magnify the bank's exposure to a coordinated default by stretching its limited budget even more. Debt is optimal because it is the security that minimizes the bank's monitoring costs (and, consequently, its exposure to coordinated defaults).

$$\varepsilon\left[(1-H(s^D))D-\int_0^{s^D}c(s,k)h(s)ds\right]. \tag{2-7}$$

The first term inside the brackets is the amount raised from the creditworthy entrepreneurs. The second term is the total cost incurred when auditing projects in default. As a consequence of Assumption 14, the whole expression is positive. Therefore, if the bank has enough resources to provide incentives for truthful reporting from entrepreneurs in group n, truthful reporting in group $n+1$ is also assured. After all, the bank's financial strength is only improving and the pool of projects potentially subject to audit is decreasing. As this argument holds for an arbitrary n, no group will default strategically. The following proposition formalizes this argument.

Proposition 19 *For any $\delta > 0$, if the bank can secure δ of capital to audit and uses a sequential audit strategy, then truth-telling is the unique equilibrium of the standard debt contract. Moreover, truthful reporting is obtained through the process of iterative deletion of strictly dominated strategies.*

The intuition behind this simple solution is the following. Coordinated defaults exist because of the strategic complementarity between entrepreneurs' actions. The sequential audit strategy is successful precisely because it breaks this strategic complementarity. The default by a given group of entrepreneurs does not affect the probability of those in g_1 being audited. An entrepreneur in this group who declares default will be audited for sure, regardless of the other entrepreneurs' decisions. Therefore, his incentives to default are not affected by the default of others. Once g_n pays up, incentives for truth-telling by g_{n+1} are guaranteed and coordinated defaults unravel.

Furthermore, the unique equilibrium under a sequential audit strategy is robust (Bergemann and Morris (2009)); it is the only strategy that survives a procedure of iterated deletion of strictly dominated strategies. To see this, note that the sequential audit strategy guarantees that truth-telling is dominant for entrepreneurs in g_1. Entrepreneurs in g_2, aware that those in g_1 will not lie, will also find it dominant to report truthfully and so on.

In many instances, a bank naturally collects its non-performing loans in a given order. For example, a bank may prefer to begin auditing entrepreneurs who are geographically closer to its headquarters or are listed in jurisdictions with creditor friendly bankruptcy courts. Alternatively, audit costs might be reduced for those entrepreneurs with whom the bank has done business for a longer period of time, which would justify placing them first in line. In any case, these forms of heterogeneity among entrepreneurs may serve as an implicit ordering device, which in equilibrium may help prevent coordinated strategic defaults.

In the next sections, we will point out a few ways by which the bank can obtain the amount δ of Proposition 19.

2.3.2 Partially Coordinated Defaults

When *partially coordinated defaults* arise, the bank is able to raise resources form a group of creditworthy entrepreneurs who have chosen to report truthfully. Formally, suppose that all partially coordinated default equilibria are given by $\{(p_1, s_1^*)..., (p_K, s_K^*)\}$.[15] Let (p, s^{MAX}) be the one with the largest s_k^*.

The bank can apply the sequential audit strategy to prevent partially coordinated defaults, taking

$$\delta = (1 - H(s^{MAX}))D. \tag{2-8}$$

This is the least amount of capital that the bank will raise in any of the partially coordinated defaults. The group size ε will then be chosen according to equation (2-6). Hence, the sequential audit solution is capable of eliminating all partially coordinated defaults, without any loss in efficiency since there is no need to hoard costly capital at date 0. This solution has its shortcomings, as it is ineffective in eliminating the fully coordinated default equilibrium. Indeed, if all entrepreneurs default, the bank will not collect the needed amount of resources to perform the sequential audit.

2.3.3 Solutions for Fully Coordinated Defaults

Positive Capital

The bank can guarantee the necessary audit resources by forming a capital cushion at the financing stage. More specifically, suppose that at date 0 the bank publicly announces that it is hoarding an amount of δ in capital, that is to be invested in risk-free securities.[16]

This capital cushion, together with the sequential audit strategy, creates a mechanism that is (robust) incentive compatible. Furthermore, the inefficiency which arises from hoarding capital can be made arbitrarily small.

[15] There can be either a finite number of equilibria featuring partially coordinated defaults or an infinite number of them depending on whether 0 is a regular value of the function Γ defined in the Appendix. If 0 is a regular value then the number of equilibria with partially coordinated defaults is finite and odd. We focus on regular equilibria, which are robust to small perturbations of the set of parameters.

[16] The bank must indeed invest in risk-free securities to eliminate the possibility that an adverse shock to its securities portfolio reduces its ability to pay for the audit costs at $t = 1$. We are also assuming that the bank can costlessly and credibly disclose to entrepreneurs the amount of capital it has hoarded and its riskiness. We thank an anonymous referee for pointing this out.

Debt Forgiveness

In this section, we study an alternative form the bank can raise the necessary capital to implement the sequential audit solution, which involves granting debt forgiveness to a group of entrepreneurs. In the analysis that follows, we explicitly explore the fact that bad equilibria exist because entrepreneurs form beliefs that others will default strategically and these beliefs are correct in equilibrium.

We assume that at date 0 the bank randomly chooses a group Δ of mass δ of entrepreneurs and subsequently divides the remaining entrepreneurs in groups of size of at most ε. Entrepreneurs in Δ^c contract with the bank through the standard debt contract, whereas entrepreneurs in Δ sign a contract that is altered as follows. After the realization of the projects' returns, but before any payments are made, each entrepreneur in Δ is required to report a flag $f_i \in \{s^D, \overline{s}\}$ to the bank. This flag represents each entrepreneurs' belief about the behavior of the entrepreneurs in Δ^c. If every entrepreneur in Δ^c declares default, then those who reported $f_i = \overline{s}$, receive debt forgiveness of $D - c_0$. On the other hand, if a group of positive mass in Δ^c honors their debt, entrepreneurs in Δ who reported $f_i = \overline{s}$ are audited and fully expropriated.

Under this contract, when every entrepreneur in Δ^c declares default, entrepreneurs in Δ are indifferent between joining the coordinated default *and* suffering the non pecuniary penalty of c_0, or paying the bank that same amount. We suppose that when confronted with this situation, entrepreneurs in Δ who have at least c_0 always choose to pay the bank. As a result, the bank collects $\delta(1 - H(s^c))c_0$, where s^c is such that $f(s^c) = c_0$. Once the bank has raised this amount of capital, it can proceed with the sequential audits, provided that ε is chosen appropriately. Under this agreement, truth-telling by every entrepreneur is the unique repayment equilibrium and debt forgiveness only occurs off equilibrium.

Since no capital needs to be put aside for its implementation, the solution with debt forgiveness is less costly than hoarding capital ex-ante. However, when the bank uses the debt forgiveness solution, the strategy each entrepreneur adopts depends on his beliefs regarding other entrepreneurs' strategies. In this respect, it requires a complex decision making process on the part of entrepreneurs.

The non-pecuniary penalty plays a crucial role in the sequential audit solution with debt-forgiveness. Because the bank can impose this cost whatever the entrepreneurs' repayment decisions, it can always raise a positive amount of money through ex-post bargaining by monetizing the non-pecuniary penalty. This in turn, guarantees that, were entrepreneurs in Δ^c to coordinate on a strategic default, the bank would be able to raise a strictly positive amount of resources to kick-start the sequential audit strategy.

2.4 Robustness of Coordinated Strategic Defaults

In this section, we discuss the robustness of our previous results by modifying some of our modeling assumptions. We show that little is substantially changed if we assume that the bank only lends to a finite number of entrepreneurs or if we allow the bank to adopt more general mechanisms. We deal with these extensions one at a time.[17]

2.4.1 Finite Number of Creditors

Assume now that the bank lends to n entrepreneurs, where $n < \infty$. We keep the remaining features of the model unchanged. In particular, at $t = 0$ the bank can still credibly commit to use all it's available resources to audit entrepreneurs who eventually declare default.

Proposition 20 *Consider the standard debt contract coupled with the symmetric audit strategy. For each n there are two threshold levels of capital $E_0(n)$ and $E_1(n)$ such that:*

(i) if $E < E_0(n)$ there is no truth-telling equilibrium;

(ii) if $E \in (E_0(n), E_1(n))$, apart from the truth-telling equilibrium, there is a fully and at least one partially coordinated default equilibrium.

Proposition 20 is the analog of Proposition 18 to the case where the bank lends to a finite number of entrepreneurs, therefore we limit the following discussion to highlighting how both propositions differ. When $n < \infty$, an additional situation must be taken into account; if capital is below a threshold given by $E_0(n)$ then the standard debt contract does not provide incentives for truthful reporting. For these low capital levels an entrepreneur prefers, irrespective of other entrepreneurs' strategies, to occasionally default strategically. Because the bank is poorly diversified and many projects may simultaneously go sour, a given entrepreneur is still better off by misreporting in some states, even if he believes that other entrepreneurs always report truthfully.

The following proposition shows how threshold levels $E_0(n)$ and $E_1(n)$ vary with n.

Proposition 21 *When $n \to \infty$, $E_0(n) \to 0$ and $E_1(n) \to E_1$, where $E_1 = \frac{(D-c_0)\int_0^{\overline{s}} c(s)h(s)ds}{f(\overline{s})}$ is as in Proposition 18.*

[17]We gratefully acknowledge both referees for suggesting that we explore the issues presented in this section.

To capture the intuition behind the result of Proposition 21, consider the starkest case which occurs when $n = 1$. When lending to a single entrepreneur, the bank's capacity to engage in cross-subsidization once audits are to begin - using resources from creditworthy entrepreneurs to audit those in default - is eliminated. The bank must thus hoard enough capital ex-ante to guarantee the feasibility of any eventual audit. If the bank hoards insufficient capital, the single entrepreneur defaults strategically on his loan.

For $n > 1$, the bank may eventually raise resources at $t = 1$ from entrepreneurs who repay their loan. As n increases, the bank becomes more diversified and the fraction of projects that fail approaches their ex-ante probability of failure. For large n and if entrepreneurs report truthfully, Assumption 14 guarantees that the bank is likely to raises from creditworthy entrepreneurs the necessary resources to audit those in default. Any given entrepreneur is thus incentivized to tell the truth (even for a low E) provided that he believes other entrepreneurs will do the same. Truth-telling is therefore an equilibrium of the standard debt contract with symmetric audits. Proposition 21 shows that the case where there is a continuum of entrepreneurs serves as a good approximation to the case where n is finite but large.

Once again when capital is below a threshold, apart from the truth-telling equilibrium, coordinated default equilibria exist as well. The bank can employ the sequential audit strategy to reduce the amount of capital hoarded to provide incentives for truthful reporting. More specifically, irrespective of n, the bank must hoard sufficient capital to audit only one entrepreneur, precisely the one that was placed first in line in the sequential audit strategy.

2.4.2 General Mechanisms

So far, we have restricted the analysis to the case where the principal only uses deterministic mechanisms. This case is of special interest, since it is consistent with many features observed in financial markets (e.g. debt contracts and bankruptcy procedures). However, because the costly state verification environment has also been applied to the study of insurance and taxation - where stochastic audits are pervasive in real life - it is interesting to establish the validity of our results when the principal adopts more general mechanisms, in particular when he randomizes audits.[18]

When analyzing general mechanisms, we maintain Assumption 13 that introduces the principal's budget constraint into the mechanism design prob-

[18]Mookherjee and Png (1989) and Border Sobel (1987) study optimal mechanisms in costly state verification environments when random audits are possible.

lem that we study. Therefore the principal must still secure beforehand the resources spent in audits.

In the Appendix, we prove the following proposition:

Proposition 22 *Suppose that the principal adopts a stochastic mechanism coupled with a symmetric audit strategy. There exists a threshold capital E_S such that, if the bank sets a capital buffer $E \leq E_S$ at $t = 0$, then the stochastic mechanism also has a fully and at least one partially coordinated default equilibrium. Furthermore, the sequential audit strategy fully implements the desired equilibrium.*

Proposition 22 establishes that our previous results are unaltered under more general mechanisms. The following intuition lies behind the existence of multiple equilibria even for general mechanisms. In a costly state verification environment, audits followed by a threat of (at least partial) expropriation of the agents returns' are the only disciplining device available to the principal. In particular, in the absence of audits an agent always reports that the return from his project is 0. Therefore truthful reporting only occurs if the bank commits to audit entrepreneurs with a positive probability at a set of states with positive measure. Because audits are costly, the principal must guarantee that he raises the necessary resources to realize them. If the principal can incentivize entrepreneurs to report truthfully, then creditworthy entrepreneurs provide the principal with the resources to pay for audits. Nevertheless, once the principal sets its capital buffer at $t = 0$ and commits to a symmetric audit strategy, agents play a game of strategic complementarity. It becomes more attractive for one agent to report an audit state when other agent's are doing the same. For sufficiently low capital buffers, all agents prefer to misreport.

2.5 Conclusion

In this paper, we revisit one of the most influential models of financial contracting, the costly state verification model first developed in Townsend (1979) and Gale and Hellwig (1985). We extend their analysis to the case of multiple borrowers and show that when a bank's resources to monitor projects are bounded, financial intermediation can lead to the existence of multiple equilibria in repayment behavior. In some of these equilibria, borrowers default because they expect other borrowers to do the same.

As opposed to what has been extensively analyzed in the academic literature, we study a bank run originating in the bank's asset side, rather than from its funding structure. The analysis suggests that coordinated strategic defaults are yet another source of financial fragility in the sense that small shocks have large effects (Allen and Gale (2000, 2004)). [19]

We show that to prevent bad equilibria a bank needs to break the strategic complementarities in borrowers' default decisions, which can be done through the adoption of a sequential audit strategy.

While cast in terms of financial intermediation, the ideas we put forth in this paper can be applied to other settings in which a large number of agents have to be monitored or audited. One example is the deterrence of crime waves by a police force who faces a large population of criminals. Other examples that come to mind include the problem of a governmental agency that has to rely on income reports of individual tax payers and a CEO who relies on the reports about the profitability of a company's divisions by managers who can engage in self-dealing.

[19]We follow Allen and Gale (2004) in claiming that sunspot equilibria, where endogenous variables are influenced by variables that have no effect on fundamentals, constitute an extreme form of financial fragility.

3 Dynamic Multitask Moral-Hazard

We use the principal-agent model developed in Sannikov (2008) to examine an environment in which a principal (she) hires an agent (he) protected by limited-liability to perform multiple tasks. Several implications for optimal job design (i.e. the optimal grouping of tasks into a unique job) emerge. The most important insight is that, regardless of the agent's production technology, incentive problems in the contractual relationship lead to economies of scope.

The rationale for the principal to optimally delegate multiple tasks to a unique manager is the following: In the environment proposed by Sannikov (2008), a unique state variable representing the agent's total expected future compensation from the proposed contract is responsible for providing him with incentives. When the principal observes poor outcomes, an incentive scheme that induces the agent to exert costly effort must reduce his expected future compensation. If a sequence of bad output realizations drives this state variable to zero, the agent's limited liability constraint prevents the principal from making the agent's compensation contingent on bad outcomes, compromising the provision of incentives. The principal must then fire the agent resulting in the inefficient termination of the contractual relationship between both parties.

However, if an agent performs multiple tasks, the principal's ability to punish the agent is expanded, as she may now engage in cross-subsidization of the agent's continuation values among multiple tasks. The combination of tasks into a unique job therefore relaxes the limited-liability constraint faced by the agent and reduces inefficient termination of the contractual relationship. Intuitively, managers with no wealth of their own pay, with the good prospects arising from the successful performance of one given activity, for the poor performance of other activities that fall under his responsibilities. Economies of scope arise independently of complementarities in the agent's actions when performing multiple tasks or the improvement of the informativeness of the performance measure. Instead, it is the cross subsidization between activities that is beneficial.

The results that we obtain contrasts with those provided by other papers studying contractual relationships characterized by multi-task moral-hazard such as Holmstrom and Milgrom (1991) and Laux (2001). Holmstrom and Milgrom (1991) analyze the case where the agent has exponential utility with monetary cost of effort and show that optimal contracts are linear in output. They argue that, under independent production technologies, an arrangement with independent commissions is the most natural. Our analysis adopts a different set of assumptions and obtains a different set of results. In our main departure, we posit that the agent faces a limited liability constraint so the principal's ability to punish the agent for bad outcomes is limited. Laux (2001) also assumes that the agent has limited liability, but his focus is on a static multi-task principal-agent setting. He finds that the optimal contract prescribes the principal to pay the agent only if all projects are successful. In this note, however, the optimal contract does not necessarily alter the power of the agent's contract with respect to each individual activity.

Other papers also explore the benefits from multi-lateral contracting. For example, in the industrial organizations literature Bernheim and Whinston (1990) show that multi-market contact facilitates tacit collusion by increasing potential sanctions following deviations from collusive behavior. Levin (2002) analyzes multilateral relationship contracts in an organization, and also shows how economies of scope naturally emerge from incentive problems.

3.1 The Model

We study a multitask principal-agent problem in a continuous time setting, where a principal may hire an agent to perform a total of $k \leq n$ *activities* or *tasks* until their contractual arrangement is terminated. The contractual relationship is characterized by moral hazard; output is observed by the principal and contractible, while the agent's effort is his private information. The principal must therefore provide the agent with incentives to exert costly effort. We assume throughout the analysis that the principal has full commitment power.

Technology and Information

We fix a probability space $(\Omega, \mathcal{F}, \mathcal{Q})$, and a filtration $\{\mathcal{F}_t : t \geq 0\}$ generated by n independent standard Brownian motions $B = (B_1, ..., B_n)$. Output of activity i is a diffusion process governed by the following equation

$$dY_i = a_i dt + \sigma_i dB_i \; \forall i, \tag{3-1}$$

where a_i is a stochastic process $a_i = \{a_{it} \in \mathcal{A}_i : 0 \leq t < \infty\}$ progressively measurable with respect to $\mathcal{F}_t$ that represents the agent's effort expended towards activity i. The set $\mathcal{A}_i$ is compact and is bounded below by 0.

σ_i measures the variability of output and/or the degree of asymmetry of information in the principal-agent relationship concerning activity i. Output is measured in monetary units and accrues directly to the principal.

Preferences

The two players derive positive utility from money, that enters linearly in their instantaneous utility function. The principal discounts the future at rate $r > 0$ and the agent at rate γ. We assume $\gamma > r$, which makes the agent more impatient than the principal. This introduces a wedge between the valuation of future transfers by the principal and the agent, and rules out indefinitely postponing payments to the latter.

The agent also derives disutility from effort. Given a cumulative compensation process c_t and an effort choice a_t, the instantaneous utility of the agent is given by

$$v_t = c_t - h(a_t). \tag{3-2}$$

The function $h(\cdot) : \mathbb{R}^n \to \mathbb{R}$ represents the agent's monetary cost of effort. We assume $h(0) = 0$, $h(\cdot)$ is strictly increasing, and convex in all of its individual arguments. The sign of the cross derivatives $\partial^2 h/\partial a_i \partial a_j$ will play an important role throughout the analysis. If $\partial^2 h/\partial a_i \partial a_j > 0$, then activities are *substitutes*; higher effort exerted in activity i increases the marginal costs of exerting positive effort in activity j. Alternatively, if $\partial^2 h/\partial a_i \partial a_j < 0$, activities i and j are *complements*, and the reverse logic applies. Finally, if $\partial^2 h/\partial a_i \partial a_j = 0$, then activities i and j are *independent*.

Contract

Before the agents starts working for the principal, the principal offers him a contract, which is formally characterized by an array $\Gamma = (c, a, \tau)$. The first component of the contract c specifies a non-negative cumulative consumption flow given by $c = c_t(Y_s; 0 \leq s \leq t) \in [0, \infty)$, that depends on the entire history of output. Because the agent has limited liability c_t must be weakly increasing. This constraint excludes the possibility that the agent provide the principal with a monetary compensation after a bad outcome. The contract's second component is an effort recommendation $a = \{a_t \in A; 0 \leq t < \infty\}$, and its third component is a stopping time τ that represent the time at which the contract is terminated.

Under the above assumptions, the agent's total expected payoff at date 0 from contract $\Gamma = (c, a, \tau)$ is

$$W_0 = \mathbb{E}^a \left[\int_0^\tau e^{-\gamma s}(c_s - h(a_s))ds | \mathcal{F}_0 \right], \tag{3-3}$$

provided that the agent indeed follows the principal's suggested action plan. In the above expression, $\mathbb{E}^a$ denotes the expectation under the probability measure $\mathcal{Q}^a$ induced by the agent's strategy.

The principal's total expected profit at date 0 from having the agent abide by $\Gamma = (c, a, \tau)$ is

$$b_0 = \mathbb{E}^a \left[\sum_{j=1}^{n} \int_0^\tau e^{-rs} dY_{j,s} - c_s | \mathcal{F}_0 \right], \tag{3-4}$$

and consists of the properly discounted output the principal accrues, minus monetary payments she delivers the agent, plus the sum of payoffs from the termination of activities.

Dynamic contracts have traditionally posed significant challenges. The complexity of the contract space often renders the analysis intractable. In an important contribution, Sannikov (2008) simplifies the characterization of the contracts by conditioning the agent's incentives on a unique state variable. Let the agent's continuation value $W_t(Y)$ after an outcome history $(Y_s, 0 \leq s \leq t)$ be his total expected payoff if he follows the effort recommended in the contract.

Formally, for a fixed contract $\Gamma = (c, a, \tau)$, the agent's continuation value $W_t(Y)$ after history of outputs $\{Y_s, 0 \leq s \leq t\}$ is defined as

$$W_t(Y) = \mathbb{E}^a \left[\int_t^\tau e^{-\gamma(s-t)} (c_s - h(a_s)) ds | \mathcal{F}_t \right]. \tag{3-5}$$

The following Lemma, a straightforward application of the results presented in Sannikov (2008) to the case of multi-dimensional Brownian Motion, provides a useful representation of the evolution of the agent's continuation value.

Lemma 23 *For any progressively measurable effort process a and consumption process c, there exists a vector valued sensitivity process* $\beta_t(Y) = (\beta_{1,t}(Y), ..., \beta_{n,t}(Y))$ *such that*

$$\begin{aligned} dW_t &= \gamma W_t dt - (c_t - h(a_t)) + \beta_t(Y_t) \cdot (dY_t - a_t) \cdot dt) & \text{(3-6)} \\ &= \gamma W_t dt - (c_t - h(a_t)) + \sum_{i=1}^{n} \beta_{i,t}(Y_t)(dY_{i,t} - a_i)dt) & \text{(3-7)} \end{aligned}$$

for any $t \in (0, \tau)$.

Lemma 23 states that the agent's continuation value satisfies an inter-temporal consistency condition, the *promise keeping* constraint. If W_t is to equal the agent's expected payoff from following the proposed contract $\Gamma = (I, a, \tau)$ from time t onwards, then, with the passing of time, the agent's continuation value must be reduced by consumption flows received and increased by an amount equal to his effort costs, all in monetary units. Coefficients $\beta_i(Y)$ represent the sensitivity of the agent's continuation value to $(dY_{i,t} - a_{i,t})dt$, which has expected value of zero when the agent follows the contract's effort recommendation.

Incentive Compatibility

An effort process a is incentive compatible with respect to a contract Γ if the effort that maximizes the agent's expected utility equals the contract's effort recommendation. Nothing thus far guarantees that the agent has incentives to follow the effort process prescribed by the contract. The following Lemma characterizes the set of incentive compatible contracts.

Lemma 24 *The vector of actions $a = (a_1, ..., a_n)$ is incentive compatible if and only if*

$$\beta_i(a) = \frac{\partial h}{\partial a_i}(a). \tag{3-8}$$

The intuition for the above Lemma is the following. By making a small downward deviation of Δ from the prescribed effort in the contract, the agent's instantaneous utility rises by an amount $\partial h/\partial a \cdot \Delta$. However, the agent also reduces the mean of the output process by Δ. This in turn hurts the agent in proportion to how sensitive the contract is to output from activity i, precisely β_i. Lemma 24 therefore places a restriction on the sensitivity that the contract must have to provide the agent with the incentives to exert action a.

We let $\mathcal{B}(a)$ be the set of optimal incentive compatible contracts given action a.

The Optimal Contract

When choosing over contracts the principal wishes to find a stream of output contingent consumption c_t and an incentive-compatible effort recommendation a_t that maximizes the principal's profit, while delivering the agent an initial required payoff $W_0 \geq 0$.

Because the principal can always reduce the agent's continuation value by any amount $c > 0$ by directly transferring him this same amount, it must be the case that

$$b(W) \geq b(W - c) - c. \tag{3-9}$$

Equation (3-9) implies that $b'(W) \geq -1$ for all W, so that the marginal cost of compensating the agent with a higher continuation value can never exceed the cost of a direct transfer. Let $\overline{W}$ be the lowest value such that $b'(W) = -1$. Then it is always optimal to immediately pay the agent

$$c = \max\{W - \overline{W}, 0\}, \tag{3-10}$$

so that W can never exceed $\overline{W}$. Within the range $(0, \overline{W})$, W evolves according to Equation (3-7) with $c_t = 0$.

The optimal contract can be derived using tools from dynamic programming. Letting $b(W)$ be the principal's value function, the Hamilton Jacobi Bellman (HJB) equation is

$$rb(W) = \max_{c,a} \sum a_i - c + b'(W)(\gamma W - (c - h(a))) + \frac{1}{2} b''(W) \left[\sum_{i=1}^{n} \sigma_i^2 \beta_i^2(a) \right], \tag{3-11}$$

where $\beta_i(a) \in \mathcal{B}(a)$ for all i.

Optimal Allocation of Effort

Having derived some basic properties of the value function, we proceed to characterize the optimal choice of effort. For a fixed continuation value W and a vector of effort recommendations $a = (a_i, a_{-i})$, let

$$\phi_i(a_i, a_{-i}, W) \equiv a_i + b'(W) h(a_i, a_{-i}) + \frac{1}{2} b''(W) \left[\sum_{i=1}^{n} \sigma_i^2 \beta_i^2(a_i, a_{-i}) \right]. \tag{3-12}$$

When choosing the optimal amount of effort a_i, the principal takes W and a^*_{-i} as fixed and solves the following problem

$$\max_{a_i} \phi_i(a_i, a^*_{-i}, W). \tag{3-13}$$

The first order condition of Equation (3-13) is given by

$$1 + b'(W) \frac{\partial h}{\partial a_i}(a_i^*, a_{-i}^*) \quad \leq \quad -b''(W) \left[\sigma_i^2 \frac{\partial^2 h(a_i^*, a_{-i}^*)}{\partial a_i^2} + \sum_{j \neq i} \sigma_j^2 \frac{\partial^2 h(a_i^*, a_{-i}^*)}{\partial a_i a_j} \right] \tag{3-14}$$

with equality if a_i^* is positive.

The first term on the left hand side of Equation (3-14) reflects the fact that effort on activity i enters linearly in the drift of output Y_i. The second term on the left hand side of Equation (3-14) represents the effect on the principal's value function of a change in the agent's continuation value that results from inducing the agent to exert more effort. More specifically, a marginal change in the prescribed effort on activity i, while keeping the agent's consumption constant, must result in a change in the agent's continuation value of precisely $\partial h(a^*)/\partial a_i$. Now, changes in W have non trivial effects on the principal's value function. Because increases in the agent's continuation value makes inefficient termination less likely, $b'(W)$ is positive for low values of W. However, as W becomes sufficiently high, it becomes more likely that the principal will have to reward the agent with positive consumption, so $b'(W)$ becomes negative.

The right hand side of Equation (3-14) represents the costs, to the principal, of providing the agent with incentives to exert costly effort. Even tough the principal is risk-neutral, the concavity of the value function $b''(\cdot) < 0$ makes him risk averse. The right-hand side captures the cost to the principal of providing the agents with incentives to exert costly effort. More specifically, by making the agent's continuation value contingent on output, the principal is exposed to the risk of a sequence of bad outcomes that exhausts the agent's continuation value. When this occurs, the principal loses the capability to punish the agent after bad outcomes and the provision of incentives requires her to fire the agent. The inefficiency in the model results from the termination of activities, therefore the principal is penalized for exposing the agent to risk.

The interaction between the costs of each activity is an important component of the costs of providing incentives. For example, assume that activities i and j are substitutes. Then, when increasing the effort required on activity i, the principal must increase the incentives to exert effort on activity j as well. It is especially costly for the principal to induce higher effort in activities that are noisy or risk (high σ_i) and activities that generate negative externalities on the costs of other actions.

Three boundary conditions pin down the a solution to equation (3-11). First, because the agent has limited liability, his continuation value can never attain negative values. Therefore, when W is exhausted after a sequence of bad outcomes the contracting relationship is terminated. This implies that $b(0) = 0$. The second boundary condition is the "smooth pasting" condition $b'(\overline{W}) = -1$. The third boundary condition guarantees that $\overline{W}$ is optimally chosen and is often called to as the "higher contact" condition. It guarantees that $b''(\overline{W}) = 0$.

In the first best, that could be achieved if effort were publicly observable, the agent would be paid at $t = 0$ because he is more impatient than the principal. However, in the second-best case where there is asymmetry of information, payments are delayed and made contingent on the agent achieving a sufficiently good enough record of performance. This must be the case if the agent is to have the incentives to exert costly effort.

Using the higher contact condition in the HJB equation yields

$$rb(\overline{W}) + \gamma\overline{W} = \sum a_i^*(\overline{W}) + h(a^*(\overline{W})). \tag{3-15}$$

3.2 Optimal Grouping of Tasks into Jobs

In this section, we establish the potential benefits of grouping several tasks into a unique job. For the sake of simplicity, we initially assume that production technologies are independent, so that $h(a) = \sum h_i(a_i)$. We do so to make clear that economies of scope in the contractual relationship emerge irrespective of the presence of complementarities in the agent's production technology. Subsequently we briefly discuss the case where there are externalities among different tasks.

Independent Technologies

When technologies are independent, the principal can define independent commissions, signing n different contracts with the agent, one for each activity. Similarly, the principal can contract the n different tasks with n different agents. In either case, each individual contract resembles the one derived in Sannikov (2008).

Under independent contracting, the agent's continuation value is replaced by a continuation vector $W^I = (W_1^I, ..., W_k^I)$. Each entry of the continuation vector corresponds to a particular activity and evolves according to

$$dW_{it}^I = \gamma W_{it}^I - (c_{it} - h_i(a_{it})) + \beta_{it}(dY_i - a_i dt), \tag{3-16}$$

with $W_0^I = (W_{0,1}^I, ..., W_{0,k}^I)$. Similarly, let $g(\cdot) = (g_1(\cdot), ..., g_n(\cdot))$ be a vector whose entries correspond to the principal's value function when she contracts the n activities independently. Finally, let $\tau^I = (\tau_1^I, ..., \tau_n^I)$ be a vector of stopping times determining the random times at which activities are terminated. The principal terminates a particular activity when its corresponding continuation values is exhausted. The principal's value functions g_i on individual activities therefore satisfies

$$g_i(0) = 0. \tag{3-17}$$

The equality in equation (3-17) is a boundary condition - it is not the result of any maximizing behavior on the part of the principal - and must be satisfied because of the agent's limited liability constraint. This constraint implies that the agent's expected future compensation from following the suggested contract can never assume negative values. Independent contracting has a distinctive feature. The punishment for a bad outcome in a particular activity is entirely reflected in the continuation value of that same activity.

We now consider the case where, despite their technological independence, the principal contracts jointly on all activities with the same agent. Evidently, under joint contracting, the principal can replicate the outcome of independent contracting by keeping track of a vector of activity scores

$S = (S_1, ..., S_n)$, where $S_0 = (W_{0,1}, ..., W_{0,n})$ and

$$dS_{it} = \gamma S_{it} dt - (c_{it} - h_i(a_{it})) + \beta_{it}(dY_{it} - a_i dt), \tag{3-18}$$

and termination of activity i occurring whenever $S_i = 0$. Since individual scores satisfy the promise-keeping constraints, they represent the expected promised value to the agent in each individual activity. Therefore, the total expected profit obtained by the principal when contracting activities independently places a lower bound on the expected profits when she contracts on all activities simultaneously. Mathematically, it must be the case that for every W such that $W = \sum S_i$ we have $b(W) \geq \sum_i g_i(S_i)$.

However, independent contracting cannot be optimal. By treating each activity separately, the principal forfeits the possibility of transferring slack across activities. Under joint contracting, whenever $g_i'(S_i) > g_j'(S_j)$ the principal can transfer slack from activity j to activity i, preserving incentives, and increasing his total value $\sum g_i(S_i)$. Optimal joint contracting will correspond to a situation where the principal optimally transfers slack from one activity to the other to maximize the sum of the individual value function. Rather than having a separate limited liability constraint for each activity, joint contracting needs to only satisfy the sum of all of the limited liability constraints.

Proposition 25 *If activities are technologically independent, then the principal is better off grouping all activities into a unique job. That is* $b(W) \geq \sum g_i(W_i^I)$*. If information is not perfectly correlated then the principal is strictly better off.*

The intuition behind the results of Theorem 25 is straightforward. By jointly contracting on all activities, the principal relaxes the agent's participation constraints by transferring slack from one activity to the other at his discretion.

Independent contracting cannot be improved upon only in a very special case in which two conditions are satisfied. First, technologies must be identical. Second, output shocks must be perfectly correlated. In this case, there are no benefits from joint contracting as the continuation values of all activities will be perfectly correlated. Therefore the principal will never benefit from transferring slack from one activity to the other. However, joint contracting still attains first best.

We now derive the optimal scoring rule, that prescribes how the principal should optimally transfer slack from one activity to the other.

Definition 26 The optimal score is a vector valued function $S(W) =$

$(S_1(W), ..., S_n(W))$ that is determined by solving the following problem

$$\max_S \sum_{i=1}^n g_i(S_i) \tag{3-19}$$

$$s.a. \qquad \sum_i S_i = W \tag{3-20}$$

Because all value functions are concave, the above problem is a standard optimization problem, whose solution is fully characterized by the following first order condition

$$g_i'(S_i) = g'(S_j)\forall i, j. \tag{3-21}$$

A simple algorithm shows how the principal can, by contracting on all activities simultaneously, improve upon independent contracting. At any time t prior to termination, the sequence of events during the infinitesimal time interval $[t, t + dt]$ can informally be described as follows:

(1) the principal observes output of all activities and computes the score of each individual activity according to Equation 3-18;

(2) she then calculates $g(S_i)$ for all i;

(3) finally the principal transfers slack across activities until condition 3-21 is satisfied.

Let τ_i be the stopping time that represents the termination of activity i. Then

$$\tau_i = \inf\{t : g(S_{it}) = 0\}, \tag{3-22}$$

so that the principal terminates activity i when its score is zero. Condition (3-22) looks similar to the value matching condition, however, there is an important difference. If $\sum S_{it} > 0$, then Condition (3-22) is an optimality condition. The principal could transfer slack from other activities to activity i but chooses not to. Under the optimal scoring rule, activity i is terminated when

$$g_i'(0) = g_j'(S_j) \ \forall j \neq i. \tag{3-23}$$

Proposition 27 *If technologies are independent and identical on all dimensions, then it is optimal to fire the agent at once when* $\sum S_i = 0$. *More specifically, under the optimal contract, the scope of the agent's job always involves performing all tasks, and remains constant throughout the contracting relationship.*

When technologies are identical, $g_i(\cdot) = g_j(\cdot)$ for all i and j. The principal always equalizes the agent's scores across activities. This in turn results in the simultaneous exhaustion of slack across all activities, at which point the principal must terminate the relationship.

Dependent Technologies

We now briefly consider the case of dependent technologies. For the sake of exposition, we consider two extreme cases: in the first all technologies are complements and in the second all technologies are substitutes. Complementarities among technologies constitute - apart from the main mechanism highlighted in this paper - an important source of economies of scope among tasks. The term $\sum_{i=1}^{n} \sigma_i^2 \beta_i^2(a)$ can be seen as an overall measure of the power of the agent's contract that is necessary to implement effort profile a. When tasks are complements, the overall power of the contract to induce a given level of effort is below that which is necessary to induce that same effort when tasks are substitutes.

The reduction of the overall power of the contract has an important effect on the principal's expected value from contracting with the principal. Because the principal can induce a given effort with a lower volatility of the agent's continuation value, inefficient termination becomes less likely benefiting the principal.

3.3 Conclusion

This note makes a simple point. Economies of scope emerge in a multi-task principal-agent relationship when the agent has a limited liability constraint. In the environment proposed by Sannikov (2008), the principal must fire the agent when his continuation value approaches zero, resulting in the inefficient termination of the contractual relationship between both parties. Therefore, if an agent performs multiple tasks, the principal can engage in cross-subsidization of the agent's continuation values among multiple tasks. The combination of tasks into a unique job therefore relaxes the limited-liability constraint faced by the agent and reduces inefficient termination of the contractual relationship. Economies of scope arise independently of complementarities in the agent's actions when performing multiple tasks or the improvement of the informativeness of the performance measure. Instead, it is the cross subsidization between activities that is beneficial.

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A Proofs

Technical Assumptions for Chapter 1

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a filtered probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right continuous filtration containing all sets of measure 0 with respect to $\mathbb{P}$ and is rich enough to carry the Brownian motion B_t and the Poisson Processes N_t^i for $i \in \{1, 2\}$. Let $\{T_n^i\}_{n\geq 0}$, be the jump times of N_t^i, which have intensity λ. It follows that $(T_1^i - T_0^i)$, $(T_1^i - T_0^i)$, ..., $(T_1^i - T_0^i)$ are independent random variables exponentially distributed with parameter λ.

I also assume throughout that the following condition holds.

Condition 1: $\mu(\cdot)$ and $\sigma(\cdot)$ imply the existence of a unique strong solution of (1-2).

Condition 1 is met if μ and σ are continuously differentiable and bounded and σ is uniformly bounded below (Karatzas and Shreve 2004)).

Proof that optimal strategies are of the threshold type in the efficient benchmark

I will show that if $m(x)$ is a solution to the HJB equation given by (1-12), then the optimal policy must be a threshold policy. When $\lambda\phi_L x_C < c$ it is never optimal to search and the proof is trivial. The proof of the assertion when the previous inequality is reversed is by contradiction.

When $\lambda\phi_L x_C > c$, it is always optimal to search in an interval of the form $[x_C, x_C + \varepsilon]$, since for x sufficiently close x_C, $m(x_C) = 0$ and the continuity of $m(\cdot)$ guarantee that $\lambda[\phi_L x - m(x)] > c$. Now, let

$$\overline{x} = \inf\{x : \lambda[\phi_L x - m(x)] < c \text{ and } x \geq x_C\}, \tag{A-1}$$

so that it is optimal to search over the whole interval $[x_C, \overline{x}]$. The continuity of the value function implies $\overline{x} > x_C$.

Now, if the optimal search strategy is not of the threshold type, it is possible to define a closed interval $\mathcal{I} = [x_1, x_2] \subset \mathbb{R}$ such that (i) $[x_C, \overline{x}] \cap \mathcal{I} = \emptyset$, (ii) $m(x_i) = \phi_L x_i - c/\lambda$, for $i \in \{1, 2\}$ and (iii) $m(x) < \phi_L x - c/\lambda$ for all x in the interior of $\mathcal{I}$. For $x \in int(\mathcal{I})$, let $\tau(x) = \inf\{t : x_t \notin \mathcal{I} \text{ and } x_0 = x\}$. Then the Markov Property yields

$$\phi_L x - \frac{c}{\lambda} > m(x) \geq \mathbb{E}^x\left[\int_0^\tau e^{-rs} D(1) x_s ds\right] + \mathbb{E}^x\left[e^{-r\tau} m(x_\tau)\right] \tag{A-2}$$

$$= \mathbb{E}^x\left[\int_0^\tau e^{-rs} D(1) x_s ds\right] + \mathbb{E}^x\left[e^{-r\tau}\left(\phi_L x_\tau - \frac{c}{\lambda}\right)\right] \tag{A-3}$$

The first strict inequality follows since x belongs to the interior of $\mathcal{I}$ where search is optimal. The inequality that follows is not an equality if Player 1's opponent is searching for a buyer.

Now, Assumption 2 results in the following inequality

$$\left(\phi_L x - \frac{c}{\lambda}\right) \Pr[\tau = \infty] < \mathbb{E}^x\left[\int_0^\infty e^{-rs} D(1) x_s \mathbf{1}_{\{\tau=\infty\}} ds\right]. \tag{A-4}$$

Then

$$\left(\phi_L x - \tfrac{c}{\lambda}\right) \Pr[\tau < \infty] > \tag{A-5}$$

$$\mathbb{E}^x\left[\int_0^\tau e^{-rs} D(1) x_s \mathbf{1}_{\{\tau<\infty\}} ds\right] + \mathbb{E}^x\left[e^{-r\tau}\left(\phi_L x_\tau - \tfrac{c}{\lambda}\right)\mathbf{1}_{\{\tau<\infty\}}\right] > \tag{A-6}$$

$$\mathbb{E}^x\left[\int_0^\tau r e^{-rs} \phi_L x_s \mathbf{1}_{\{\tau<\infty\}} ds\right] + \mathbb{E}^x\left[e^{-r\tau}\left(\phi_L x_\tau - \tfrac{c}{\lambda}\right)\mathbf{1}_{\{\tau<\infty\}}\right]. \tag{A-7}$$

The last strict inequality follows once again from Assumption 2. After an integration by parts, the first term on the right hand side of the last line equals

$$\mathbb{E}^x\left[(\phi_L x - \phi_L e^{-r\tau} x_\tau + \phi_L \int_0^\tau e^{-rs} dx_s)\mathbf{1}_{\{\tau<\infty\}}\right], \tag{A-8}$$

which when plugged into the last line yields after some manipulation

$$0 > \frac{c}{\lambda}\mathbb{E}^x\left[\left(1 - e^{-r\tau}\right)\mathbf{1}_{\{\tau<\infty\}}\right] + \mathbb{E}^x\left[\phi_L \int_0^\tau e^{-rs} \mathbf{1}_{\{\tau<\infty\}} dx_s\right], \tag{A-9}$$

and we arrive at a contradiction, since both terms on the right hand side of the above expression are strictly positive.

The intuition for the above result is the following. First, if x never leaves $\mathcal{I}$, it can't be optimal to divest the division for $x \in \mathcal{I}$ since the financial constraint will never bind. Now assume that x eventually *does* leave $\mathcal{I}$. Given that, upon x leaving $\mathcal{I}$, it is optimal to stop searching, the only way for it to be optimal to divest the division while x is in the interior of $\mathcal{I}$ is if the opportunity costs of keeping it operational is too high. But Assumption 2 guarantees that a firm always earns a higher flow of profits from operating the division than from selling it and investing the proceeds at the risk-free rate.

A similar argument can be applied to prove that $d(x)$ only crosses $\phi_H x$ once from below.

Proof of Proposition 4

In this section, I prove that the solution to the HJB equation derived in the text is the value function of the optimization problem faced by the social planner. Furthermore, I show that the search strategy previously derived is optimal. I proceed in a series of steps. I initially formalize the optimal control problem faced by the social planner and then show through a standard verification argument that the HJB equation is a sufficient condition for optimality.

The set of admissible controls when there remains only one active firm is defined as

$$\mathcal{A}_M = \{u = (u_t, \mathcal{F}_t) \ : \ u \text{ is measurable and adapted, } u_t \in \{0,1\} \ \forall t\}. \tag{A-10}$$

The social planner's payoff function from having one active division and adopting an admissible strategy u is

$$\begin{aligned} J_M(x;u) &\equiv \mathbb{E}^{x,u}\left[\int_0^{\tau\wedge\tau_C} e^{-rt}(D(1)x_t - cu_t)dt + e^{-r\tau}\phi_L x_\tau \mathbf{1}_{\{\tau<\tau_\mathbf{C}\}}\right] && \text{(A-11)} \\ &= \mathbb{E}^{x}\left[\int_0^{\tau_C} e^{-rt}D(1)x_t e^{-\lambda\int_0^t u_s ds}dt + \int_0^{\tau_C} u_t\lambda e^{-rt-\lambda\int_0^t u_s ds}\phi_L x_t dt\right] && \text{(A-12)} \end{aligned}$$

where $\tau = \tau(u)$ is the time when a buyer is found and and $\tau_C = \inf\{t : x_t \leq x_C\}$.

To understand (A-12), note that the probability of no sale taking place up to time t when search is characterized by the process u is given by $e^{-\lambda\int_0^t u_s ds}$. Therefore, the first term on the right hand side of (A-12) is just a sum of accrued profits, after considering the probability that they may not actually occur if the division has already been sold.

The intuition for the second term is similar. Since, a sale takes place the first time that $u = 1$ and the Poisson process makes a jump, the unit is sold during the interval of time $[t, t + dt)$ if (i) no sale has taken place until time t and (ii) the Poisson process jumps while the social planner is searching, which occurs with probability $u_t\lambda dt$. Hence (A-12) is the expected profit to the social planner from adopting the admissible strategy u.

The social planner wishes to maximize the expected payoff by choosing among controls in $\mathcal{A}_M$. Formally,

$$m(x) \equiv \sup_{u\in\mathcal{A}_M} J_M(x;u). \tag{A-13}$$

Now I show that $m(x) = \overline{m}(x)$.

For an admissible control u, define

$$S_t = e^{-rt-\lambda\int_0^t u_s ds} m(x_t). \tag{A-14}$$

Applying Itô's Lemma to S_t, yields

$$dS_t = e^{-rt-\lambda\int_0^t u_s ds}[\mathcal{L}m(x_t) - (r+\lambda u_t)m(x_t)] + dM_t, \tag{A-15}$$

where

$$M_T = \int_0^T e^{-rt-\lambda\int_0^t u_s ds} m'(x_t)\sigma(x_t)dB_t. \tag{A-16}$$

Then

$$S_T = m(x) + \int_0^T e^{-rt-\lambda\int_0^t u_s ds}[\mathcal{L}m(x_t) - (r+\lambda u_t)m(x_t)]dt + M_T \tag{A-17}$$

Since $m'(\cdot)$ is bounded by assumption, M_T is a martingale. Furthermore it is an uniformly integrable martingale since

$$\mathbb{E}^x\left[\int_0^T \left(e^{-rt-\lambda\int_0^t u_s ds} m'(t)\sigma(x_t)\right)^2 dt\right] < \infty \tag{A-18}$$

This implies that, for any stopping-time τ,

$$\mathbb{E}^x\left[\int_0^T e^{-rt-\lambda\int_0^t u_s ds} m'(t)\sigma(x_t)dB_t\right] = 0 \tag{A-19}$$

Taking expectations and using the fact that, for any admissible control, $m(x)$ satisfies

$$\mathcal{L}m(x) - (r+\lambda u)m(x) \leq -(D(1)x + u\lambda\phi_L x) \tag{A-20}$$

then

$$\mathbb{E}^x[S_T] + \mathbb{E}^x\left[\int_0^T e^{-rt-\lambda\int_0^t u_z dz} D(1)x_t dt\right] + \mathbb{E}^x\left[\int_0^T e^{-rt-\lambda\int_0^t u_s ds} u_t\lambda\phi_L x_t dt\right] \leq m(x). \tag{A-21}$$

Therefore $m(x) \geq J(x;u)$ for any admissible control and the solution to the HJB equation is an upper bound for the social planner's payoff function from operating one division when the state variable is x_0. All the above inequalities become equalities when the optimal control is employed.

When both firms are still active, the social planner must choose, at any time t, the number of firms that search for an outside buyer. Therefore, the social planner's strategy can be summarized by a control process $w = (w_t, \mathcal{F}_t)$ taking values in $\{0, 1, 2\}$ and represents the number of firms searching at each moment. Let

$$\mathcal{A}_D = \{w = (w_t, \mathcal{F}_t) \ : \ w \text{ is measurable and adapted, } w_t \in \{0,1,2\} \ \forall t\} \tag{A-22}$$

be the set of admissible controls when two firms are active.

$$J_D(x;u) \equiv \mathbb{E}^x \left[2\int_0^{\tau_C} e^{-\lambda \int_0^t u_s ds} D(2) x_t dt + \int_0^{\tau_C} e^{-rt-\lambda \int_0^t u_s ds} u_t \lambda (m(x_t) + \phi_H x_t) dt \right] \tag{A-23}$$

The optimal control problem when the social planner is operating two divisions can be defined similarly. I will now show that $x_D > x_M$. Note that, for any value of x, it must be the case that

$$(\phi_H - \phi_L)x + m(x) > d(x) - m(x). \tag{A-24}$$

The intuition for the above inequality is the following. The value of running the marginal division, given by $d(x) - m(x)$, might actually be higher than the value of operating as a monopolist if the price discount given by $(\phi_H - \phi_L)$ is large enough. But, if the monopolist were to receive a compensation of $(\phi_H - \phi_L)x$, then it must be strictly better to operate as a monopolist.

The inequality given by (A-24) then implies:

$$\begin{aligned} \phi_H x_D - (d(x_D) - m(x_D)) \quad &> \quad \phi_L x_D - m(x_D) \Rightarrow && \text{(A-25)} \\ \lambda[\phi_H x_D - d(x_D) + m(x_D)] = c \quad &> \quad \lambda[\phi_L x_D - m(x_D)], && \text{(A-26)} \end{aligned}$$

so that $x_D > x_M$.

Proof of Proposition 5

Comparative statics with respect to $D(1)$

Since, for any fixed x, $m(x)$ is increasing in $D(1)$ equation (1-11) implies that x_M is decreasing in $D(1)$. Furthermore, let τ_D and τ_M be optimal times where a buyer is met for the first and second divisions respectively. It is the case that $\tau_D < \tau_M$ must hold almost surely. Therefore, writing

$$m(x) = \mathbb{E}^x \left[\int_0^{\tau_D \wedge \tau_C} e^{-rs} D(1) x_S ds + e^{-r\tau_D} m(\tau_D) \mathbf{1}_{\{\tau_D < \tau_C\}} \right] \tag{A-27}$$

and subtracting this expression from equation (1-13), yields:

$$(d-m)(x) \quad = \quad \mathbb{E}\left[\int_0^{\tau_D \wedge \tau_C} e^{-rs}[2D(2)-D(1)]x_S ds + e^{-r\tau_D}\phi x_{\tau_D}\mathbf{1}_{\{\tau_D<\tau_C\}}\right].$$

Now, an application of the envelope theorem guarantees that $(d-m)(x)$ is decreasing in $D(1)$, since the optimal search strategy remains unaltered. Therefore x_D is increasing in $D(1)$ according to (1-15).

Comparative statics with respect to ϕ and c

A change in ϕ_L affects value functions of the monopolist through two channels. The first is through the increase in prices fetched at liquidation, keeping optimal search strategies fixed, and the second is through the induced change in search strategies. By the Envelope Theorem the second effect is of second order for small changes of ϕ_L. Then

$$m(x,\phi_L) = \mathbb{E}^{x,x_M}\left[\int_0^{\tau\wedge\tau_C} e^{-rt}(\delta(x_t,1) - c\mathbf{1}_{\{x_t<x_M\}})dt + e^{-r\tau}\phi_L x_\tau \mathbf{1}_{\{\tau<\tau_C\}}\right], \tag{A-28}$$

where the probability measure underlying the expectation operator explicitly depends on the search strategy adopted. Differentiation of the above expression with respect to ϕ yields

$$\frac{\partial m}{\partial \phi}(x,\phi_L) = \mathbb{E}^{x,u}\left[e^{-r\tau}x_\tau\mathbf{1}_{\{\tau<\tau_C\}}\right] < x_M. \tag{A-29}$$

Now, recall that the optimal search strategy is characterized by

$$\phi_L x_M - m(x_M,\phi) = \frac{c}{\lambda} \tag{A-30}$$

The left hand side of the above expression is increasing in ϕ_L which implies that x_M must increase when ϕ_L increases so that the search strategy satisfies the equality. A similar argument can be applied to show that x_D is increasing in both ϕ_L and ϕ_D. The comparative statics on c is similar.

Comparative Statics with respect to λ

By the Implicit Function Theorem,

$$\frac{dx}{d\lambda} = -\frac{[\phi x - m(x,\lambda)] - \lambda\partial m/\partial\lambda}{\lambda[\phi - \partial m/\partial x]}. \tag{A-31}$$

When the above expression is evaluated at $x = x_M$, the denominator is always negative. So the sign of $dx/d\lambda$ depends on the sign of $[\phi x - m(x,\lambda)] - \lambda\partial m/\partial\lambda$, which is positive when $\lambda \approx 0$ and becomes negative for sufficiently high λ.

Parts (4) and (5) of the proposition are an immediate consequence of part (1).

Definitions of the game played by firms

The state space of the game is given by $\mathcal{S} = [x_C, \infty) \times \{1, 2\}$, with a typical element given by $x \times Q \in \mathcal{S}$, where Q is the number of divisions that are still operating in the widget market. I now define the strategies of the agents in the divestiture game. If firm i has not divested up to time t, its action set is $A_t^i = \{0, 1\}$, where 0 and 1 denotes *no search* and *search* respectively. After firm i divests, A_t^i becomes $\{\emptyset\}$, which corresponds to *don't move*.

A strategy for firm i is a mapping $\sigma_t^i : H_t \to A_t^i$, where H_t is the set of possible histories of the game. Without the Markovian assumption, an equilibrium would allow strategies $\sigma_t^i(h^t)$ to be functions of the entire history $h^t \in H_t$. When the Markovian restriction is imposed, it must be the case that $\sigma_t^i(h^t) = \sigma_t^i(\hat{h}^t)$ for two histories $h^t, \hat{h}^t \in H_t$ where the state variable $x_t \times Q$ is the same. Furthermore, when the stationarity restriction is imposed, it must be the case that $\sigma_t^i(\cdot) = \sigma_{t'}^i(\cdot)$ for every t and t'.

Proof of Lemma 6

Optimal search strategies must be part of a Nash Equilibrium. Each firm chooses its search strategy so as to maximize its value, taking as given the search strategy of its competitor. Recall that I am restricting the analysis to Markov Perfect Equilibrium, so a search strategy for firm i can be fully characterized by a partition of the state space into a *search*and *no-search* region. Once we fix the strategy of its competitor, firm i faces a standard (unique agent) stochastic optimization problem. Therefore, to prove this Lemma one can use arguments similar to those used to prove that a monopolist adopts a threshold strategy.

If firm i does not adopt a threshold strategy, then one can define an interval set $\mathcal{I}$, with lower bound greater than x_C, such that (i) firm i searches in the interior of $\mathcal{I}$, and (ii) firm i stops searching as soon as x leaves $\mathcal{I}$. But this generates a contradiction. First, if x never leaves $\mathcal{I}$, it can't be optimal to divest the division for $x \in \mathcal{I}$ since the financial constraint will never bind. Now assume that x eventually *does* leave $\mathcal{I}$. Given that, upon x leaving $\mathcal{I}$, it is optimal to stop searching, the only way for it to be optimal to divest the division while x is in the interior of $\mathcal{I}$ is if the opportunity costs of keeping it operational is too high. But Assumption 2 guarantees that a firm always earns a higher flow of profits from operating the division than from selling it and investing the proceeds at the risk-free rate. Since strategies are Markovian, this threshold remains constant through time.

After one firm exits the market, the remaining firm searches according to the optimal search strategy of the central planner when there is only one division left.

Recall that this search strategy is characterized by the threshold x_M.

Proof of Proposition 7

I will first prove that assumption 1 implies that $m(x) > f(x)$ for all x.

Let $h(x) = m(x) - f(x)$. The following set of ODE's describe the evolution of $h(x)$ in the relevant regions:

$$rh(x) = x[D(1) - D(2)] + \mathcal{L}h(x) \;\; \forall x \in [x_L, \infty) \tag{A-32}$$

$$rh(x) = x[D(1) - D(2)] + \mathcal{L}h(x) - \lambda h(x) \;\; \forall x \in [x_F, x_L) \tag{A-33}$$

$$rh(x) = x[D(1) - D(2)] - \{\lambda[\phi_H x - m(x)] - c\} + \mathcal{L}h(x) - \lambda h(x) \;\; \forall x \in [x_M, x_F) \tag{A-34}$$

$$rh(x) = x[D(1) - D(2) - \lambda(\phi_H - \phi_L)] + \mathcal{L}h(x) - 2\lambda h(x) \;\; \forall x \in [x_C, x_M) \tag{A-35}$$

Now, $h(x)$ can be interpreted as asset that (i) equals zero at x_C and (ii) always pays positive dividends. Together, these two conditions imply that $h(x) = m(x) - f(x)$ must always be positive.

With the above result, search on the part of one firm exerts an externality of $\lambda[m(x) - g(x)]$, where $g(x)$ is equal to $f(x)$, $l(x)$ or $s(x)$, depending on the equilibrium that ensues and the role adopted by firms. Now, this externality is always a positive one, so that more search on the part of one firm increases the value of its rival and leads to less search on its part.

Proof of Proposition 8

Except for the existence of a symmetric equilibrium, all other results are a straightforward application of Theorem 5 (and its Corollaries) of Milgrom and Roberts (1990). Now to prove that there exists a symmetric equilibrium, we can apply Tarski's intersection point theorem.

Proof that the value of the Follower is always greater than that of the Leader

Define $h(x) = f(x) - l(x)$. Then $h(x)$ satisfies the following set of ODE's in the relevant regions

$$rh(x) = \mathcal{L}h(x) \;\; \forall x \in [x_L, \infty) \tag{A-36}$$

$$rh(x) = \mathcal{L}h(x) + \lambda[m(x) - \phi x] - c - \lambda h(x) \;\; \forall x \in [x_F, x_L) \tag{A-37}$$

$$rh(x) = \mathcal{L}h(x) - 2\lambda h(x) \;\; \forall x \in [x_C, x_F) \tag{A-38}$$

Therefore $h(x)$ can be seen as an "asset" that is "killed" with intensities λ and 2λ in the regions $[x_F, x_L)$and $[x_C, x_F)$ respectively.[1]

Furthermore $h(x)$ only pays dividends when $x \in [x_F, x_L)$.

If these dividends are always non-negative and are strictly positive in a set of positive measure, then the price of this asset must be strictly positive as well. Note that, in any equilibrium it must be the case that $x_F > x_M$, which implies that

$$\lambda[m(x) - \phi x] - c > 0 \;\; \forall x \in [x_F, x_C). \tag{A-39}$$

Hence the value of being the Follower is always greater than that of the Leader as long as $x_F \neq x_L$.

Proof of Proposition 9

Let (x_A^1, x_B^1) and (x_A^2, x_B^2) be any two equilibria of the game, where $j \in \{A, B\}$ is an index for firms. The fact that best replies are strictly decreasing yields:

$$x_A^1 > x_A^1 \Rightarrow x_B^2 = BR(x_A^1) < BR(x_A^2) = x_B^2.$$

Proof of Proposition 10

Equations (1-11),(1-19), together with the fact that $m(x) > f(x)$ for all x, implies that $x_M < x_F$. Furthermore, $x_F < x_S < x_L$ since the games is characterized by strategic substitutability. The last inequality $x_L < x_D$ will be true if it is the case that $d(x) - m(x) < l(x)$, which must hold since the Leader can always adopt the social planner's strategy towards the first firm it sells.

Proof of Proposition 11

I restrict the proof to the situation where best-replies are decreasing in the parameter of interest. This is the case for example when we are performing comparative statics in monopoly rents, given by $D(1)$. A proof with obvious modifications applies when best replies are increasing.

By way of contradiction, assume that for parameters $t^2 > t^1$ there exists $x_j^i = BR(x_{-j}, t^i)$ for $j \in \{A, B\}$ and $i \in \{1, 2\}$, where

$$(x_A^2, x_B^2) > (x_A^1, x_B^1) \tag{A-40}$$

[1]See Oksendal (2003), page 145 about the killing of diffusions.

in the natural product order. When t corresponds to $D(1)$, Equation (A-40) states that an increase in the monopoly rents leads both firms to search earlier in equilibrium. But then

$$x_A^2 - x_A^1 = BR(x_B^2, t_2) - BR(x_B^1, t_1) \quad \text{(A-41)}$$
$$= [BR(x_B^2, t_2) - BR(x_B^2, t_1)] + [BR(x_B^2, t_1) - BR(x_B^1, t_1)] \quad \text{(A-42)}$$

Now the first term on the right of the last line is negative since best replies are order-reversing in the parameter. This implies that if $x_A^2 - x_A^1$ is positive, then $BR(x_B^2, t_1) - BR(x_B^1, t_1)$ must be positive as well. But then best-response functions must be increasing in rival's strategies, a contradiction. Therefore, as monopoly rents increase, at least one firm must expend less search efforts in the new equilibrium. As an immediate Corollary, x_S, the optimal search threshold in the symmetric equilibrium, is decreasing in $D(1)$.

Example: Brownian Motion

In this section, I solve the model explicitly assuming that the diffusion process governing the evolution of x is a geometric Brownian motion with drift μ and volatility $\sigma^2 > 0$. Let $\gamma_0 = \frac{-m-\sqrt{m^2+2r\sigma^2}}{\sigma^2}$, $\gamma_1^+ = \frac{-m+\sqrt{m^2+2(r+\lambda)\sigma^2}}{\sigma^2}$, $\gamma_1^- = \frac{-m-\sqrt{m^2+2(r+\lambda)\sigma^2}}{\sigma^2}$, $\gamma_2^+ = \frac{-m+\sqrt{m^2+2(r+2\lambda)\sigma^2}}{\sigma^2}$, $\gamma_2^- = \frac{-m-\sqrt{m^2+2(r+2\lambda)\sigma^2}}{\sigma^2}$ and $m = \mu - \frac{\sigma^2}{2}$.

Monopolist

The value of being a monopolist is given by

$$m(x) = \begin{cases} \frac{xD(1)}{r-\mu} + A_{M,0}x^{\gamma_0} & \text{if } x \geq x_M \\ \frac{x\{D(1)+\lambda\phi_L\}}{r+\lambda-\mu} + A_{M,1}^+ x^{\gamma_1^+} + A_{M,1}^- x^{\gamma_1^-} & \text{if } x_M > x \geq x_C. \end{cases}$$

Coefficients $A_{M,0}$, $A_{M,1}^+$, and $A_{M,1}^-$ and the optimal threshold x_M are determined by the following conditions:

1. $m(x_C) = 0$
2. $m(x_M+) = m(x_M-)$
3. $m'(x_M+) = m'(x_M-)$.
4. $\lambda[\phi_L x_M - m(x_M)] = c$

Duopolist

The value of being a duopolist is given by

$$d(x) = \begin{cases} \frac{2D(2)x}{r-\mu} + A_{D,0}x^{\gamma_0} & \text{if } x \geq x_D \\ \frac{x}{r+2\lambda-\mu}\left\{2D(2) + \frac{2\lambda D(1)}{r-\mu} + 2\lambda\phi_H\right\} + \frac{2\lambda A_{M,0}x^{\gamma_0}}{r+2\lambda-\mu\gamma_0-0.5\sigma^2\gamma_0(\gamma_0-1)} & \\ -\frac{2c}{r+2\lambda} + A^+_{D,2}x^{\gamma_2^+} + A^-_{D,2}x^{\gamma_2^-} & \text{if } x_D > x \geq x_C. \end{cases}$$

Coefficients $A_{D,0}$, $A^+_{D,2}$, and $A^-_{D,2}$ and the optimal threshold x_D are determined by the following conditions:

1. $d(x_C) = 0$
2. $d(x_D+) = d(x_D-)$
3. $d'(x_D+) = d'(x_D-)$.
4. $\lambda[\phi_H x_D + m(x_D) - d(x_D)] = c$

Symmetric Equilibrium

The value of a division in the competitive equilibrium when firms adopt the same search strategy is given by

$$s(x) = \begin{cases} \frac{D(2)x}{r-\mu} + A_{S,0}x^{\gamma_0} & \text{if } x \geq x_S \\ \frac{x}{r+2\lambda-\mu}\left\{D(2) + \frac{\lambda D(1)}{r-\mu} + \lambda\phi_H\right\} + \frac{\lambda A_{M,0}x^{\gamma_0}}{r+2\lambda-\mu\gamma_0-0.5\sigma^2\gamma_0(\gamma_0-1)} & \\ -\frac{c}{r+2\lambda} + A^+_{S,2}x^{\gamma_2^+} + A^-_{S,2}x^{\gamma_2^-} & \text{if } x_S \geq x \geq x_C \end{cases}$$

Coefficients $A_{S,0}$, $A^+_{S,2}$, and $A^-_{S,2}$ and the optimal threshold x_S are determined by the following conditions:

1. $s(x_C) = 0$
2. $s(x_S+) = s(x_S-)$
3. $s'(x_S+) = s'(x_S-)$.
4. $\lambda[\phi_H x_S - s(x_S)] = c$

Asymmetric Equilibrium: Follower

$$f(x) = \begin{cases} \frac{xD(2)}{r-\mu} + A_{F,0}x^{\gamma_0} & \textbf{if } x \geq x_F \\ \frac{x}{r+\lambda-\mu}\left\{D(2) + \frac{\lambda D(1)}{r-\mu}\right\} + \frac{\lambda A_{M,0}x^{\gamma_0}}{r+\lambda-\mu\gamma_0-0.5\sigma^2\gamma_0(\gamma_0-1)} + A^+_{F,1}x^{\gamma_1^+} + A^-_{F,1}x^{\gamma_1^-} & \textbf{if } x_L > x \geq x_F \\ \frac{x}{r+2\lambda-\mu}\left\{D(2) + \frac{\lambda D(1)}{r-\mu} + \lambda\phi_H\right\} + \frac{\lambda A_{M,0}x^{\gamma_0}}{r+2\lambda-\mu\gamma_0-0.5\sigma^2\gamma_0(\gamma_0-1)} & \\ -\frac{c}{r+2\lambda} + A_{F,2}x^{\gamma_2^+} + A^-_{F,2}x^{\gamma_2^-} & \textbf{if } x_F > x \geq x_C \end{cases}$$

Given x_L, coefficients $A_{F,0}$, $A^+_{F,1}$, $A^-_{F,1}$,$A^+_{F,2}$, and $A^-_{F,2}$and the optimal threshold x_F are determined by the following conditions:

1. $f(x_C) = 0$
2. $f(x_F+) = f(x_F-)$
3. $f'(x_F+) = f'(x_F-)$.
4. $f(x_L+) = f(x_L-)$
5. $f'(x_L+) = f'(x_L-)$.
6. $\lambda[\phi_H x_F - f(x_F)] = c$

Asymmetric Equilibrium: Leader

$$l(x) = \begin{cases} \frac{xD(2)}{r-\mu} + Ax^{\gamma} & \textbf{if } x \geq x_L \\ \frac{x}{r+\lambda-\mu}\left\{D(2) + \lambda\phi_H\right\} - \frac{c}{r+\lambda} + Ax^{\gamma_1^+} + Ax^{\gamma_1^-} & \textbf{if } x_L > x \geq x_F \\ \frac{x}{r+2\lambda-\mu}\left\{D(2) + \frac{\lambda D(1)}{r-\mu} + \lambda\phi_H\right\} + \frac{\lambda A_{M,0}x^{\gamma_0}}{r+2\lambda-\mu\gamma_0-0.5\sigma^2\gamma_0(\gamma_0-1)} & \\ -\frac{c}{r+2\lambda} + Ax^{\gamma_2^+} + Ax^{\gamma_2^-} & \textbf{if } x_F > x \geq x_C \end{cases}$$

Given x_F, coefficients $A_{L,0}$, $A^+_{L,1}$, $A^-_{L,1}$,$A^+_{L,2}$, and $A^-_{L,2}$and the optimal threshold x_L are determined by the following conditions:

1. $l(x_C) = 0$
2. $l(x_F+) = l(x_F-)$
3. $l'(x_F+) = l'(x_F-)$.
4. $l(x_L+) = l(x_L-)$
5. $l'(x_L+) = l'(x_L-)$.
6. $\lambda[\phi_H x_L - l(x_L)] = c$

Proof of Proposition 15

When faced with a probability of audit p, entrepreneur i declares default if and only if

$$(1-p)f(s_i) - c_0 \geq f(s_i) - D. \qquad \text{(A-43)}$$

The left hand side of Equation (A-43) is the entrepreneur's expected payoff when he declares default, while the right-hand side is the expected payoff when he honors the debt contract. Equation (A-43) is equivalent to $D - c_0 \geq p \cdot f(s_i)$. If $D - c_0 > p \cdot f(\overline{s})$, then $s^* = \overline{s}$ since even the entrepreneur with the highest possible return would rather default than honor the debt contract. If $D - c_0 = p \cdot f(s)$ for some s, define $s^* = s$. Uniqueness is due to the fact that $f(\cdot)$ is strictly increasing. The result then follows.

Proof of Proposition 17

If Assumption 14 holds, then $p(s^D) = 1$ and the bank can credibly commit to audit all projects in default. If default automatically triggers an audit, then the best response for each entrepreneur is to report truthfully. Therefore $s^* = s^D$. This holds for any capital level E.

Proof of Proposition 18

Define

$$\Gamma(s) \equiv p(E,s) \cdot f(s) - (D - c_0). \tag{A-44}$$

The function $\Gamma(s)$ is continuous. If $\Gamma(s) > 0$, then an entrepreneur that faces an audit probability of $p(E,s)$ prefers to repay his loan when the return from his project is $f(s)$. When $\Gamma(s) < 0$, the opposite holds.

To prove the existence of a full coordinated default, note that if $E < E_1$ and $s^* = \overline{s}$ then $p(E,\overline{s}) < \frac{D-c_0}{f(\overline{s})}$. Therefore, $\Gamma(\overline{s}) < 0$. From Proposition 15, it is optimal for every entrepreneur to declare default (irrespective of his project's return), if he expects all other entrepreneurs to do the same. Therefore $\mathcal{E} = (\overline{s}, p(E,\overline{s}))$ is a repayment equilibrium.

We now show the existence of a partially coordinated default. At a partially coordinated default with threshold s^*, an entrepreneur $s_i = s^*$ should be indifferent between repaying or defaulting, therefore $\Gamma(s^*) = 0$. Let $s^L = \sup\{s; p(E,s) = 1\}$. From Assumption 14, we have $s^L > s^D$. Since $\Gamma(s^L) > 0$ and $\Gamma(\overline{s}) < 0$, the Intermediate Value Theorem guarantees that there exists $s^* \in (s^L, \overline{s})$ such that $\Gamma(s^*) = 0$.

Proof of Proposition 20

To simplify the following proof, we assume that $c(s) = c$. First, recall that $H(s^D)$ is the probability that a truthful entrepreneur declares default after he signs a debt contract with notional value D. When i declares default along with other $k < n$ entrepreneurs (who are reporting truthfully), the bank collects a total of $(n-(k+1))D$ in resources and must audit $(k+1)$ projects. The conditional (on k other defaults) probability that i is subsequently audited is given by

$$q(k,E,n) \equiv \min\left\{\frac{1}{k+1}\left\lfloor\frac{E+(n-(k+1))D}{c}\right\rfloor, 1\right\}, \tag{A-45}$$

where $\lfloor x \rfloor$ is the highest integer smaller than x. Now the probability that k truthful entrepreneurs declare default among a total of $(n-1)$ entrepreneurs is $\binom{n-1}{k}H(s^D)^k(1-H(s^D))^{n-1-k}$. Therefore the unconditional probability that entrepreneur i is audited if he declares default is given by

$$p(E,n,s^D) \equiv \sum_{k=0}^{n-1} \binom{n-1}{k} H(s^D)^k(1-H(s^D))^{n-1-k} q(k,E,n) \tag{A-46}$$

Equation (A-46) is the equivalent of Equation (2-5) to the case of a finite number of entrepreneurs.

Given $p(E,n,s^D)$, entrepreneur i repays his loan when $s_i = s^D$ if and only if

$$\Gamma(s^D) = p(E,n,s^D) \cdot f(s^D) - (D - c_0) \geq 0. \tag{A-47}$$

Because $p(E,n,s^D)$ is left-continuous in E, there is a minimum capital $E_0(n)$ such that inequality (A-47) is satisfied.

If on the other hand $E < E_0(n)$, an entrepreneur does not have incentives to tell the truth for all $s \in (s^D, \overline{s}]$, even if he believes that other entrepreneurs are reporting truthfully. Therefore, truth-telling cannot be an equilibrium.

We now show that if $E < E_1(n)$, then there is a full and at least one partial coordinated default. Let $E_1(n) \equiv \frac{(D-c_0)nc}{f(\overline{s})}$. If $E < E_1$, then $p(E,n,\overline{s}) < \frac{D-c_0}{f(\overline{s})}$ and $\Gamma(\overline{s}) < 0$, and there is a full coordinated default equilibrium. To see that there is a partial coordinated default as well, fix entrepreneur i. Assume that, for each $j \neq i$, entrepreneur j declares default if and only if $s_j \leq s$. Then the unconditional probability that entrepreneur i is audited if he declares default is given by $p(E,n,s)$, where E is the amount of capital that the bank sets aside at $t = 0$. Once again, at a coordinated default, an entrepreneur with $s_i = s^*$ should be indifferent between repayment and declaring default, so we must have $\Gamma(s^*) = 0$. The function $\Gamma(s)$ is continuous in s. Since $\Gamma(s^D) > 0$ and $\Gamma(\overline{s}) < 0$, the Intermediate Value Theorem guarantees once again that there exists an s^* such that $\Gamma(s^*) = 0$.

Proof of Proposition 21

To show that $E_0(n)$ is decreasing in n, note that $p(E,n,s)$ is weakly increasing in E and n. Therefore $E_0(n)$ must decrease as n increases so as to maintain the equality in Equation (A-47). To prove that $E_0(n) \to 0$, let X_n be a random variable representing the total resources the bank collects from entrepreneurs that report truthfully when lending to n different entrepreneurs. By the Strong Law of Large Numbers

$$\frac{X_n}{n} \to D(1-H(s^D)) \;\; a.s. \quad \text{as } n \to \infty. \tag{A-48}$$

Similarly, let total average audit costs be given by C_n. Then

$$\frac{C_n}{n} \to \int_0^{s^D} c(s)h(s)ds \;\; a.s. \;\; \text{as } n \to \infty. \tag{A-49}$$

A direct application of Slutsky's Theorem (Casella and Berger (1990) pg. 239) yields $X_n/C_n \to \frac{D(1-H(s^D))}{\int_0^{S^D} c(s)h(s)ds} > 1$. Therefore, all entrepreneurs in default are audited, even if the bank sets $E = 0$ at $t = 0$. Therefore $E_0(n) \to 0$ as $n \to \infty$.

Proof that General Mechanism are also subject to Coordinated Defaults

A direct mechanism can be fully characterized by an array (R_b, R_e, μ), where R_b and R_e are the returns to the bank and entrepreneurs respectively, and $\mu = \mu(s)$ is the probability that an audit is realized when the agent reports the message s to the principal. We assume that, for each message, the lottery that determines whether an audit takes place is independent of the distribution function of the states of nature.

A direct mechanism partitions the message space $\mathcal{M} = [0, \bar{s}]$ into the regions $\mathcal{A}$ and $\mathcal{A}^c$, such that

$$\mathcal{A} = \{s \in \mathcal{M}; \mu(s) > 0\} \text{ and } \mathcal{A}^c = \{s \in \mathcal{M}; \mu(s) = 0\}. \tag{A-50}$$

For the mechanism to be incentive compatibility, the agent's transfer in the no-audit region $\mathcal{A}^c$ must be a constant given by D. In the audit region $\mathcal{A}$, the entrepreneur's payment may depend on the message $\hat{s}$ and whether the entrepreneur is found to have reported truthfully if audited.

Before audits begin, the principal collects $D\int_{\mathcal{A}^c} h(s)ds$ in resources from agents that report a non-audit message to the principal. Total audit costs are given by $\int_{\mathcal{A}} h(\hat{s})d\hat{s} \int_{\mathcal{A}} c(s)\mu(s)ds$. Two remarks are in order. First we are formally using the fact that the lottery that decides whether a given agent is audited is independent of the realization of the state. Second, we do not require that audit costs be increasing in s.[2] The probability that entrepreneur is audited after reporting state $\hat{s}$ is given by

$$p(E, \hat{s}) = \mu(\hat{s}) \cdot \min\left\{\frac{E + D\int_{\mathcal{A}^c} h(s)ds}{\int_{\mathcal{A}} h(\hat{s})d\hat{s}\int_{\mathcal{A}} c(s)\mu(s)ds}, 1\right\}. \tag{A-51}$$

An entrepreneur announces $\hat{s} \in \mathcal{A}$ when his state is $s \in \mathcal{A}^c$ if and only if

$$(1 - p \cdot \mu(\hat{s}))f(s_i) - c_0 > f(s_i) - D.$$

[2] In particular, the second remark implies that the results obtained along the exposition do not rely on audit costs being increasing in the state variable.

Now, if an agent anticipates that all other agents will default on their loans $p(0, \hat{s}) = 0$, and. So there is a full coordinated default.

Technical Assumptions

The proofs of Lemmas 23 and 24 are a straightforward extensions of the results presented in DeMarzo and Sannikov (2006) and Sannikov (2008). We include them however for the sake of completeness.

Proof of Lemma 23

Given information up to time t, the agent's total expected payoff from contract $\Gamma = (I, a, \tau)$ is given by

$$V_t = \int_0^t e^{-\gamma s}(c_s - h(a_s))ds + e^{-\gamma t}W_t(c, a, \tau). \tag{A-52}$$

Since V_t is $\mathcal{Q}^a$martingale, the Martingale Representation Theorem for a multi-dimensional martingale yields the following representation

$$\begin{aligned} V_t &= V_0 + \int_0^t e^{-\gamma s}\beta_s \cdot dB^a && \text{(A-53)} \\ &= V_0 + \int_0^t e^{-\gamma s}\sum_{i=1}^n \beta_{i,s} dB^a_{i,s}, \quad 0 \le t < \infty, && \text{(A-54)} \end{aligned}$$

where

$$B^a_{i,t} = \frac{1}{\sigma_i}\left(Y_t - \int_0^t a_s ds\right) \tag{A-55}$$

is a Brownian Motion under $\mathcal{Q}^a$. Differentiation of equations (A-52) and (A-54) with respect to t and some simple algebra yields

$$dW = \gamma W - (c - h(a)) + \sum_{i=1}^n \beta_i dB^a_i. \tag{A-56}$$