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A
Cálculo dos Invariantes Diretamente

Este apêndice tem por objetivo escrever os invariantes geométricos de uma superfície implícita $S = \{(x, y, z) \in \mathbb{R}^3 / f(x, y, z) = 0\}$, onde f é de classe C^4 e 0 é valor regular de f, a partir da própria função f e reforçar a ideia que isto inclui um longo cálculo. Sabemos que toda superfície regular pode ser vista localmente como um gráfico $\mathcal{G} = \{(x, y, g(x, y)/(x, y) \in U\}$, em particular na seção 3.1 encontramos o plano tangente, a métrica afim escrita em termos da função f. Outros elementos geométricos importantes são o vetor normal $\mathbf{N}_e$ e a curvatura K_e que são dados, respectivamente, por

$$
\begin{aligned}
\mathbf{N}_e &= \frac{(-g_x, -g_y, 1)}{\sqrt{g_x^2 + g_y^2 + 1}} = (f_z^2 + f_x^2 + f_y^2)^{-1/2}(f_x, f_y, f_z) \\
K_e &= \frac{\left(f_{zz} f_{yy} - f_{yz}^2\right) f_x^2 + \left(-2 f_{xy} f_{zz} + 2 f_{xz} f_{yz}\right) f_y f_x}{\left(f_z^2 + f_x^2 + f_y^2\right)^2} \\
&+ \frac{2\left(-f_{xz} f_{yy} + f_{xy} f_{yz}\right) f_x f_z + \left(f_{xx} f_{zz} - f_{xz}^2\right) f_y^2}{\left(f_z^2 + f_x^2 + f_y^2\right)^2} \\
&+ \frac{-2\left(-f_{xz} f_{xy} + f_{xx} f_{yz}\right) f_y f_z + \left(f_{xx} f_{yy} - f_{xy}^2\right) f_z^2}{\left(f_z^2 + f_x^2 + f_y^2\right)^2}.
\end{aligned}
$$

Notamos que é preciso fazer um escalonamento do vetor normal $\mathbf{N}_e$ usando a curvatura K_e para obtermos um vetor contravariante o co-normal ν (ver seção 3.2) cuja fórmula em função de f é

$$\nu = \frac{1}{f_z \mathsf{d}^{1/4}} \ (\ f_x, f_y, f_z \ ) \ ,$$

onde

$$
\begin{aligned}
\mathsf{d} = \frac{1}{f_z^4} \cdot \Big(\ & \left(f_{yy} f_{zz} - f_{yz}^2\right) f_x^2 + 2\left(f_{xz} f_{xy} - f_{xx} f_{yz}\right) f_y f_z + \\
& \left(f_{zz} f_{xx} - f_{xz}^2\right) f_y^2 + 2\left(f_{xy} f_{yz} - f_{yy} f_{xz}\right) f_z f_x + \\
& \left(f_{xx} f_{yy} - f_{xy}^2\right) f_z^2 + 2\left(f_{yz} f_{xz} - f_{zz} f_{xy}\right) f_x f_y \Big).
\end{aligned}
\tag{A-1}
$$

Encontraremos a expressão do normal afim a partir da função f.

Utilizando os cálculos e definições da seção 3.2, temos as componentes do vetor normal afim em função das derivadas da função g. Usando a regra da cadeia obtemos as expressões das derivadas da função g até a terceira ordem. Substituindo estas expressões nas fórmulas explícitas de ξ dadas na seção 3.2, obtemos as coordenadas de ξ

$$
\begin{aligned}
\xi_1 = a\Big[\, & f_x^2\ (-f_z f_{xy} f_{zz} f_{yyy} - 2 f_y f_{yz} f_{yyz} f_{xz} + f_y f_{zz} f_{xz} f_{yyy} + f_{yy} f_y f_{yzz} f_{xz} \\
& - \ f_z f_{yy} f_{xy} f_{yzz} - f_{yy} f_y f_{xzz} f_{yz} - 4 f_{zz} f_{xz} f_{yy}^2 - 4 f_{xy} f_{yz}^3 + 4 f_{yy} f_{xz} f_{yz}^2 \\
& + \ 2 f_z f_{xy} f_{yz} f_{yyz} + 4 f_{yy} f_{xy} f_{yz} f_{zz} - 2 f_z f_{yy} f_{yz} f_{xyz} - f_y f_{zz} f_{xyy} f_{yz} \\
& + \ 2 f_y f_{yz}^2 f_{xyz} + f_z f_{yy}^2 f_{xzz} + f_z f_{yy} f_{zz} f_{xyy}\big) \\
+ \ & f_x\ (8 f_{yy} f_{xz} f_{xy} f_y f_{zz} - 2 f_{xx} f_{yy} f_y f_{zz} f_{yz} - 2 f_z f_{yy} f_y f_{zz} f_{xxy} + 4 f_z f_{yy} f_y f_{xxz} f_{yz} \\
& + \ 2 f_z f_{xyy} f_{xy} f_y f_{zz} + 4 f_z f_y f_{xz} f_{xyy} f_{yz} - 4 f_z f_y f_{xy} f_{xyz} f_{yz} - 2 f_z f_{yy} f_{xy} f_y f_{xzz} \\
& - \ 12 f_z f_{yy} f_{xy} f_{xz} f_{yz} - 2 f_y^2 f_{zz} f_{xyy} f_{xz} - 2 f_z^2 f_{yy} f_{xz} f_{xyy} - 2 f_z^2 f_{xyy} f_{xy} f_{yz} \\
& + \ 4 f_z^2 f_{yy} f_{xy} f_{xyz} + 2 f_{xy} f_y^2 f_{xzz} f_{yz} + 2 f_z^2 f_{xy} f_{xz} f_{yyy} - 2 f_z f_y f_{xz}^2 f_{yyy} \\
& - \ 2 f_{xy} f_{yzz} f_y^2 f_{xz} - 6 f_{yy} f_{xz}^2 f_{yz} f_y + 2 f_z f_{xx} f_{zz} f_{yy}^2 - 2 f_z f_{yy} f_{xy}^2 f_{zz} \\
& - \ 6 f_y f_{zz} f_{yz} f_{xy}^2 - 2 f_z f_y f_{xxy} f_{yz}^2 + 2 f_y^2 f_{xxy} f_{yz} f_{zz} + 2 f_z^2 f_{yy} f_{xxy} f_{yz} \\
& + \ 4 f_y f_{xz} f_{yz}^2 f_{xy} - 2 f_z f_{xx} f_{yy} f_{yz}^2 - 2 f_z^2 f_{xxz} f_{yy}^2 + 2 f_{xx} f_y f_{yz}^3 \\
& + \ 2 f_y^2 f_{xz}^2 f_{yyz} - 2 f_y^2 f_{xxz} f_{yz}^2 + 6 f_z f_{yy}^2 f_{xz}^2 - 2 f_z^2 f_{xy}^2 f_{yyz} + 8 f_z f_{xy}^2 f_{yz}^2\big) \\
+ \ & f_y^3\ (2 f_{xz} f_{xxz} f_{yz} - 2 f_{xyz} f_{xz}^2 + f_{xxy} f_{xz} f_{zz} - f_{xx} f_{xzz} f_{yz} - f_{xxx} f_{zz} f_{yz} + f_{xx} f_{yzz} f_{xz}) \\
+ \ & f_y^2\ \big(f_z f_{xx} f_{yy} f_{xzz} - 4 f_z f_{xxy} f_{xz} f_{yz} - 2 f_z f_{xx} f_{yyz} f_{xz} - 2 f_z f_{xy} f_{xxz} f_{yz} \\
& - \ f_z f_{xx} f_{xy} f_{yzz} - 2 f_{xx} f_{yy} f_{xz} f_{zz} - 2 f_z f_{yy} f_{xz} f_{xxz} + 4 f_{xx} f_{xy} f_{yz} f_{zz} \\
& + \ f_z f_{yy} f_{xxx} f_{zz} - 2 f_{xx} f_{yz}^2 f_{xz} + 4 f_z f_{xyz} f_{xy} f_{xz} - 2 f_{xy}^2 f_{xz} f_{zz} \\
& + \ 2 f_z f_{xxx} f_{yz}^2 + 2 f_z f_{xyy} f_{xz}^2 + 2 f_{yy} f_{xz}^3 - f_z f_{xy} f_{xxy} f_{zz} + 2 f_z f_{xx} f_{xyz} f_{yz}\big) \\
+ \ & f_y\ \big(-f_z^2 f_{xx} f_{xyy} f_{yz} + 8 f_z f_{xx} f_{yy} f_{xz} f_{yz} + 4 f_z f_{xy}^2 f_{xz} f_{yz} - 6 f_z f_{yy} f_{xy} f_{xz}^2 \\
& + \ f_z^2 f_{xx} f_{yyy} f_{xz} + 2 f_z^2 f_{xx} f_{xy} f_{yyz} - 2 f_z f_{xx} f_{yy} f_{xy} f_{zz} + 4 f_z^2 f_{xy} f_{yz} f_{xxy} \\
& - \ 3 f_z^2 f_{yy} f_{xxx} f_{yz} - 2 f_z^2 f_{xy}^2 f_{xyz} + 2 f_z f_{xy}^3 f_{zz} + 3 f_z^2 f_{yy} f_{xxy} f_{xz} \\
& - \ 2 f_z^2 f_{xx} f_{yy} f_{xyz} + 2 f_z^2 f_{yy} f_{xy} f_{xxz} - 4 f_z^2 f_{xy} f_{xyy} f_{xz} - 6 f_z f_{xx} f_{xy} f_{yz}^2\big) \\
+ \ & 4\, f_z^2 f_{xx} f_{yy} f_{xy} f_{yz} + 2 f_z^3 f_{xy}^2 f_{xyy} + 4 f_z^2 f_{yy} f_{xz} f_{xy}^2 - f_z^3 f_{xx} f_{xy} f_{yyy} \\
- \ & 3\, f_z^3 f_{xy} f_{xxy} f_{yy} - 4 f_z^2 f_{xy}^3 f_{yz} + f_z^3 f_{xxx} f_{yy}^2 + f_z^3 f_{xx} f_{xyy} f_{yy} - 4 f_z^2 f_{xx} f_{xz} f_{yy}^2 \Big],
\end{aligned}
$$

onde

$$
a = (4(f_z^3 \mathsf{d}^{3/4})\,(-f_{xz} f_{yy} + f_{xy} f_{yz})\, f_x + f_z \left(f_{xx} f_{yy} - f_{xy}^2\right) + (-f_{xx} f_{yz} + f_{xy} f_{xz})\, f_y)^{-1},
$$

$$
\begin{aligned}
\xi_2 = a\Big[\; & f_x^3 \;\big(-2f_{xyz}f_{yz}^2 + f_{zz}f_{xyy}f_{yz} + f_{yy}f_{xzz}f_{yz} + 2f_{xz}f_{yz}f_{yyz} - f_{xz}f_{zz}f_{yyy} - f_{yy}f_{xz}f_{yzz}\big)\\
+ \;& f_x^2 \;\big(-2f_{xx}f_{yz}f_{zz}f_{yy} + 4f_zf_{xyz}f_{xy}f_{yz} - 2f_{xxy}f_{yz}f_yf_{zz} + 4f_{yy}f_{xz}f_{xy}f_{zz}\\
& - 2f_{xzz}f_{xy}f_{yz}f_y + 2f_{xz}f_yf_{zz}f_{xyy} + f_zf_{xx}f_{yy}f_{yzz} - 2f_zf_{yy}f_{xxz}f_{yz}\\
& - 2f_{yy}f_{xz}^2f_{yz} - 2f_zf_{xx}f_{yz}f_{yyz} - f_zf_{yy}f_{xzz}f_{xy} + 2f_zf_{yy}f_{xz}f_{xyz} + 2f_{xx}f_{yz}^3\\
& - 2f_zf_{xz}f_{xy}f_{yyz} + f_zf_{xx}f_{zz}f_{yyy} - 2f_{xy}^2f_{yz}f_{zz} + 2f_{xxz}f_yf_{yz}^2 - 2f_{xz}^2f_{yyz}f_y\\
& + 2f_zf_{xz}^2f_{yyy} - 4f_zf_{xz}f_{xyy}f_{yz} + 2f_{xz}f_{xy}f_yf_{yzz} + 2f_zf_{yz}^2f_{xxy} - f_zf_{zz}f_{xyy}f_{xy}\big)\\
+ \;& f_x \;\big(2f_{yy}f_{xz}^3f_y + 2f_zf_{xy}^3f_{zz} - 2f_z^2f_{xy}^2f_{xyz} + 2f_{xz}^2f_{xyz}f_y^2 - 2f_zf_{xz}^2f_yf_{xyy}\\
& + 4f_z^2f_{xz}f_{xy}f_{xyy} + 3f_z^2f_{xx}f_{xyy}f_{yz} - 2f_z^2f_{xx}f_{yy}f_{xyz} + 2f_zf_{xy}^2f_yf_{xzz}\\
& + f_{xx}f_{xzz}f_y^2f_{yz} - 3f_z^2f_{xx}f_{xz}f_{yyy} + 2f_z^2f_{xx}f_{xy}f_{yyz} - f_{xx}f_{xz}f_{yzz}f_y^2\\
& - 6f_{xz}f_yf_{xy}^2f_{zz} + f_{xxx}f_{yz}f_y^2f_{zz} - 2f_zf_{xxx}f_yf_{yz}^2 + f_z^2f_{yy}f_{xxx}f_{yz}\\
& - f_{xz}f_{xxy}f_y^2f_{zz} - f_z^2f_{yy}f_{xz}f_{xxy} - 4f_z^2f_{xxy}f_{yz}f_{xy} + 2f_z^2f_{yy}f_{xxz}f_{xy}\\
& - 2f_y^2f_{xz}f_{xxz}f_{yz} - 6f_zf_{yy}f_{xz}^2f_{xy} - 6f_zf_{xx}f_{yz}^2f_{xy} - 6f_{xx}f_{xz}f_{yz}^2f_y\\
& + 4f_zf_{xz}f_{xy}^2f_{yz} + 4f_{xz}^2f_yf_{yz}f_{xy} - 2f_zf_{xx}f_{yy}f_{zz}f_{xy} - 2f_{xx}f_{yy}f_{xz}f_yf_{zz}\\
& + 8f_{xx}f_{yz}f_{xy}f_yf_{zz} + 2f_zf_{xxy}f_{xy}f_yf_{zz} + 4f_zf_{xz}f_yf_{xxy}f_{yz} - 2f_zf_{xx}f_{xyy}f_yf_{zz}\\
& - 4f_zf_{xz}f_yf_{xy}f_{xyz} + 4f_zf_{xx}f_{xz}f_{yyz}f_y - 2f_zf_{xx}f_{xy}f_yf_{yzz} + 8f_zf_{xx}f_{yy}f_{yz}f_{xz}\big)\\
+ \;& f_y^2 \;\big(-f_zf_{xx}f_{xzz}f_{xy} + 4f_{xx}f_{yz}f_{xz}^2 + 4f_{xx}f_{xz}f_{xy}f_{zz} - 2f_zf_{xx}f_{xz}f_{xyz}\\
& - f_zf_{xxx}f_{zz}f_{xy} + 2f_zf_{xz}f_{xxz}f_{xy} + f_zf_{xx}f_{xxy}f_{zz} - 4f_{yz}f_{zz}f_{xx}^2\\
& + f_zf_{xx}^2f_{yzz} - 4f_{xz}^3f_{xy}\big)\\
+ \;& f_y \;\big(-2f_z^2f_{xx}^2f_{yyz} + 2f_z^2f_{xx}f_{xyy}f_{xz} + 2f_zf_{xx}^2f_{zz}f_{yy} - 2f_zf_{xx}f_{xy}^2f_{zz}\\
& - 2f_z^2f_{xxy}f_{xy}f_{xz} + 2f_z^2f_{xxx}f_{yz}f_{xy} - 12f_zf_{xx}f_{xz}f_{xy}f_{yz} + 4f_z^2f_{xx}f_{xyz}f_{xy}\\
+ \;& 6f_zf_{xx}^2f_{yz}^2 - 2f_z^2f_{xx}f_{yz}f_{xxy} - 2f_zf_{xx}f_{yy}f_{xz}^2 - 2f_z^2f_{xy}^2f_{xxz} + 8f_zf_{xz}^2f_{xy}^2\big)\\
- \;& 4f_z^2f_{xz}f_{xy}^3 - 3f_z^3f_{xx}f_{xyy}f_{xy} + 2f_z^3f_{xy}^2f_{xxy} + 4f_z^2f_{xx}f_{yy}f_{xy}f_{xz}\\
+ \;& 4f_z^2f_{xx}f_{yz}f_{xy}^2 - f_z^3f_{xxx}f_{xy}f_{yy} + f_z^3f_{yyy}f_{xx}^2 + f_z^3f_{xx}f_{xxy}f_{yy} - 4f_z^2f_{xx}^2f_{yz}f_{yy}.
\end{aligned}
$$

E finalmente,

$$
\begin{aligned}
\xi_3 = a\Big[\; & f_x^3 \;\big(-f_{yy}^2f_{xzz} + 2f_{yy}f_{yz}f_{xyz} - f_{yy}f_{zz}f_{xyy} - 2f_{xy}f_{yz}f_{yyz} + f_{xy}f_{zz}f_{yyy} + f_{yy}f_{xy}f_{yzz}\big)\\
+ \;& f_x^2 \;\big(2f_zf_{yy}f_{xz}f_{xyy} + 2f_{yy}^2f_{xz}^2 + 2f_{xx}f_yf_{yz}f_{yyz} + 2f_zf_{xyy}f_{xy}f_{yz}\\
& - 4f_zf_{yy}f_{xy}f_{xyz} + 2f_{xz}f_yf_{xy}f_{yyz} - 2f_{yy}f_{xz}f_{xyz}f_y - f_{zz}f_yf_{xy}f_{xyy}\\
& - f_{xx}f_yf_{yzz}f_{yy} + 4f_{xy}^2f_{yz}^2 - f_{xx}f_yf_{zz}f_{yyy} - 2f_{yy}f_{xxz}f_{yz}f_y - 2f_zf_{xy}f_{xz}f_{yyy}\\
& + 3f_{yy}f_{xy}f_yf_{xzz} - 2f_zf_{yy}f_{xxy}f_{yz} + 2f_zf_{xxz}f_{yy}^2 - 2f_{xx}f_{yy}f_{yz}^2 - 2f_{yy}f_{xy}^2f_{zz}\\
& + 2f_{yy}f_{xxy}f_yf_{zz} - 2f_{xy}^2f_yf_{yzz} + 2f_{xx}f_{zz}f_{yy}^2 + 2f_zf_{xy}^2f_{yyz} - 4f_{yy}f_{xz}f_{yz}f_{xy}\big)\\
+ \;& f_x \;(2f_zf_{yy}f_{xxx}f_{yz}f_y - 2f_zf_{yy}f_yf_{xxy}f_{xz} - 4f_zf_{yy}f_{xy}f_yf_{xxz} - 2f_zf_{xx}f_{xyy}f_{yz}f_y\\
& + 4f_zf_{xx}f_{yy}f_{xyz}f_y + 2f_zf_{xx}f_{xz}f_yf_{yyy} - 4f_zf_{xx}f_yf_{xy}f_{yyz} + 8f_zf_{xx}f_{yy}f_{xy}f_{yz}
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad 8f_{xx}f_{yy}f_{xz}f_{yz}f_y - 4f_{xx}f_{yy}f_{xy}f_yf_{zz} - 8f_zf_{xy}^3f_{yz} + 4f_{xy}^3f_yf_{zz}\\
&- \quad 2f_z^2f_{xyy}f_{xy}^2 - 2f_{xzz}f_y^2f_{xy}^2 + 4f_zf_{xy}^2f_{xyz}f_y - 2f_{xx}f_{xyz}f_{yz}f_y^2\\
&+ \quad f_z^2f_{xx}f_{xy}f_{yyy} - f_{xx}f_{yy}f_{xzz}f_y^2 - 2f_{xx}f_{xz}f_{yyz}f_y^2 + 3f_{xx}f_{xy}f_y^2f_{yzz}\\
&- \quad f_{yy}f_{xxx}f_{zz}f_y^2 - f_{xy}f_y^2f_{zz}f_{xxy} + 3f_z^2f_{yy}f_{xy}f_{xxy} + 2f_{yy}f_y^2f_{xz}f_{xxz}\\
&+ \quad 2f_{xy}f_y^2f_{xxz}f_{yz} + 2f_{xx}f_{xyy}f_y^2f_{zz} - f_z^2f_{xx}f_{yy}f_{xyy} - 8f_zf_{xx}f_{xz}f_{yy}^2\\
&- \quad 4f_{yy}f_{xy}f_yf_{xz}^2 + 8f_zf_{yy}f_{xz}f_{xy}^2 - 4f_{xx}f_{xy}f_{yz}^2f_y - f_z^2f_{xxx}f_{yy}^2\big)\\
+ \quad & f_y^3\,(-f_{xx}f_{xxy}f_{zz}+f_{xxx}f_{zz}f_{xy}+2f_{xx}f_{xz}f_{xyz}+f_{xx}f_{xzz}f_{xy}-f_{xx}^2f_{yzz}\\
&- \quad 2f_{xz}f_{xxz}f_{xy})\\
+ \quad & f_y^2\,\big(4f_{xy}^2f_{xz}^2 - 2f_zf_{xx}f_{xyy}f_{xz} - 4f_zf_{xx}f_{xy}f_{xyz} + 2f_zf_{xx}f_{xxy}f_{yz}\\
&+ \quad 2f_zf_{xy}^2f_{xxz}2f_{xx}^2f_{yz}^2 + 2f_zf_{xxy}f_{xy}f_{xz} - 4f_{xx}f_{xz}f_{xy}f_{yz}\\
&+ \quad 2f_zf_{xx}^2f_{yyz} - 2f_{xx}f_{xz}^2f_{yy} + 2f_{xx}^2f_{zz}f_{yy} - 2f_{xx}f_{xy}^2f_{zz} - 2f_zf_{xxx}f_{yz}f_{xy}\big)\\
+ \quad & f_y\,\big(-f_z^2f_{xx}f_{yy}f_{xxy} + f_z^2f_{yy}f_{xxx}f_{xy} + 3f_z^2f_{xx}f_{xy}f_{xyy} + 8f_zf_{xx}f_{yy}f_{xy}f_{xz}\\
&- \quad 8f_zf_{xy}^3f_{xz} - 8f_zf_{xx}^2f_{yz}f_{yy} - f_z^2f_{xx}^2f_{yyy} + 8f_zf_{xx}f_{xy}^2f_{yz} - 2f_z^2f_{xy}^2f_{xxy}\big)\\
+ \quad 4 \quad & f_z^2f_{xx}^2f_{yy}^2 + 4f_z^2f_{xy}^4 - 8f_z^2f_{xx}f_{yy}f_{xy}^2.
\end{aligned}
$$

Notemos que ao substituirmos $f_x = 0 = f_y = f_{xy}$ e $f_z = 1$ em ξ temos exatamente a expressão do ξ dada no capítulo 4.

Agora, queremos calcular as expressões para as curvaturas afins, para isso é necessário calcularmos as derivadas do co-normal afim ν e do normal afim ξ. Denotaremos por $\nu_x = (\partial_x\nu_1, \partial_x\nu_2, \partial_x\nu_3)$, $\nu_y = (\partial_y\nu_1, \partial_y\nu_2, \partial_y\nu_3)$ e $\xi_x = (\partial_x\xi_1, \partial_x\xi_2, \partial_x\xi_3)$, $\xi_y = (\partial_y\xi_1, \partial_y\xi_2, \partial_y\xi_3)$. Obtemos,

$$
\begin{aligned}
\partial_x\nu_1 &= \frac{1}{4\mathsf{d}^{5/4}}\cdot(g_xg_{xxx}g_{yy} + g_xg_{xx}g_{xyy} - 2g_xg_{xy}g_{xxy} - 4g_{xx}^2g_{yy} + 4g_{xx}g_{xy}^2),\\
\partial_x\nu_2 &= \frac{1}{4\mathsf{d}^{5/4}}\cdot(g_yg_{xxx}g_{yy} + g_yg_{xx}g_{xyy} - 2g_yg_{xy}g_{xxy} - 4g_{xy}g_{xx}g_{yy} + 4g_{xy}^3),\\
\partial_x\nu_3 &= \frac{1}{4\mathsf{d}^{5/4}}\cdot(-g_{xxx}g_{yy} - g_{xx}g_{xyy} + 2g_{xy}g_{xxy})
\end{aligned}
$$

e

$$
\begin{aligned}
\partial_y\nu_1 &= \frac{1}{4\mathsf{d}^{5/4}}\cdot(g_xg_{xxy}g_{yy} + g_xg_{xx}g_{yyy} - 2g_xg_{xy}g_{xyy} - 4g_{xy}g_{xx}g_{yy} + 4g_{xy}^3),\\
\partial_y\nu_2 &= \frac{1}{4\mathsf{d}^{5/4}}\cdot(g_yg_{xxy}g_{yy} + g_yg_{xx}g_{yyy} - 2g_yg_{xy}g_{xyy} - 4g_{xx}g_{yy}^2 + 4g_{yy}g_{xy}^2),\\
\partial_y\nu_3 &= -\frac{1}{4\mathsf{d}^{5/4}}\cdot(g_{xxy}g_{yy} + g_{xx}g_{yyy} - 2g_{xy}g_{xyy}),
\end{aligned}
$$

onde as expressões de g_x, g_y, g_{xx}, g_{xy} e g_{yy} foram dadas no capítulo 4 e as demais derivadas são

$$
\begin{aligned}
g_{xxx} &= -\frac{f_{xxx}}{f_z} + \frac{3(f_{xxz}f_x + f_{xz}f_{xx})}{f_z^2} + \frac{3(-f_{zz}f_xf_{xx} - 2f_{xz}^2f_x - f_{xzz}f_x^2)}{f_z^3} \\
&+ \frac{9f_{xz}f_{zz}f_x^2f_x^3f_{zzz}}{f_z^4} - \frac{3f_{zz}^2f_x^3}{f_z^5} \\
g_{xxy} &= -\frac{f_{xxy}}{f_z} + \frac{f_{yz}f_{xx} + 2f_xf_{xyz} + 2f_{xz}f_{xy} + f_{xxz}f_y}{f_z^2} \\
&+ \frac{-2f_{xz}^2f_y - f_x^2f_{yzz} - f_{zz}f_yf_{xx} - 2f_{zz}f_xf_{xy} - 4f_{xz}f_{yz}f_x - 2f_xf_{xzz}f_y}{f_z^3} \\
&+ \frac{f_x^2f_{zzz}f_y + 6f_{xz}f_yf_{zz}f_x + 3f_{zz}f_x^2f_{yz}}{f_z^4} - \frac{3f_{zz}^2f_x^2f_y}{f_z^5} \\
g_{xyy} &= -\frac{f_{xyy}}{f_z} + \frac{2f_{yz}f_{xy} + f_xf_{yyz} + f_{xz}f_{yy} + 2f_{xyz}f_y}{f_z^2} \\
&- \frac{4f_{yz}f_yf_{xz} + 2f_{zz}f_yf_{xy} + f_{xzz}f_y^2 + 2f_xf_{yzz}f_y + 2f_{yz}^2f_x + f_{zz}f_xf_{yy}}{f_z^3} \\
&+ \frac{3f_{xz}f_{zz}f_y^2 + 6f_{yz}f_yf_{zz}f_x + f_y^2f_xf_{zzz}}{f_z^4} - \frac{3f_{zz}^2f_xf_y^2}{f_z^5} \\
g_{yyy} &= -\frac{f_{yyy}}{f_z} + \frac{3f_{yyz}f_y + 3f_{yz}f_{yy}}{f_z^2} + \frac{-3f_{zz}f_yf_{yy} - 6f_{yz}^2f_y - 3f_{yzz}f_y^2}{f_z^3} \\
&+ \frac{9f_{yz}f_{zz}f_y^2 + f_y^3f_{zzz}}{f_z^4} - \frac{3f_{zz}^2f_y^3}{f_z^5}.
\end{aligned}
$$

As derivadas do normal afim com relação a x e y são

$$
\begin{aligned}
\partial_x\xi_1 = \frac{1}{16\mathsf{d}^{5/4}} \big(&-12g_{xy}^3g_{xxxy}g_{yy} + 3g_{xx}^2g_{xyy}^2g_{yy} - 39g_{xx}g_{xyy}g_{yy}g_{xy}g_{xxy} \\
&- 3g_{xx}g_{xy}g_{yyy}g_{xxx}g_{yy} + 7g_{xxx}^2g_{yy}^3 + 8g_{xy}^4g_{xxyy} + 2g_{xx}g_{xyy}g_{yy}^2g_{xxx} \\
&- 7g_{xx}^2g_{xy}g_{yyy}g_{xyy} + 10g_{xx}^2g_{xy}g_{yyy}g_{xxy} - 35g_{xxx}g_{yy}^2g_{xy}g_{xxy} \\
&+ 26g_{xy}^2g_{xyy}g_{xxx}g_{yy} - 4g_{xx}g_{xxyy}g_{yy}g_{xy}^2 + 4g_{xx}^2g_{xxy}g_{yyy}g_{yy} \\
&+ 4^2g_{xx}g_{xy}g_{xyyy}g_{yy} + 12g_{xy}g_{xxxy}g_{yy}^2g_{xx} + 30g_{xy}^2g_{xxy}^2g_{yy} \\
&+ 18g_{xy}^2g_{xyy}^2g_{xx} - 24g_{xy}^3g_{xyy}g_{xxy} - 4g_{xx}^2g_{xxyy}g_{yy}^2 - 4g_{xxx}g_{xy}^3g_{yyy} \\
&- 4g_{xx}g_{xy}^3g_{xyyy} - 4g_{xxxx}g_{yy}^3g_{xx} + 4g_{xxxx}g_{yy}^2g_{xy}^2 + 12g_{xxy}^2g_{yy}^2g_{xx}\big)\,, \\
\partial_x\xi_2 = \frac{1}{16\mathsf{d}^{5/4}} \big(&-7g_{xy}g_{xxx}^2g_{yy}^2 - 12g_{xy}^3g_{xx}g_{xxyy} - 12g_{xxy}^2g_{xy}^3 + 8g_{xxxy}g_{xy}^4 \\
&- 21g_{xy}g_{xx}^2g_{xyy}^2 + 7g_{xx}^3g_{yyy}g_{xyy} - 4g_{xy}^3g_{xxxx}g_{yy} - 16g_{xy}^3g_{xxx}g_{xyy} \\
&- 4g_{xx}^3g_{xyyy}g_{yy} + 4g_{xx}^2g_{xyyy}g_{xy}^2 - 4g_{xxxy}g_{xx}^2g_{yy}^2 + 28g_{xxy}g_{xy}^2g_{xxx}g_{yy} \\
&+ 48g_{xxy}g_{xy}^2g_{xx}g_{xyy} - g_{xx}^2g_{yyy}g_{xxx}g_{yy} - 14g_{xx}^2g_{yyy}g_{xy}g_{xxy} + 7g_{xxy}g_{xx}g_{yy}^2g_{xxx} \\
&+ 15g_{xxy}g_{xx}^2g_{yy}g_{xyy} - 30g_{xxy}^2g_{xx}g_{yy}g_{xy} - 4g_{xxxy}g_{xy}^2g_{xx}g_{yy} + 4g_{xy}g_{xxxx}g_{yy}^2g_{xx} \\
&+ 12g_{xy}g_{xx}^2g_{xxyy}g_{yy} + 8g_{xx}g_{yyy}g_{xxx}g_{xy}^2 - 12g_{xy}g_{xxx}g_{yy}g_{xx}g_{xyy}\big), \\
\partial_x\xi_3 = \frac{1}{16\mathsf{d}^{5/4}} \big(&4g_xg_{xxxx}g_{yy}^2g_{xy}^2 - 4g_xg_{xxxx}g_{yy}^3g_{xx} + 4g_{xx}^2g_yg_{xyyy}g_{xy}^2
\end{aligned}
$$

$$
\begin{aligned}
&- 4g_{xx}^3 g_y g_{xyyy} g_{yy} + 12g_{yy}^2 g_x g_{xxy}^2 g_{xx} - 24g_x g_{xyy} g_{xy}^3 g_{xxy} + 18g_x g_{xyy}^2 g_{xy}^2 g_{xx}\\
&- 7g_{yy}^2 g_y g_{xxx}^2 g_{xy} + 30g_{yy} g_x g_{xy}^2 g_{xxy}^2 - 21g_{xx}^2 g_y g_{xy} g_{xyy}^2 + 3g_{xx}^2 g_{yy} g_x g_{xyy}^2\\
&+ 7g_{xx}^3 g_y g_{yyy} g_{xyy} - 12g_{yyy} g_x g_{xy}^3 g_{xxxy} - 12g_{xx} g_y g_{xy}^3 g_{xxyy} - 16g_{xxx} g_y g_{xy}^3 g_{xyy}\\
&- 4g_{xx}^2 g_{yy}^2 g_y g_{xxxy} - 4g_{xx}^2 g_{yy}^2 g_x g_{xxyy} - 4g_{yy} g_y g_{xxxx} g_{xy}^3 - 4g_{xx} g_{xy}^3 g_x g_{xyyy}\\
&- 4g_{xxx} g_{xy}^3 g_x g_{yyy} - 14g_{xx}^2 g_y g_{yyy} g_{xy} g_{xxy} - g_{xx}^2 g_y g_{yyy} g_{xxx} g_{yy}\\
&+ 2g_{xx} g_{yy}^2 g_x g_{xyy} g_{xxx} + 15g_{xx}^2 g_{yy} g_y g_{xxy} g_{xyy} + 7g_{xx} g_{yy}^2 g_y g_{xxy} g_{xxx}\\
&- 30g_{xx} g_{yy} g_y g_{xxy}^2 g_{xy} + 48g_{xx} g_y g_{xy}^2 g_{xyy} g_{xxy} + 8g_{xy}^4 g_y g_{xxxy}\\
&- 39g_{xx} g_{yy} g_x g_{xyy} g_{xy} g_{xxy} - 12g_{xx} g_y g_{xy} g_{xyy} g_{xxx} g_{yy} - 3g_{xx} g_{xy} g_x g_{yyy} g_{xxx} g_{yy}\\
&- 35g_x g_{xxx} g_{yy}^2 g_{xy} g_{xxy} + 10g_{xx} g_{xy}^2 g_x g_{yyy} g_{xxy} - 7g_{xx}^2 g_{xy} g_x g_{yyy} g_{xyy}\\
&+ 28g_{yy} g_y g_{xxx} g_{xy}^2 g_{xxy} + 7g_x g_{xxx}^2 g_{yy}^3 + 4g_{xx}^2 g_{xxy} g_x g_{yyy} g_{yy}\\
&+ 4g_{xx}^2 g_{xy} g_x g_{xyyy} g_{yy} + 4g_{yy}^2 g_y g_{xxxx} g_{xy} g_{xx} + 8g_{xx} g_y g_{yyy} g_{xxx} g_{xy}^2\\
&+ 12g_{xx}^2 g_y g_{xy} g_{xxyy} g_{yy} + 12g_{yy}^2 g_x g_{xy} g_{xxxy} g_{xx} + 26g_x g_{xyy} g_{xy}^2 g_{xxx} g_{yy}\\
&- 12g_{xy}^3 g_y g_{xxy}^2 + 8g_x g_{xxyy} g_{xy}^4 - 4g_{xy}^2 g_y g_{xxxy} g_{xx} g_{yy} - 4g_x g_{xxyy} g_{xy}^2 g_{xx} g_{yy})
\end{aligned}
$$

e

$$
\begin{aligned}
\partial_y \xi_1 = \frac{1}{16\mathsf{d}^{5/4}}(&-12g_{xy}^3 g_{xxyy} g_{yy} - 12g_{xx} g_{xy} g_{yyy} g_{xxy} g_{yy} - 12g_{xy}^3 g_{xyy}^2 + 8g_{xy}^4 g_{xyyy}\\
&- 7g_{xx}^2 g_{xy} g_{yyy}^2 + 7g_{xxx} g_{yy}^3 g_{xxy} - 21g_{xy} g_{xxy}^2 g_{yy}^2 - 4g_{xx}^2 g_{xyyy} g_{yy}^2 - 16g_{xxy} g_{xy}^3 g_{yyy}\\
&- -4g_{xx} g_{xy}^3 g_{yyyy} - 4g_{xxxy} g_{yy}^3 g_{xx} + 4g_{xxxy} g_{yy}^2 g_{xy}^2 + 15g_{xx} g_{xyy} g_{yy}^2 g_{xxy}\\
&+ 7g_{xx}^2 g_{xyy} g_{yy} g_{yyy} - 30g_{xx} g_{xyy}^2 g_{yy} g_{xy} + 28g_{xx} g_{xy}^2 g_{yyy} g_{xyy} - g_{xxx} g_{yy}^2 g_{xx} g_{yyy}\\
&- 14g_{xxx} g_{yy}^2 g_{xy} g_{xyy} + 48g_{xy}^2 g_{xxy} g_{yy} g_{xyy} - 4g_{xx} g_{xyyy} g_{yy} g_{xy}^2\\
&+ 4g_{xx}^2 g_{xy} g_{yyyy} g_{yy} + 8g_{xxx} g_{yy} g_{yyy} g_{xy}^2 + 12g_{xy} g_{xxyy} g_{yy}^2 g_{xx}),\\
\partial_y \xi_2 = \frac{1}{16\mathsf{d}^{5/4}}(&-4g_{xy}^3 g_{xxxy} g_{yy} + 12g_{xx}^2 g_{xyy}^2 g_{yy} - 39g_{xx} g_{xyy} g_{yy} g_{xy} g_{xxy}\\
&- 3g_{xx} g_{xy} g_{yyy} g_{xxx} g_{yy} + 8g_{xy}^4 g_{xxyy} + 4g_{xx} g_{xyy} g_{yy}^2 g_{xxx} - 35g_{xx}^2 g_{xy} g_{yyy} g_{xyy}\\
&+ 26g_{xx} g_{xy}^2 g_{yyy} g_{xxy} - 7g_{xxx} g_{yy}^2 g_{xy} g_{xxy} + 10g_{xy}^2 g_{xyy} g_{xxx} g_{yy} - 4g_{xx} g_{xxyy} g_{yy} g_{xy}^2\\
&+ 2g_{xx}^2 g_{xxy} g_{yyy} g_{yy} + 12g_{xx}^2 g_{xy} g_{xyyy} g_{yy} + 4g_{xy} g_{xxxy} g_{yy}^2 g_{xx} + 18g_{xy}^2 g_{xxy}^2 g_{yy}\\
&+ 30g_{xy}^2 g_{xyy}^2 g_{xx} - 24g_{xy}^3 g_{xyy} g_{xxy} - 4g_{xx}^2 g_{xxyy} g_{yy}^2 - 4g_{xxx} g_{xy}^3 g_{yyy} + 7g_{xx}^3 g_{yyy}^2\\
&+ 3g_{xxy}^2 g_{yy}^2 g_{xx} - 4g_{xx}^3 g_{yyyy} g_{yy} + 4g_{xx}^2 g_{yyyy} g_{xy}^2 - 12g_{xx} g_{xy}^3 g_{xyyy}),\\
\partial_y \xi_3 = \frac{1}{16\mathsf{d}^{5/4}}(&-21g_{yy}^2 g_x g_{xy} g_{xxy}^2 + 7g_x g_{xxx} g_{yy}^3 g_{xxy} - 7g_{xx}^2 g_{xy} g_x g_{yyy}^2\\
&+ 30g_{xx} g_y g_{xy}^2 g_{xyy}^2 + 3g_{xx} g_{yy}^2 g_y g_{xxy}^2 - 12g_{xx} g_y g_{xy}^3 g_{xyyy} - 4g_{xx}^2 g_{yy}^2 g_y g_{xxyy}\\
&- 4g_{xx}^2 g_{yy}^2 g_x g_{xyyy} - 4g_{yy} g_y g_{xxxy} g_{xy}^3 - 4g_{yyy} g_y g_{xxx} g_{xy}^3 - 4g_{xx} g_{xy}^3 g_x g_{yyyy}\\
&+ 4g_x g_{xxxy} g_{yy}^2 g_{xy}^2 - 4g_x g_{xxxy} g_{yy}^3 g_{xx} + 4g_{xx}^2 g_y g_{yyyy} g_{xy}^2 - 4g_{xx}^3 g_y g_{yyyy} g_{yy}\\
&+ 12g_{xx}^2 g_y g_{xyyy}^2 g_{yy} - 24g_{xy}^3 g_y g_{xxy} g_{xyy} + 18g_{xy}^2 g_y g_{xxy}^2 g_{yy} - 12g_{yyy} g_x g_{xy}^3 g_{xxyy}\\
&- 16g_{xxy} g_{xy}^3 g_x g_{yyy} + 8g_{xy}^4 g_y g_{xxyy} + 8g_x g_{xyyy} g_{xy}^4 - 39g_{xx} g_{yy} g_y g_{xxy} g_{xy} g_{xyy}
\end{aligned}
$$

$$
\begin{aligned}
&- \ 12 g_{xx} g_{xy} g_x g_{yyy} g_{xxy} g_{yy} - 3 g_{yy} g_y g_{xxx} g_{xy} g_{xx} g_{yyy} - 12 g_x g_{xyy}^2 g_{xy}^3 \\
&+ \ 2 g_{xx}^2 g_y g_{yyy} g_{xxy} g_{yy} - 35 g_{xx}^2 g_y g_{yyy} g_{xy} g_{xyy} + 15 g_{xx} g_{yy}^2 g_x g_{xyy} g_{xxy} \\
&+ \ 7 g_{xx}^2 g_{yy} g_x g_{xyy} g_{yyy} - 30 g_{xx} g_{yy} g_x g_{xyy}^2 g_{xy} + 28 g_{xx} g_{xy}^2 g_x g_{yyy} g_{xyy} \\
&- \ g_x g_{xxx} g_{yy}^2 g_{xx} g_{yyy} - 14 g_x g_{xxx} g_{yy}^2 g_{xy} g_{xyy} + 48 g_{yy} g_x g_{xy}^2 g_{xxy} g_{xyy} \\
&- \ 7 g_{yy}^2 g_y g_{xxx} g_{xy} g_{xxy} + 10 g_{yy} g_y g_{xxx} g_{xy}^2 g_{xyy} + 26 g_{xy}^2 g_y g_{xxy} g_{xx} g_{yyy} \\
&- \ 4 g_{xy}^2 g_y g_{xxyy} g_{xx} g_{yy} - 4 g_x g_{xyyy} g_{xy}^2 g_{xx} g_{yy} + 4 g_{xx}^2 g_{xy} g_x g_{yyyy} g_{yy} \\
&+ \ 4 g_{yy}^2 g_y g_{xxxy} g_{xy} g_{xx} + 4 g_{yy}^2 g_y g_{xxx} g_{xyy} g_{xx} + 12 g_{xx}^2 gy g_{xy} g_{xyyy} g_{yy} \\
&+ \ 8 g_x g_{xxx} g_{yy} g_{yyy} g_{xy}^2 + 12 g_{yy}^2 g_x g_{xy} g_{xxyy} g_{xx} + 7 g_{xx}^3 g_y g_{yyy}^2).
\end{aligned}
$$

Notemos que agora precisamos determinar as derivadas até a quarta ordem de g, o que implica que temos que calcular as derivadas de f também até essa ordem. A seguir, exibiremos apenas uma dessas derivadas de g até a quarta ordem para exemplificar o quanto o cálculo direto é caro.

$$
\begin{aligned}
g_{xxyy} \ &= \ -\frac{f_{xxyy}}{f_z} + \frac{f_{xxz} f_{yy} + 2 f_{xz} f_{xyy} + 2 f_{xxy} f_{yz} + 2 f_{xxyz} f_y + 4 f_{xy} f_{xyz}}{f_z^2} \\
&+ \ \frac{f_{xx} f_{yyz} + 2 f_x f_{xyyz}}{f_z^2} \\
&- \ \frac{2 f_{xxz} f_{yz} f_y + 2 f_{xx} f_{yzz} f_y + f_{xx} f_{zz} f_{yy} + 4 f_x f_{xyzz} f_y + f_{xxzz} f_y^2}{f_z^3} \\
&- \ \frac{2(f_{yz} f_{xx} + 2 f_x f_{xyz} + 2 f_{xz} f_{xy} + f_{xxz} f_y) f_{yz} + 4 f_{xy} f_{xzz} f_y + 2 f_{xxy} f_{zz} f_y}{f_z^3} \\
&- \ \frac{2 f_{xz} f_x f_{yyz} - 4(f_{yz} f_x + f_y f_{xz}) f_{xyz} - 2 f_{zz} f_x f_{xyy} + 2 f_{xzz} f_x f_{yy} + f_x^2 f_{yyzz}}{f_z^3} \\
&- \ \frac{2 f_{xz}(2 f_{yz} f_{xy} + f_x f_{yyz} + f_{xz} f_{yy} + 2 f_{xyz} f_y) - 4 f_x f_{yzz} f_{xy} - 2 f_{zz} f_{xy}^2}{f_z^3} \\
&+ \ \frac{4 f_{zz}(f_{yz} f_x + f_y f_{xz}) f_{xy} + 2 f_{xz} f_x f_{zz} f_{yy} + 2 f_{xx} f_{zz} f_{yz} f_y + 4 f_y f_{zz} f_x f_{xyz}}{f_z^4} \\
&+ \ \frac{2(f_{yz} f_{xx} + 2 f_x f_{xyz} + 2 f_{xz} f_{xy} + f_{xxz} f_y) f_{zz} f_y + 4 f_x f_{yzz}(f_{yz} f_x + f_y f_{xz})}{f_z^4} \\
&+ \ \frac{f_x^2 f_{zzz} f_{yy} + 4 f_{xz} f_x f_{yzz} f_y + 2 f_x^2 f_{yzzz} f_y + f_{xxz} f_{zz} f_y^2}{f_z^4} \\
&+ \ \frac{2 f_{zz} f_x(2 f_{yz} f_{xy} + f_x f_{yyz} + f_{xz} f_{yy} + 2 f_{xyz} f_y) + 4(f_{yz} f_x + f_y f_{xz}) f_{xzz} f_y}{f_z^4} \\
&+ \ \frac{-2 f_{xz}(-4 f_{yz} f_y f_{xz} - 2 f_{zz} f_y f_{xy} - f_{xzz} f_y^2 - 2 f_x f_{yzz} f_y - 2 f_{yz}^2 f_x - f_{zz} f_x f_{yy})}{f_z^4} \\
&+ \ \frac{-2(-2 f_{xz}^2 f_y - f_x^2 f_{yzz} - f_{zz} f_y f_{xx} - 2 f_{zz} f_x f_{xy} - 4 f_{xz} f_{yz} f_x - 2 f_x f_{xzz} f_y) f_{yz}}{f_z^4} \\
&+ \ \frac{f_{zz} f_x^2 f_{yyz} + 2 f_y^2 f_x f_{xzzz} + 4 f_{xzz} f_x f_{yz} f_y + f_{xx} f_y^2 f_{zzz} + 4 f_x f_{zzz} f_y f_{xy}}{f_z^4}
\end{aligned}
$$

$$
\begin{aligned}
&+ \frac{2f_{zz}f_x(-4f_{yz}f_yf_{xz} - 2f_{zz}f_yf_{xy} - f_{xzz}f_y^2 - 2f_xf_{yzz}f_y - 2f_{yz}^2f_x - f_{zz}f_xf_{yy})}{f_z^5} \\
&+ \frac{-6f_{zz}f_x^2f_{yzz}f_y - 2f_x^2f_{zzz}f_{yz}f_y - 6f_y^2f_{zz}f_xf_{xzz}}{f_z^5} \\
&- \frac{2f_{zz}^2(2f_yf_{zz}f_xf_{xy} + f_{yz}f_x + f_yf_{xz}) - 2(f_x^2f_{zzz}f_y + 6f_{xz}f_yf_{zz}f_x + 3f_{zz}f_x^2f_{yz})f_{yz}}{f_z^5} \\
&+ \frac{-2f_{xz}f_xf_y^2f_{zzz} - f_{zz}^2f_x^2f_{yy} - 4f_{xz}f_xf_{zz}f_{yz}f_y - f_{xx}f_{zz}^2f_y^2}{f_z^5} \\
&+ \frac{-4f_xf_{zzz}f_y(f_{yz}f_x + f_yf_{xz}) - 2f_{xz}(3f_{xz}f_{zz}f_y^2 + 6f_{yz}f_yf_{zz}f_x + f_y^2f_xf_{zzz})}{f_z^5} \\
&+ \frac{2(-2f_{xz}^2f_y - f_x^2f_{yzz} - f_{zz}f_yf_{xx} - 2f_{zz}f_xf_{xy} - 4f_{xz}f_{yz}f_x - 2f_xf_{xzz}f_y)f_{zz}f_y}{f_z^5} \\
&+ \frac{-f_x^2f_y^2f_{zzzz}}{f_z^5} \\
&+ \frac{6f_x^2f_{zzz}f_y^2f_{zz} + 8f_{xz}f_{zz}^2f_xf_y^2 + 8f_{zz}^2f_x^2f_yf_{yz}}{f_z^6} \\
&+ \frac{4f_{zz}^2f_yf_x(f_{yz}f_x + f_yf_{xz}) + 2f_{zz}f_x(3f_{xz}f_{zz}f_y^2 + 6f_{yz}f_yf_{zz}f_x + f_y^2f_xf_{zzz})}{f_z^6} \\
&+ \frac{2(f_x^2f_{zzz}f_y + 6f_{xz}f_yf_{zz}f_x + 3f_{zz}f_x^2f_{yz})f_{zz}f_y}{f_z^6} - \frac{15f_y^2f_{zz}^3f_x^2}{f_z^7}.
\end{aligned}
$$

E por fim, resta calcular os coeficientes $(b_{i,j})_{1\leq i,j\leq 2}$ do operador forma $\mathcal{S}$ definidos no capítulo 2

$$
\begin{aligned}
b_{11} &= 12g_{xy}^3g_{xxxy}g_{yy} - 3g_{xx}^2g_{xyy}^2g_{yy} + 39g_{xx}g_{xyy}g_{yy}g_{xy}g_{xxy} + 3g_{xx}g_{xy}g_{yyy}g_{xxx}g_{yy} \\
&- 7g_{xxx}^2g_{yy}^3 - 8g_{xy}^4g_{xxyy} - 2g_{xx}g_{xyy}g_{yy}^2g_{xxx} + 7g_{xx}^2g_{xy}g_{yyy}g_{xyy} - 10g_{xx}g_{xy}^2g_{yyy}g_{xxy} \\
&+ 35g_{xxx}g_{yy}^2g_{xy}g_{xxy} - 26g_{xy}^2g_{xyy}g_{xxx}g_{yy} + 4g_{xx}g_{xxyy}g_{yy}g_{xy}^2 - 4g_{xx}^2g_{xxy}g_{yyy}g_{yy} \\
&- 4g_{xx}^2g_{xy}g_{xyyy}g_{yy} - 12g_{xy}g_{xxxy}g_{yy}^2g_{xx} - 30g_{xy}^2g_{xxy}^2g_{yy} - 18g_{xy}^2g_{xyy}^2g_{xx} \\
&+ 24g_{xy}^3g_{xyy}g_{xxy} + 4g_{xx}^2g_{xxyy}g_{yy}^2 + 4g_{xxx}g_{xy}^3g_{yyy} + 4g_{xx}g_{xy}^3g_{xyyy} \\
&+ 4g_{xxxx}g_{yy}^3g_{xx} - 4g_{xxxx}g_{yy}^2g_{xy}^2 - 12g_{xxy}^2g_{yy}^2g_{xx}, \\
b_{12} &= 7g_{xy}g_{xxx}^2g_{yy}^2 + 12g_{xy}^3g_{xx}g_{xxyy} + 12g_{xxy}^2g_{xy}^3 - 8g_{xxxy}g_{xy}^4 + 21g_{xy}g_{xx}^2g_{xyy}^2 \\
&- 7g_{xx}^3g_{yyy}g_{xyy} + 4g_{xy}^3g_{xxxx}g_{yy} + 16g_{xy}^3g_{xxx}g_{xyy} + 4g_{xx}^3g_{xyyy}g_{yy} - 4g_{xx}^2g_{xyyy}g_{xy}^2 \\
&+ 4g_{xxxy}g_{xx}^2g_{yy}^2 - 28g_{xxy}g_{xy}^2g_{xxx}g_{yy} - 48g_{xxy}g_{xy}^2g_{xx}g_{xyy} + g_{xx}^2g_{yyy}g_{xxx}g_{yy} \\
&+ 14g_{xx}^2g_{yyy}g_{xy}g_{xxy} - 7g_{xxy}g_{xx}g_{yy}^2g_{xxx} - 15g_{xxy}g_{xx}^2g_{yy}g_{xyy} + 30g_{xxy}^2g_{xx}g_{yy}g_{xy} \\
&+ 4g_{xxxy}g_{xy}^2g_{xx}g_{yy} - 4g_{xy}g_{xxxx}g_{yy}^2g_{xx} - 12g_{xy}g_{xx}^2g_{xxyy}g_{yy} \\
&- 8g_{xx}g_{yyy}g_{xxx}g_{xy}^2 + 12g_{xy}g_{xxx}g_{yy}g_{xx}g_{xyy}, \\
b_{21} &= 12g_{xy}^3g_{xxyy}g_{yy} + 12g_{xx}g_{xy}g_{yyy}g_{xxy}g_{yy} + 12g_{xy}^3g_{xyy}^2 - 8g_{xy}^4g_{xyyy} + 7g_{xx}^2g_{xy}g_{yyy}^2 \\
&- 7g_{xxx}g_{yy}^3g_{xxy} + 21g_{xy}g_{xxy}^2g_{yy}^2 + 4g_{xx}^2g_{xyyy}g_{yy}^2 + 16g_{xxy}g_{xy}^3g_{yyy} + 4g_{xx}g_{xy}^3g_{yyyy}
\end{aligned}
$$

$$
\begin{aligned}
&+ \; 4g_{xxxy}g_{yy}^3g_{xx} - 4g_{xxxy}g_{yy}^2g_{xy}^2 - 15g_{xx}g_{xyy}g_{yy}^2g_{xxy} - 7g_{xx}^2g_{xyy}g_{yy}g_{yyy} \\
&+ \; 30g_{xx}g_{xyy}^2g_{yy}g_{xy} - 28g_{xx}g_{xy}^2g_{yyy}g_{xyy} + g_{xxx}g_{yy}^2g_{xx}g_{yyy} + 14g_{xxx}g_{yy}^2g_{xy}g_{xyy} \\
&- \; 48g_{xy}^2g_{xxy}g_{yy}g_{xyy} + 4g_{xx}g_{xyyy}g_{yy}g_{xy}^2 - 4g_{xx}^2g_{xy}g_{yyyy}g_{yy} \\
&- \; 8g_{xxx}g_{yy}g_{yyy}g_{xy}^2 - 12g_{xy}g_{xxyy}g_{yy}^2g_{xx}
\end{aligned}
$$

e

$$
\begin{aligned}
b_{22} \; = \; & 4g_{xy}^3g_{xxxy}g_{yy} - 12g_{xx}^2g_{xyy}^2g_{yy} + 39g_{xx}g_{xyy}g_{yy}g_{xy}g_{xxy} + 3g_{xx}g_{xy}g_{yyy}g_{xxx}g_{yy} \\
- \; & 8g_{xy}^4g_{xxyy} - 4g_{xx}g_{xyy}g_{yy}^2g_{xxx} + 35g_{xx}^2g_{xy}g_{yyy}g_{xyy} - 26g_{xx}g_{xy}^2g_{yyy}g_{xxy} \\
+ \; & 7g_{xxx}g_{yy}^2g_{xy}g_{xxy} - 10g_{xy}^2g_{xyy}g_{xxx}g_{yy} + 4g_{xx}g_{xxyy}g_{yy}g_{xy}^2 - 2g_{xx}^2g_{xxy}g_{yyy}g_{yy} \\
- \; & 12g_{xx}^2g_{xy}g_{xyyy}g_{yy} - 4g_{xy}g_{xxxy}g_{yy}^2g_{xx} - 18g_{xy}^2g_{xxy}^2g_{yy} - 30g_{xy}^2g_{xyy}^2g_{xx} \\
+ \; & 24g_{xy}^3g_{xyy}g_{xxy} + 4g_{xx}^2g_{xxyy}g_{yy}^2 + 4g_{xxx}g_{xy}^3g_{yyy} + 12g_{xx}g_{xy}^3g_{xyyy} - 3g_{xxy}^2g_{yy}^2g_{xx} \\
- \; & 7g_{xx}^3g_{yyy}^2 + 4g_{xx}^3g_{yyyy}g_{yy} - 4g_{xx}^2g_{yyyy}g_{xy}^2.
\end{aligned}
$$

Observemos que ao trabalharmos com o *método direto* temos um enorme número de derivadas e isso acarreta erros numéricos. O que pode ser visto claramente nas expressões do cálculo dos coeficientes b_{ij} antes e depois da transformação (seção 3.3.2).