5
Towards a transitive closure approach to $\mathcal{DS}_3$

In order to present a more natural and intuitive way to deal with Stochastic Petri Nets we present a variation of the $\mathcal{DS}_3$ system regarding a transitive closure operator to express the firing sequence of the Stochastic Petri Net program, the $\mathcal{DS}_3^\star$. This operator is primitive from PDL and we show how to use it with Stochastic Petri Net programs.

With the usage of the transitive closure operator it is possible to achieve shorter proofs than in $\mathcal{DS}_3$. The transitive closure behaviour is also naturally mapped into CTMCs.

5.1
Basic definitions

The language of $\mathcal{DS}_3^\star$ is defined as follows.

Propositional symbols: p, q..., where Φ is the set of all propositional symbols

Place names: e.g.: $a, b, c, d \ldots$

Petri Net Composition symbol: $\odot$

PDL operator: * (iteration)

Sequence of names: $S = \{\epsilon, s_1, s_2, \ldots\}$, where ϵ is the empty sequence. We use the notation $s \prec s'$ to denote that all names occurring in s also occur in s'.

Definition 80 $\mathcal{DS}_3^\star$ program

We use π to denote a Stochastic Petri Net program and s a sequence of names (the markup of π).

Basic programs: $\pi_b ::= at_1b \mid at_2bc \mid abt_3c$ *where* t_i *is of type* $T_i, i = 1, 2, 3$

Stochastic Petri Net Programs: $\pi ::= s, \pi_b \mid \pi \odot \pi \mid \pi^\star$

Definition 81 $\mathcal{DS}_3^\star$ formula

A $\mathcal{DS}_3^\star$ formula is defined as

$$\varphi ::= p \mid \top \mid \neg\varphi \mid \varphi \wedge \varphi \mid \langle s, \pi\rangle\varphi.$$

We use the standard abbreviations $\bot \equiv \neg\top$, $\varphi \vee \phi \equiv \neg(\neg\varphi \wedge \neg\phi)$, $\varphi \rightarrow \phi \equiv \neg(\varphi \wedge \neg\phi)$ and $[s,\pi]\varphi \equiv \neg\langle s,\pi\rangle\neg\varphi$, and π is a Stochastic Petri Net program.

The firing of a transition in $\mathcal{DS}_3^\star$ is defined according the firing function in Definition 25.

Definition 82 $\mathcal{DS}_3^\star$ frame

A frame for $\mathcal{DS}_3^\star$ is a 5-tuple $\mathcal{F}_3^\star = \langle W, R_\pi, M, (\Pi, \Lambda), \delta\rangle$ such as in $\mathcal{DS}_3$ frame definition (Definition 60) including (let $\pi \in \Pi$) the following.

- *$R_{\pi^\star} = R_\pi^\star$, where $R_\pi^\star$ denotes the reflexive transitive closure of R_π.*
- *$\pi = \pi_1 \odot \pi_2 \odot \cdots \odot \pi_n$, as $R_\pi = \{(w,v) \mid \exists\eta_i, \exists u$ such that $s_i \prec M(u)$ and $wR_{\pi_i}u$ and $\delta(w,\pi_i) = \min(\delta(w,\Pi))$ and $uR_\pi v\}$ where $s_i = f(s,\pi_i)$, for all $1 \leq i \leq n$.*

Definition 83 $\mathcal{DS}_3^\star$ model

A model for $\mathcal{DS}_3^\star$ is a pair $\mathcal{M}_3^\star = \langle \mathcal{F}_3^\star, \mathbf{V}\rangle$, where $\mathcal{F}_3^\star$ is a $\mathcal{DS}_3^\star$ frame and $\mathbf{V}$ is a valuation function $\mathbf{V}\colon \Phi \rightarrow 2^W$.

Lemma 84 *Truth Probability of a Modality*

The probability of $\mathcal{M}_3^\star, w \Vdash \langle s, \pi_b\rangle\varphi$ is (where $s = M(w)$)

$$\Pr(\mathcal{M}_3^\star, w \Vdash \langle s,\pi_b\rangle\varphi \mid \delta(w,\Pi)) = \frac{\delta(w,\pi_b)}{\sum\limits_{\pi_b \in \Pi : f(s,\pi_b) \neq \epsilon} \delta(w,\pi_b)}$$

Proof: This proof is straightforward from relation (4-4) *and Definition 82.* ■

Definition 85 $\mathcal{DS}_3^\star$ semantical notion

Let $\mathcal{M}_3^\star = \langle W, R_\pi, M, (\Pi,\Lambda), \delta, \mathbf{V}\rangle$ be a model for $\mathcal{DS}_3^\star$. The notion of satisfaction of a formula φ in $\mathcal{M}_3^\star$ at a state w, namely $\mathcal{M}_3^\star, w \Vdash \varphi$ is inductively defined as follows.

- *$\mathcal{M}_3^\star, w \Vdash p$ iff $w \in \mathbf{V}(p)$*
- *$\mathcal{M}_3^\star, w \Vdash \top$ always*
- *$\mathcal{M}_3^\star, w \Vdash \neg\varphi$ iff $\mathcal{M}_3^\star, w \nVdash \varphi$*

- $\mathcal{M}_3^\star, w \Vdash \varphi_1 \wedge \varphi_2$ *iff* $\mathcal{M}_3^\star, w \Vdash \varphi_1$ *and* $\mathcal{M}_3^\star, w \Vdash \varphi_2$
- $\mathcal{M}_3^\star, w \Vdash \langle s, \eta \rangle \varphi$ *if there exists* $v \in W$, $wR_\eta v$ *and* $\Pr(\mathcal{M}_3^\star, v \Vdash \langle s, \eta_b \rangle \varphi \mid \delta(v, \Pi)) > 0$
- $\mathcal{M}_3^\star, w \Vdash \langle s, \eta^\star \rangle \varphi$ *iff* $\mathcal{M}_3^\star, w \Vdash \varphi$ *or if there exists* $v \in W$, $wR_\eta^\star v$ *and* $\mathcal{M}_3^\star, v \Vdash \langle s, \eta \rangle \varphi$

If φ *is satisfied in all states of* $\mathcal{M}_3^\star$ *then* φ *is satisfied in* $\mathcal{M}_3^\star$, *namely* $\mathcal{M}_3^\star \Vdash \varphi$; *and if* φ *is valid in any model then* φ *is valid, denoted by* $\Vdash \varphi$.

5.2 Axiomatic system

We consider the following set of axioms and rules, where p and q are propositional symbols, φ and ψ are formulae, $\eta = \eta_1 \odot \eta_2 \odot \cdots \odot \eta_n$ is a Petri Net program and π is a Marked Petri Net program.

(PL) Enough propositional logic tautologies

(K) $[s, \pi](p \to q) \to ([s, \pi]p \to [s, \pi]q)$

(Rec) $\langle s, \pi^\star \rangle p \leftrightarrow p \vee \langle s, \pi \rangle \langle s, \pi^\star \rangle p$

(FP) $p \wedge [s, \pi^\star](p \to [s, \pi]p) \to [s, \pi^\star]p$

($\mathbf{PC}_3^\star$) $\langle s, \pi \rangle \varphi \leftrightarrow \langle s, \pi_1 \rangle \langle s_1, \pi^\star \rangle \varphi \vee \langle s, \pi_2 \rangle \langle s_2, \pi^\star \rangle \varphi \vee \cdots \vee \langle s, \pi_n \rangle \langle s_n, \pi^\star \rangle \varphi$, where $s_i = f(s, \pi_i)$, for all $1 \leq i \leq n$

($\mathbf{R}_{3_\epsilon}^\star$) $[s, \pi] \bot$, if $f(s, \pi) = \epsilon$

(Sub) If $\Vdash \varphi$, then $\Vdash \varphi^\sigma$, where σ uniformly substitutes propositional symbols by arbitrary formulae.

(MP) If $\Vdash \varphi$ and $\Vdash \varphi \to \psi$, then $\Vdash \psi$.

(Gen) If $\Vdash \varphi$, then $\Vdash [s, \pi]\varphi$.

5.3
Soundness and completeness

In this section we prove soundness and completeness of $\mathcal{DS}_3^\star$ for normalised Stochastic Petri Nets as in section 4.4. The axioms **(PL)** and **(K)** and the rules **(Sub)**, **(MP)** and **(Gen)** are standard in the modal logic literature. We present the cases that are not proved for $\mathcal{DS}_3$, that is, proofs for $\mathcal{DS}_3^\star$ that are not proofs for $\mathcal{DS}_3$ (section 4.4). Notice that a proof for $\mathcal{DS}_3$ is a proof for $\mathcal{DS}_3^\star$.

Lemma 86 *Validity of $\mathcal{DS}_3^\star$ axioms*

1. $\Vdash$ **Rec**

   Proof: Supose that there is a world w from a model $\mathcal{M}_3^\star = \langle W, R_\pi, M, (\Pi, \Lambda), \delta, \mathbf{V}\rangle$ where Rec is false. For Rec to be false in w, there are two cases:

   (a) *Suppose $\mathcal{M}_3^\star$,$w \Vdash \langle s, \pi^\star\rangle p$ (1) and*
   $\mathcal{M}_3^\star$,$w \nVdash p \vee \langle s, \pi\rangle\langle s, \pi^\star\rangle p$ (2)
   Applying Definition 85 in (1) we have that $\mathcal{M}_3^\star$,$w \Vdash \langle s, \pi\rangle\langle s, \pi^\star\rangle p$ (3).
   Applying Definition 85 again we have that $\mathcal{M}_3^\star$,$w \Vdash p \vee \langle s, \pi\rangle\langle s, \pi^\star\rangle p$, which contradicts (2).

   (b) *Suppose $\mathcal{M}_3^\star$,$w \nVdash \langle s, \pi^\star\rangle p$ (1) and*
   $\mathcal{M}_3^\star$,$w \Vdash p \vee \langle s, \pi\rangle\langle s, \pi^\star\rangle p$ (2)
   Applying Definition 85 in we have $\mathcal{M}_3^\star$,$w \Vdash p \vee \langle s, \pi\rangle p$ (3).
   Using the axiom (Gen) and then (K) in (3) we have that $\mathcal{M}_3^\star$,$w \Vdash [s, \pi]p \vee \langle s, \pi\rangle p$, that, using Definition 82, we have that $\mathcal{M}_3^\star$,$w \Vdash \langle s, \pi\rangle p \vee \langle s, \pi\rangle p$, which by Definition 85 implies that $\mathcal{M}_3^\star$,$w \Vdash \langle s, \pi\rangle p$. (4)
   But by (1) and Definition 85 we can not have (4).
   Then, there is a contradiction.

   So, Rec is valid. ■

2. $\Vdash$ **FP**

   Proof: Supose that there is a world w from a model $\mathcal{M}_3^\star = \langle W, R_\pi, M, (\Pi, \Lambda), \delta, \mathbf{V}\rangle$ where FP is false.
   So, $\mathcal{M}_3^\star$,$w \Vdash p \wedge [s, \pi^\star](p \rightarrow [s, \pi]p)$ (1) and
   $\mathcal{M}_3^\star$,$w \nVdash [s, \pi^\star]p$ (2).
   By (1) and Definition 85 we have that $\mathcal{M}_3^\star$,$w \Vdash p$ and $\mathcal{M}_3^\star$,$w \Vdash [s, \pi^\star](p \rightarrow [s, \pi]p)$ (4)

Applying (MP) in (4) we have that $\mathcal{M}_3^\star, w \Vdash [s,\pi^\star]([s,\pi]p)$*, which contradicts (2).*
So, FP is valid. ■

3. $\Vdash \mathbf{PC}_3^\star$

Proof: Supose that there is a world w *from a model* $\mathcal{M}_3^\star = \langle W', R_\pi, M, (\Pi, \Lambda), \delta, \mathbf{V}\rangle$ *where* $PC_3^\star$ *is false. For* $PC_3^\star$ *to be false in* w*, there are two cases:*

(a) Suppose $\mathcal{M}_3^\star, w \Vdash \langle s,\pi\rangle\varphi$ *(1) and*
$\mathcal{M}_3^\star, w \nVdash \langle s,\pi_1\rangle\langle s_1,\pi^\star\rangle\varphi \vee \langle s,\pi_2\rangle\langle s_2,\pi^\star\rangle\varphi \vee \cdots \vee \langle s,\pi_n\rangle\langle s_n,\pi^\star\rangle\varphi$ *(2)*
(1) iff there is a world v *such that* $wR_\pi v$ *and* $\Pr(\mathcal{M}_3^\star, v \Vdash \langle s,\pi_b\rangle\varphi \mid \delta(v,\Pi)) > 0$ *(3).*
By Definition 82 $R_\pi = (R_{\pi_1} \circ R_{\pi^\star}) \cup \cdots \cup (R_{\pi_n} \circ R_{\pi^\star})$ *which implies that for some* $1 \leq i \leq n$*,* $w(R_{\pi_i} \circ R_{\pi^\star})v$*. Using Definition 85 twice we obtain* $\mathcal{M}_3^\star, w \Vdash \langle s,\pi_i\rangle\langle s_i,\pi^\star\rangle\varphi$*.*
This implies $\mathcal{M}_3^\star, w \Vdash \langle s,\pi_1\rangle\langle s_1,\pi^\star\rangle\varphi \vee \langle s,\pi_2\rangle\langle s_2,\pi^\star\rangle\varphi \vee \cdots \vee \langle s,\pi_n\rangle\langle s_n,\pi^\star\rangle\varphi$*, which contradicts (2).*

(b) Suppose $\mathcal{M}_3^\star, w \nVdash \langle s,\pi\rangle\varphi$ *(1) and*
$\mathcal{M}_3^\star, w \Vdash \langle s,\pi_1\rangle\langle s_1,\pi^\star\rangle\varphi \vee \langle s,\pi_2\rangle\langle s_2,\pi^\star\rangle\varphi \vee \cdots \vee \langle s,\pi_n\rangle\langle s_n,\pi^\star\rangle\varphi$ *(2)*
(2) iff $\Pr(\mathcal{M}_3^\star, w \Vdash \langle s,\pi_i\rangle\langle s_i,\pi^\star\rangle\varphi \mid \delta(w,\Pi)) > 0$*, for some* i *such that* $1 \leq i \leq n$ *and* $s_i = f(s,\pi_i)$ *(3).*
(3) iff there is a world u *such that* $wR_{\pi_i}u$ *and* $\Pr(\mathcal{M}_3^\star, w \Vdash \langle s_i,\pi^\star\rangle\varphi \mid \delta(w,\Pi)) > 0$*, for some* $1 \leq i \leq n$ *and* $s_i = f(s,\pi_i)$*.*
By (3) there is a world v *such that* $uR_\pi v$*,* $s_i \preceq M(u)$ *and* $\mathcal{M}_3^\star, v \Vdash \varphi$ *(4).*
But this implies that $w(R_{\pi_i} \circ R_{\pi^\star})v$ *and consequently* $w((R_{\pi_1} \circ R_{\pi^\star}) \cup \cdots \cup (R_{\pi_n} \circ R_{\pi^\star}))v$ *(5).*
By Definition 82, (4) and (5) we have $wR_\pi v$ *and* $s \preceq M(w)$ *and* $M_3^\star, v \Vdash \varphi$*. Thus,* $\mathcal{M}_3^\star, w \Vdash \langle s,\pi\rangle\varphi$*, which contradicts (1).*

So, PC_3 *is valid.* ■

4. $\Vdash \mathbf{R}_{3_\epsilon}^\star$

Proof: Suppose a $\mathcal{DS}_3^\star$ *model* $\mathcal{M}_3^\star = \langle W', R_\pi, M, (\Pi, \Lambda), \delta, \mathbf{V}\rangle$*.*
Suppose a program π *and a sequence* s *such that* $f(s,\pi) = \epsilon$ *and that* $\mathcal{M}_3^\star, \nVdash [s,\pi]\bot$ *(1).*
Then $\Pr(\mathcal{M}_3^\star, w \Vdash [s,\pi]\bot \mid \delta(w,\Pi)) \neq 1$*, so in some world* v *such that* $wR_\pi v$ *we have that* $\mathcal{M}_3^\star, v \Vdash \top$ *(2).*
But $f(s,\pi) = \epsilon$ *and thus by Definition 82* $(w,v) \notin R_\pi$*, which is a*

contradiction.
So, R_{3_ϵ} is valid. ■

As for $\mathcal{DS}_3$, the completeness proof goes as in the work of Blackburn et al. (2001); Harel et al. (2000) and Goldblatt (1992*b*).

Definition 87 Atoms

Let Γ be a set of formulae. A set of formulae $\mathcal{A}$ is said to be an atom of Γ if it is a maximal consistent subset of $FL(\Gamma)$. The set of all atoms of Γ is denoted by $At(\Gamma)$.

Lemma 88 *Let Γ be a set of formulae. If $\varphi \in FL(\Gamma)$ and φ is consistent then there exists an atom $\mathcal{A} \in At(\Gamma)$ such that $\varphi \in \mathcal{A}$.*
Proof: We can construct the atom $\mathcal{A}$ as follows. First, we enumerate the elements of $FL(\Gamma)$ as $\phi_1, \ldots, \phi_n$. We start the construction making $\mathcal{A}_1 = \{\varphi\}$, then for $1 < i < n$, we know that $\vdash \bigwedge \mathcal{A}_i \leftrightarrow (\bigwedge \mathcal{A}_i \wedge \phi_{i+1}) \vee (\bigwedge \mathcal{A}_i \wedge \neg\phi_{i+1})$ is a tautology and therefore either $\mathcal{A}_i \wedge \phi_{i+1}$ or $\mathcal{A}_i \wedge \neg\phi_{i+1}$ is consistent. We take $\mathcal{A}_{i+1}$ as the union of $\mathcal{A}_i$ with the consistent member of the previous disjunction. At the end, we make $\mathcal{A} = \mathcal{A}_n$. ■

Definition 89 Canonic relations

Let Γ be a set of formulae and $\langle s, \eta\rangle\varphi \in At(\Gamma)$. The canonic relations over Γ S^Γ_η on $At(\Gamma)$ are defined as $\mathcal{A}S^\Gamma_\eta\mathcal{B}$, iff $\bigwedge \mathcal{A} \wedge \langle s, \eta\rangle \bigwedge \mathcal{B}$ is consistent.

Definition 90 Canonic marking

Let $\{\langle s_1, \eta_1\rangle\varphi_1, \ldots, \langle s_n, \eta_n\rangle\varphi_n\}$ be the set of all formulae in the form $\langle r, \eta\rangle\varphi$ occurring in an atom $\mathcal{A}$. We define the canonic marking of $\mathcal{A}$, says $M(\mathcal{A})$, as follows

1. *$M(\mathcal{A}) := s_1; s_2; \ldots; s_n$;*
2. *for all basic programs π_b, if $\mathcal{A}S^\Gamma_\eta\mathcal{B}$ and $f(M(\mathcal{A}), \pi_b) \not\preceq M(\mathcal{B})$, then add to $M(\mathcal{B})$ as few as possible names to make $f(M(\mathcal{A}), \pi) \preceq M(\mathcal{B})$.*

Lemma 91 *Suppose $\mathcal{A}$ and $\mathcal{B}$ two atoms of a model $\mathcal{M}^\star_3 = \langle W, R_\pi, M, (\Pi, \Lambda), \delta\rangle$. There is an $R \preceq S$ such that $f(M(\mathcal{A}), \pi_b) \preceq M(\mathcal{B}) \circ R$ where $\pi_b \in \Pi$ is a basic program.*
Proof: For $M(\mathcal{A})$ and $M(\mathcal{B})$ there are three cases.

(i) *$f(M(\mathcal{A}), \pi_b) \preceq M(\mathcal{B})$ and $R = \emptyset$.*

(ii) *$\{f(M(\mathcal{A}), \pi_b)\} \cap M(\mathcal{B}) = \emptyset$ and $R = \{f(M(\mathcal{A}), \pi_b)\}$.*

(iii) $\{f(M(\mathcal{A}),\pi_b)\} \cap M(\mathcal{B}) \neq \emptyset$ *and* $R = \{f(M(\mathcal{A}),\pi_b)\} \setminus (\{f(M(\mathcal{A}),\pi_b)\} \cap M(\mathcal{B}))$.

■

Definition 92 Canonic Model

Let Γ *be a set of formulae. The canonic model over* Γ *is a tuple* $\mathcal{M}_3^{\star^\Gamma} = \langle At(\Gamma), S_\eta^\Gamma, M^\Gamma, (\Pi^\Gamma, \Lambda^\Gamma), \delta^\Gamma, \mathbf{V}^\Gamma\rangle$, *where for all propositional symbols* p *and for all atoms* $\mathcal{A} \in At(\Gamma)$ *we have as follows.*

- S_η^Γ *the canonic relations*
- $M^\Gamma : At(\Gamma) \to S$, *the canonic marking*
- $(\Pi^\Gamma, \Lambda^\Gamma)$ *is a Stochastic Petri Net program*
- δ^Γ: $At(\Gamma) \times \Pi^\Gamma \to \vec{\mathbb{R}}^+$.
- $\mathbf{V}^\Gamma(p) = \{\mathcal{A} \in At(\Gamma) \mid p \in \mathcal{A}\}$ *the canonic valuation*

Lemma 93 *For all basic programs* π_b, *let* $s = M(\mathcal{A})$, S_{π_b} *satisfies*

1. *if* $f(s,\pi_b) \neq \epsilon$, *if* $\mathcal{A}S_{\pi_b}\mathcal{B}$ *then* $f(s,\pi_b) \preceq M(\mathcal{B})$
2. *if* $f(s,\pi) = \epsilon$, *then* $(\mathcal{A},\mathcal{B}) \notin S_{\pi_b}$

Proof: The proof of 1. is straightforward from the Definition 90 and Lemma 91. The proof of 2. is straightforward from the soundness of axiom $R^\star_{3_\epsilon}$. ■

Lemma 94 *Existence Lemma for Canonic Models*

Let $\mathcal{A} \in At(\Gamma)$ *and* $\langle s,\eta\rangle\varphi \in FL(\Gamma)$. *Then,* $\langle s,\eta\rangle\varphi \in \mathcal{A}$ *iff there exists* $\mathcal{B} \in At(\Gamma)$ *such that* $\mathcal{A}S_\eta\mathcal{B}$, $s \preceq M(\mathcal{A})$ *and* $\varphi \in \mathcal{B}$.

Proof: This proof goes in two steps.

1. *Suppose* $\langle s,\eta\rangle\varphi \in \mathcal{A}$. *By Definition 90 and Lemma 91* $s \preceq M(\mathcal{A})$ *By Definition 87, we have that* $\bigwedge\mathcal{A}\wedge\langle s,\eta\rangle\varphi$ *is consistent. Using the tautology* $\vdash \varphi \leftrightarrow ((\varphi\wedge\phi)\vee(\varphi\wedge\neg\phi))$, *we have that either* $\bigwedge\mathcal{A}\wedge\langle s,\eta\rangle(\varphi\wedge\phi)$ *is consistent or* $\bigwedge\mathcal{A}\wedge\langle s,\eta\rangle(\varphi\wedge\neg\phi)$ *is consistent. So, by the appropriate choice of* ϕ, *for all formulae* $\phi \in FL(\Gamma)$, *we can construct an atom* $\mathcal{B}$ *such that* $\varphi \in \mathcal{B}$ *and* $\bigwedge\mathcal{A}\wedge\langle s,\eta\rangle(\varphi\wedge\bigwedge\mathcal{B})$ *is consistent and by Definition 89 we have that* $\mathcal{A}S_\eta\mathcal{B}$.

2. *Suppose there is* $\mathcal{B}$ *such that* $\varphi \in \mathcal{B}$ *and* $\mathcal{A}S_\eta\mathcal{B}$ *and* $s \preceq M(\mathcal{A})$. *Then* $\bigwedge\mathcal{A}\wedge\langle s,\eta\rangle\bigwedge\mathcal{B}$ *is consistent and also* $\bigwedge\mathcal{A}\wedge\langle s,\eta\rangle\varphi$ *is consistent. But* $\langle s,\eta\rangle\varphi \in FL(\Gamma)$ *and, by maximality,* $\langle s,\eta\rangle\varphi \in \mathcal{A}$.

So, this lemma holds. ■

Lemma 95 *Truth Lemma for Canonic Models*

Let $\mathcal{M}_3^\star = \langle W, S_\eta, M, (\Pi, \Lambda), \delta, \mathbf{V}\rangle$ *be a finite canonic model constructed over a formula* ϕ*. For all atoms* $\mathcal{A}$ *and all* $\varphi \in FL(\phi)$*,* $\mathcal{M}_3^\star, \mathcal{A} \Vdash \varphi$ *iff* $\varphi \in \mathcal{A}$*.*
Proof: The proof is by induction on the construction of φ*:* $\mathcal{M}_3^\star, \mathcal{A} \Vdash \varphi$ *iff* $\varphi \in \mathcal{A}$*.*

- *Atomic formulae: the proof is straightforward from the definition of* $\mathbf{V}$*.*
- *Boolean formulae: the proof is straightforward from the semantical notion of* $\mathcal{DS}_3^\star$ *(Definition 85).*
- *Modality* $\langle s, \tau\rangle$*, for* $\tau \in \{\pi, \pi_1 \odot \cdots \odot \pi_n, \eta^\star\}$*: there are two cases.*

  1. *Suppose* $\mathcal{M}_3^\star, \mathcal{A} \Vdash \langle s, \tau\rangle\varphi$*, then there exists* $\mathcal{A}'$ *such that* $\mathcal{A}S_\tau\mathcal{A}'$*,* $s \preceq M(\mathcal{A})$ *and* $\mathcal{M}_3^\star, \mathcal{A}' \Vdash \varphi$*. By the induction hypothesis we know that* $\varphi \in \mathcal{A}'$*, and by Lemma 94 we have* $\langle s, \tau\rangle\varphi \in \mathcal{A}$*.*
  2. *Suppose* $\mathcal{M}_3^\star, \mathcal{A} \nVdash \langle s, \tau\rangle\varphi$*, by the definition of satisfaction (Definition 85) we have* $\mathcal{M}_3^\star, \mathcal{A} \Vdash \neg\langle s, \tau\rangle\varphi$*. Then for all* $\mathcal{A}'$*,* $\mathcal{A}S_\tau\mathcal{A}'$ *and* $s \preceq M(\mathcal{A})$ *implies* $\mathcal{M}_3^\star, \mathcal{A}' \nVdash \varphi$*. By the induction hypothesis we know that* $\varphi \notin \mathcal{A}'$*, and by Lemma 94 we have* $\langle s, \tau\rangle\varphi \notin \mathcal{A}$*.*

■

Lemma 96 *Let* $\mathcal{A}, \mathcal{B} \in At(\Gamma)$*. Then if* $\mathcal{A}S_{\eta^\star}\mathcal{B}$ *then* $\mathcal{A}S_\eta^\star\mathcal{B}$*.*
Proof: Suppose $\mathcal{A}S_{\eta^\star}\mathcal{B}$*. Let* $\mathbf{C} = \{\mathcal{C} \in At(\Gamma) \mid \mathcal{A}S_\eta^\star\mathcal{C}\}$ *where* $C_1, C_2, \ldots, C_n$ *is an enumeration of* $\mathbf{C}$*. We want to show that* $\mathcal{B} \in \mathbf{C}$*. Let* $\mathbf{C}_\vee^\wedge = (\bigwedge \mathcal{C}_1 \vee \cdots \vee \bigwedge \mathcal{C}_n)$ *and* $s = s_1, \ldots, s_n$*, where* $s_i = M(\mathcal{C}_i)$*.*

We have that, $\mathbf{C}_\vee^\wedge \wedge \langle s, \eta\rangle\neg\mathbf{C}_\vee^\wedge$ *is inconsistent, otherwise for some* $\mathcal{D} \in At(\Gamma)$ *not reachable from* $\mathcal{A}$*,* $\mathbf{C}_\vee^\wedge \wedge \langle s, \eta\rangle \bigwedge \mathcal{D}$ *would be consistent, and for some* $\mathcal{C}_i$*,* $\bigwedge \mathcal{C}_i \wedge \langle s_i, \eta\rangle \bigwedge \mathcal{D}$ *was also consistent, which would mean that* $\mathcal{D} \in \mathbf{C}$*, which is not the case. From a similar reasoning we know that* $\bigwedge \mathcal{A} \wedge \langle s, \eta\rangle\neg\mathbf{C}_\vee^\wedge$ *is also inconsistent and hence* $\Vdash \bigwedge \mathcal{A} \to [s, \eta]\mathbf{C}_\vee^\wedge$ *is a theorem.*

As $\mathbf{C}_\vee^\wedge \wedge \langle s, \eta\rangle\neg\mathbf{C}_\vee^\wedge$ *is inconsistent, so its negation is a theorem* $\Vdash \neg(\mathbf{C}_\vee^\wedge \wedge \langle s, \eta\rangle\neg\mathbf{C}_\vee^\wedge)$ *and also* $\Vdash (\mathbf{C}_\vee^\wedge \to [s, \eta]\mathbf{C}_\vee^\wedge)$ *(1),*
therefore we can apply generalization, $\Vdash [s, \eta^\star](\mathbf{C}_\vee^\wedge \to [s, \eta]\mathbf{C}_\vee^\wedge)$*. Using axiom (FP), we have* $\Vdash ([s, \eta]\mathbf{C}_\vee^\wedge \to [s, \eta^\star]\mathbf{C}_\vee^\wedge)$ *and by (1) we obtain* $\Vdash (\mathbf{C}_\vee^\wedge \to [s, \eta^\star]\mathbf{C}_\vee^\wedge)$*. As* $\Vdash \bigwedge \mathcal{A} \to [s, \eta]\mathbf{C}_\vee^\wedge$ *is a theorem, then* $\Vdash \bigwedge \mathcal{A} \to [s, \eta^\star]\mathbf{C}_\vee^\wedge$*. By supposition,* $\bigwedge \mathcal{A} \wedge \langle s, \eta^\star\rangle \bigwedge \mathcal{B}$ *is consistent and so is* $\bigwedge \mathcal{B} \wedge \mathbf{C}_\vee^\wedge$*. Therefore, for at least one* $\mathcal{C} \in \mathbf{C}$*, we know that* $\bigwedge \mathcal{B} \wedge \bigwedge \mathcal{C}$ *is consistent. By maximality, we have that* $\mathcal{B} = \mathcal{C}$*. And by the definition of* $\mathbf{C}_\vee^\wedge$*, we have* $\mathcal{A}S_\eta^\star\mathcal{B}$*.* ■

Definition 97 Proper Canonic Model

Let Γ be a set of formulae. The proper canonic model over Γ is a tuple $\mathcal{N}^\Gamma = \langle At(\Gamma), R^\Gamma_\eta, M^\Gamma, (\Pi^\Gamma, \Lambda^\Gamma), \delta^\Gamma, \mathbf{V}^\Gamma\rangle$, where for all propositional symbols p and for all atoms $\mathcal{A} \in At(\Gamma)$ we have

- $\mathbf{V}^\Gamma(p) = \{\mathcal{A} \in At(\Gamma) \mid p \in \mathcal{A}\}$, *the canonic valuation;*
- M^Γ *is the canonic marking;*
- $R^\Gamma_{\eta_b} := S^\Gamma_{\eta_b}$, *for every basic program η_b;*
- *we inductively define the binary relation R^Γ_η follows (for the sake of clarity we ommit the Γ subscripts)*
  - $R^\Gamma_{\eta^\star} = R^{\Gamma\star}_\eta$
  - $\eta = \eta_1 \odot \eta_2 \odot \cdots \odot \eta_n$
  - $R^\Gamma_\eta = (R^\Gamma_{\eta_1} \circ R^\Gamma_{\eta^\star}) \cup \cdots \cup (R^\Gamma_{\eta_n} \circ R^\Gamma_{\eta^\star})$.
- $(\Pi^\Gamma, \Lambda^\Gamma)$ *is a Stochastic Petri Net program*
- $\delta^\Gamma\colon At(\Gamma) \times \Pi^\Gamma \to \vec{\mathbb{R}}^+$ *such that (let $\eta \in \Pi^\Gamma$, $w \in At(\Gamma)$, $v \in At(\Gamma)$, $s = M^\Gamma(w)$ and $r = M^\Gamma(v)$)*
  - *if $wR^\Gamma_\eta v$ then $f(r,\eta) = \epsilon$ and $\delta^\Gamma(w,\eta) = \delta^\Gamma(v,\eta)$ (once there was no firing);*
  - *if $f(s,\eta) = \epsilon$, $f(r,\eta) \neq \epsilon$ and $wR^\Gamma_\eta v$, $\delta^\Gamma(v,\eta)$ is an occurrence of a random variable of exponential distribution with parameter $\Lambda^\Gamma(\eta)$*
  - *if $f(s,\eta) \neq \epsilon$ then $f(r,\eta) \neq \epsilon$ and $wR^\Gamma_\eta v$, $\delta^\Gamma(v,\eta) < \delta^\Gamma(w,\eta)$*

Lemma 98 *Let $\mathcal{A}$ be an atom, a set χ of formulae in the form$\langle s,\eta\rangle\varphi$ and a sequence of names s. Then, there is an atom $\mathcal{C}$ such that $\bigwedge \mathcal{A} \wedge \langle s, \pi_i\rangle \bigwedge \mathcal{C}$ is consistent.*

Proof: By Definition 89 we can construct $\mathcal{C}$ and apply (PC) over χ, such that, according to Lemma 88 we can force a choice to make $\bigwedge \mathcal{A} \wedge \langle s, \pi_i\rangle \bigwedge \mathcal{C}$ consistent. ■

Lemma 99 *For all programs η, $S_\eta \subseteq R_\eta$ as in definition 97.*

Proof: Induction on the length of programs η.

- *For basic programs π_b, $S_{\eta_b} = R_{\eta_b}$ (Definition 97)*
- *$\eta = \theta^\star$. We have that $R_{\theta^\star} = R^\star_\theta$. By the induction hypothesis $S_\theta \subseteq R_\theta$. But we know that if $S_\theta \subseteq R_\theta$ then $S^\star_\theta \subseteq R^\star_\theta$. So $S^\star_\theta \subseteq R^\star_\theta$.*
  By Lemma 96, $S_{\theta^\star} \subseteq S^\star_\theta$, and thus $S_{\theta^\star} \subseteq S^\star_\theta \subseteq R^\star_\theta = R_{\theta^\star}$

– $\eta = \pi_1 \odot \pi_2 \odot \cdots \odot \pi_n$. *We have that* $R_\eta = (R_{\pi_1} \circ R_{\eta^\star}) \cup \cdots \cup (R_{\pi_n} \circ R_{\eta^\star})$. *By the previous item we know* $S_{\theta^\star} \subseteq R_{\theta^\star}$, *and by the induction hypothesis* $S_{\pi_i} \subseteq R_{\pi_i}$ *and thus* $(S_{\pi_1} \circ S_{\eta^\star}) \cup \cdots \cup (S_{\pi_n} \circ S_{\eta^\star}) \subseteq R_\theta$ *(1).*
Suppose $\mathcal{A} S_\eta \mathcal{B}$, *iff* $\bigwedge \mathcal{A} \wedge \langle s, \eta \rangle \bigwedge \mathcal{B}$ *is consistent (from the distribution and semantics of disjunction).*
Using axiom (PC) $\bigwedge \mathcal{A} \wedge \langle s, \pi_1 \rangle \langle s_1, \eta^\star \rangle \bigwedge \mathcal{B} \vee \langle s, \pi_2 \rangle \langle s_2, \eta^\star \rangle \bigwedge \mathcal{B} \vee \cdots \vee \langle s, \pi_n \rangle \langle s_n, \eta^\star \rangle \bigwedge \mathcal{B}$ *is consistent.*
For at least one i, $\bigwedge \mathcal{A} \wedge \langle s, \pi_i \rangle \langle s_i, \eta^\star \rangle \bigwedge \mathcal{B}$ *is consistent.*
By Lemma 98 we can construct a $\mathcal{C}$ *such that* $\bigwedge \mathcal{A} \wedge \langle s, \pi_i \rangle \bigwedge \mathcal{C}$ *is consistent (2) and* $\bigwedge \mathcal{C} \wedge \langle s_i, \eta^\star \rangle \bigwedge \mathcal{B}$ *is consistent.*
Let $s' = M(\mathcal{C})$. *As* $s_i \preceq s'$, *then* $\bigwedge \mathcal{C} \wedge \langle s', \eta^\star \rangle \bigwedge \mathcal{B}$ *is consistent (3).*
From (2) and (3) we have $\mathcal{A} S_{\pi_i} \mathcal{C}$ *and* $\mathcal{C} S_{\eta^\star} \mathcal{B}$, *and*
$\mathcal{A}(S_{\pi_i} \circ S_{\eta^\star})\mathcal{B}$. *Thus*
$\mathcal{A}(S_{\pi_1} \circ S_{\eta^\star}) \cup \cdots \cup (S_{\pi_n} \circ S_{\eta^\star})\mathcal{B}$.
By (1), $\mathcal{A} R_\eta \mathcal{B}$. *Therefore,* $S_\eta \subseteq R_\eta$.

Thus, this lemma holds. ■

Lemma 100 *Existence Lemma for Proper Canonic Models*

Let $\mathcal{A} \in At(\Gamma)$ *and* $\langle s, \eta \rangle \varphi \in FL(\Gamma)$. *Then,* $\langle s, \eta \rangle \varphi \in \mathcal{A}$ *iff there exists* $\mathcal{B} \in At(\Gamma)$ *such that* $\mathcal{A} R_\eta \mathcal{B}$, $s \preceq M(\mathcal{A})$ *and* $\varphi \in \mathcal{B}$.

Proof: This proof follows in two steps.

– *Suppose* $\langle s, \eta \rangle \varphi \in \mathcal{A}$. *By the Existence Lemma for Canonic Models, Lemma 94, we have that there exists* $\mathcal{B} \in At(\Gamma)$ *such that* $\mathcal{A} S_\eta \mathcal{B}$ *and* $\varphi \in \mathcal{B}$. *By Lemma 99,* $S_\eta \subseteq R_\eta$. *Thus, there exists* $\mathcal{B} \in At(\Gamma)$ *such that* $\mathcal{A} R_\eta \mathcal{B}$ *and* $\varphi \in \mathcal{B}$.

– *Programs* x, *for* $x \in \{\pi, \pi_1 \odot \cdots \odot \pi_n, \eta^\star\}$. *Suppose there exists* $\mathcal{B} \in At(\Gamma)$ *such that* $\mathcal{A} R_x \mathcal{B}$ *and* $\varphi \in \mathcal{B}$. *This part of the proof follows by induction on the structure of* x.

(base) $x = \pi_b$: *this is straightforward once* $R_{\pi_b} = S_{\pi_b}$ *and, by the existence lemma for canonic models, Lemma 94,* $\langle s, \pi \rangle \varphi \in \mathcal{A}$.

⋆ $x = \eta^\star$. *By definition* $R_{\eta^\star} = R_\eta^\star$ *Suppose that for some* $\mathcal{B}$, $\mathcal{A} R_\eta^\star \mathcal{B}$ *and* $\varphi \in \mathcal{B}$. *Then, for some* n $\mathcal{A} = \mathcal{A}_1 R_\eta \cdots R_\eta \mathcal{A}_n = \mathcal{B}$. *We can prove by sub-induction on* $1 \leq k \leq n$.

$k = 1$: $\mathcal{A} R_\eta \mathcal{B}$ *and* $A \in \mathcal{B}$. *By induction hypothesis,* $\langle s, \eta \rangle \varphi \in \mathcal{A}$. *By axiom (Rec), we know that* $\vdash \langle s, \eta \rangle \varphi \rightarrow \langle s, \eta^\star \rangle \varphi$ *and by the definition of* $FL(\Gamma)$ *and maximality we have* $\langle s, \eta^\star \rangle \varphi \in \mathcal{A}$.

$k > 1$ *: By the sub-induction hypothesis* $\langle s, \eta^\star\rangle\varphi \in \mathcal{A}_2$ *and* $\langle s,\eta\rangle\langle s,\eta^\star\rangle\varphi \in \mathcal{A}_1$*. By axiom (Rec), we know that* $\vdash \langle s,\eta\rangle\langle s,\eta^\star\rangle\varphi \rightarrow \langle s,\eta^\star\rangle\varphi$ *and by the definition of* $FL(\Gamma)$ *and maximality we have* $\langle s,\eta^\star\rangle\varphi \in \mathcal{A}$*.*

$\odot$ $x = \pi_1 \odot \cdots \odot \pi_n$*:* $\mathcal{A}R_{\pi_1\odot\cdots\odot\pi_n}\mathcal{B}$ *and* $\varphi \in \mathcal{B}$ *iff* $\mathcal{A}(R_{\pi_1} \circ R_{x^\star}) \cup \cdots \cup (R_{\pi_n} \circ R_{x^\star})\mathcal{B}$ *and* $\varphi \in \mathcal{B}$*. For some* $1 \leq i \leq n$*,* $\mathcal{A}(R_{\pi_i} \circ R_{x^\star})\mathcal{B}$ *and* $\varphi \in \mathcal{B}$*. There exists a* $\mathcal{C}$ *such that* $\mathcal{A}R_{\pi_i}\mathcal{C}$ *and* $\mathcal{C}R_{x^\star}\mathcal{B}$ *and* $\varphi \in \mathcal{B}$*. By the previous case* $\langle s_i, x^\star\rangle\varphi \in \mathcal{C}$*, where* $s_i = f(s,\pi_i)$*, and by the induction hypothesis* $\langle s,\pi_i\rangle\langle s_i,x^\star\rangle\varphi \in \mathcal{A}$*. But this implies that* $\langle s,\pi_1\rangle\langle s_1,x^\star\rangle\varphi \vee \cdots \vee \langle s,\pi_n\rangle\langle s_n,x^\star\rangle\varphi \wedge \bigwedge\mathcal{A}$ *is consistent. By axiom (PC),* $\langle s,\pi_1\odot\cdots\odot\pi_n\rangle\varphi \wedge \bigwedge\mathcal{A}$ *is consistent. By maximality,* $\langle s,\pi_1\odot\cdots\odot\pi_n\rangle\varphi \in \mathcal{A}$*.*

So, there exists a proper canonic model. ■

Lemma 101 *Truth Lemma for Proper Canonic Models*

Let $\mathcal{N}_3^\star = \langle W, R_\eta, M, (\Pi,\Lambda), \delta, \mathbf{V}\rangle$ *be a finite proper canonic model constructed over a formula* ϕ*. For all atoms* $\mathcal{A}$ *and all* $\varphi \in FL(\phi)$*,* $\mathcal{N}_3^\star$,$\mathcal{A} \Vdash \varphi$ *iff* $\varphi \in \mathcal{A}$*.*

Proof: The proof is by induction on the construction of φ*.*

- *Atomic formulae: the proof is straightforward from the definition of* $\mathbf{V}$*.*
- *Boolean operators: the proof is straightforward from the semantical notion of* $\mathcal{DS}_3^\star$ *(Definition 85).*
- *Modality* $\langle s,x\rangle$*, for* $x \in \{\pi, \pi_1 \odot \cdots \odot \pi_n, \eta^\star\}$*.*
  - *Suppose* $\mathcal{N}_3^\star, \mathcal{A} \Vdash \langle s,x\rangle\varphi$*, then there exists* $\mathcal{A}'$ *such that* $\mathcal{A}S_x\mathcal{A}'$ *and* $\mathcal{N}_3^\star$,$\mathcal{A}' \Vdash \varphi$*. By the induction hypothesis we know that* $\varphi \in \mathcal{A}'$*, and by Lemma 94 we have* $\langle s,x\rangle\varphi \in \mathcal{A}$*.*
  - *Suppose* $\mathcal{N}_3^\star, \mathcal{A} \nVdash \langle s,x\rangle\varphi$*. By the semantical notion we have* $\mathcal{N}_3^\star, \mathcal{A} \Vdash \neg\langle s,x\rangle\varphi$*. Then for all* $\mathcal{A}'$*,* $\mathcal{A}R_x\mathcal{A}'$ *implies* $\mathcal{N}_3^\star$,$\mathcal{A}' \nVdash \varphi$*. By the induction hypothesis we know that* $\varphi \notin \mathcal{A}'$*, and by Lemma 100 we have* $\langle s,x\rangle\varphi \notin \mathcal{A}$*.*

So, this lemma holds. ■

Theorem 102 *Completeness for Proper Canonic Models*

$\mathcal{DS}_3^\star$ *programs are complete with respect to the class of Proper Canonic Models.*

Proof: If φ *is valid then it is valid in all models, including in a Proper Canonic Model. So it is valid in all worlds of* $\mathcal{N}_3^\star$*. By Lemma 101,* φ *is derivable. Therefore if* $\Vdash \varphi$*, then* $\vdash \varphi$*. Thus,* $\mathcal{DS}_3^\star$ *is complete.* ■

Definition 103 The Fischer-Ladner closure

It is inductively defined as follows, where $FL(\varphi)$ denotes the smallest set containing φ which is closed and closed under sub formulae.

$FL\colon \Upsilon \to 2^{\Upsilon}$, where Υ is the set of all formulae

1. *$FL(\varphi)$ is closed under subformulae;*
2. *if $\langle s, \eta^\star\rangle\phi \in FL(\varphi)$, then $\phi \in FL(\varphi)$;*
3. *if $\langle s, \eta^\star\rangle\phi \in FL(\varphi)$, then $\langle s, \eta\rangle\phi \in FL(\varphi)$;*
4. *if $\langle s, \eta^\star\rangle\phi \in FL(\varphi)$, then $\langle s, \eta\rangle\langle s, \eta^\star\rangle\phi \in FL(\varphi)$;*
5. *if $\langle s, \eta\rangle\phi \in FL(\varphi)$, then $\langle s, \eta_i\rangle\langle s_i, \eta^\star\rangle\phi \in FL(\varphi)$, where $\eta = \eta_1 \odot \eta_2 \odot \cdots \odot \eta_n$ and $s_i = f(s, \eta_i)$, for all $1 \leq i \leq n$.*
6. *if $\langle s, \eta\rangle\phi \in FL(\varphi)$, then $\Pr(\mathcal{M}_3, v \Vdash \langle s, \eta_b\rangle\psi \mid \delta(v, \Pi)) > 0$, according to the semantical notion of $\mathcal{DS}_3^\star$ (Definition 85), so $\langle s, \eta_i\rangle\langle s_i, \eta\rangle\psi \in FL(\varphi)$, where $\eta = \eta_1 \odot \eta_2 \odot \cdots \odot \eta_n$ and $s_i = f(s, \eta_i)$, for all $1 \leq i \leq n$;*

Lemma 104 *$FL(\varphi)$ is finite.*

Proof: The proof is the same of Lemma 73 including the transitive closure case. If $\langle s, \kappa^\star\rangle\Psi \succ \langle s, \kappa\rangle\langle f(s, \kappa_i), \kappa^\star\rangle\Psi$, for κ a normalised Stochastic Petri Net program, then, as there are no places that can accumulate tokens indefinitely, there are two cases (where $S = \{s_1, s_2, \ldots, s_n\}$ a list of all sequences returned by applications of f).

(a) After consecutive applications of f over basic programs and their current markups f will return ϵ for any $\kappa_i \in \kappa$ (i.e. the SPN program stopped) and there will be no changes over the markup anymore.

(b) After j applications of f over basic programs and their current markups f will return a value s_j such that s_j occurs more then once in S (i.e. the markup repeats).

Therefore there can be no infinitely-long $\succ$-sequences.

So, $FL(\varphi)$ is finite. ■

Lemma 105

(i) If $\sigma \in FL(\varphi)$, then $FL(\sigma) \subseteq FL(\varphi)$

(ii) If $\sigma \in FL(\langle s, \pi^\star\rangle\varphi)$, then $FL(\sigma) \subseteq FL(\langle s, \pi^\star\rangle\varphi) \cup FL(\varphi)$

(iii) If $\sigma \in FL(\langle s, \pi \rangle \varphi)$*, then* $FL(\sigma) \subseteq FL(\langle s, \pi \rangle \varphi) \cup FL(\varphi)$

Proof: This proof is the same than for Petri-PDL, regarding $\mathcal{DS}_3^\star$ *FL Definition 104* ■

The Filtration definition is the same of $\mathcal{DS}_3$ (Definition 75) regarding the same properties.

5.4 Computational complexity

As the language of $\mathcal{DS}_3$ is subsumed by the language of $\mathcal{DS}_3^\star$, $\mathcal{DS}_3^\star$ satisfiability computational complexity is stated bellow.

Theorem 106 *$\mathcal{DS}_3^\star$ satisfiability problem is EXPTime-hard.*
Proof: Taking Lemma 45, we use the Petri Net which models the game stating the same firing rate to each transition. As the $\mathcal{DS}_3$ language subsumed by $\mathcal{DS}_3^\star$ language then the reduction procedure presented in Lemma 45 is also valid for $\mathcal{DS}_3^\star$. So $\mathcal{DS}_3^\star$ SAT is EXPTime-hard. ■

Notice that the proof of EXPTime-hardness for $\mathcal{DS}_3^\star$ may be composed using the transitive closure operator which leads to a more intuitive way to model the flow of the game. The flow does not need to be modelled as a sequence of disjunctions, but may be a sequence of repetitions denoted by the transitive closure operator.