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8. Appendix A

Here follows the demonstration of Proposition 1. It is similar to the one that appears in Skrzypacz and Hopenhayn (2004a, b), which is done in 6 steps. Notice that the scheme described in the present article does not differ from the one by Skrzypacz and Hopenhayn along the equilibrium path on the bids being done, therefore the efficiency demonstration is identical to the one utilized by them. The difference is in the sustainability demonstration, and for this 2 new steps have to be added.

Step 1: Because the scheme is stationary and symmetric, included participants will make bids following symmetric, strictly increasing functions over an interval $(v^H - c, v^H)$. Additionally, the sure bidder offers bid $b = 0$ when his valuation is less or equal to c.

Denote the continuation utility of an included participant as $\overline{w}$ and that of an excluded participant of index T as $\underline{w}$, and have $\overline{w} > \underline{w}$ (the values of these variables will be determined in step 4, this inequality will be trivial after that). Let $\widehat{w} = \alpha \times \overline{w} + (1 - \alpha)\underline{w}$ be the expected continuation value after an included participant wins, but before knowing whether he will become excluded or stay as an included participant. Notice also that an included participant, when choosing his bid, observes his valuation v and chooses his bid to obtain utility $vQ(b) - P(b) + \delta\big(Q(b)\widehat{w} + \big(1 - Q(b)\big)\overline{w}\big)$, where b is his bid, $Q(b)$ is the probability of a participant winning an auction when he offers b and $P(b)$ is his expected payment. This equation makes explicit a participants' decision between: **(1)** attempting to win an auction and obtain the good, but receiving a smaller continuation value in the future ($\widehat{w}$); and **(2)** not winning the auction, therefore not receiving the good, but having a higher continuation value, $\overline{w}$ in the future.

Thus, each participant solves:

$$\max_{b\geq 0} vQ(b) - P(b) + \delta\big(Q(b)\widehat{w} + \big(1 - Q(b)\big)\overline{w}\big)$$

Which is equivalent to:

$$\max_{b\geq 0}\big(v - \delta\,(\overline{w} - \widehat{w})\big)Q(b) - P(b) + \delta\overline{w}$$

or:

$$\max_{b\geq 0}(v - c)Q(b) - P(b) + \delta\overline{w}$$

where $c = \delta(\overline{w} - \widehat{w})$. Notice that, when we discard the constant $\delta\overline{w}$, this problem is equivalent to an auction in which the participant's valuation for the good is

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$v - c$. For this reason, and for the symmetry of the problem, we can assume that the bidding functions will be symmetric and strictly increasing over the interval $(v^H - c, v^H)$, where c is defined as above. Also, in case some included participant has valuation less or equal to c, then $v - c < 0$ and the participant will minimize $Q(b) \geq 0$ by not making an offer. However, the sure bidder, having to offer a bid, will offer the lowest possible bid, $b = 0$, when he has valuation lower or equal to c (notice that, if he does not, he will obtain continuation value equal to the one he would obtain in a myopic Nash Equilibrium, w_{NE}; it will be necessary, then, that $\widehat{w} \geq w_{NE}$).

Step 2: There exists $c = v^H - \epsilon$ and M_2 such that for all $M \geq M_2$, if M players bid according to the above strategies, then the equilibrium satisfies conditions (1) to (3) from Definition 2 by Skrzypacz e Hopenhayn on Optimal Collusive Schemes (which are quoted in subsection 4.1).

The probability that highest realized value among the M included players is higher than c is:

$$1 - F^M(c)$$

In that event the winner is a player with a value at most ϵ lower than the highest possible value (that is, a valuation at least $v^H - \epsilon$). Therefore, to satisfy condition (3) it is sufficient that this probability is at least as high as p. For any $c < v^H$ clearly exists M_2 such that for all $M \geq M_2$ it is the case.

Given c, the expected payments are bounded above by ϵ, as the bidding strategies correspond to an equilibrium in which players have values bounded above by ϵ. So condition (2) is satisfied. Finally, condition (1) is satisfied as the sure bidder always bids.

Step 3: Show that there exists M_3 such that for all $M \geq M_3$ the per-period expected total surplus (per-period payoffs summed over all players) from such bidding functions is higher than $c < v^H$.

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Introduce notation:

$v_M^{(1)}$ is the first order statistic from M draws and

$v_M^{(2)}$ is the second order statistic from M draws.

Given that the M players bid symmetrically, we can use revenue equivalence theorem to calculate expected payoffs and payments for all standard auctions. Without loss of generality consider a second-price auction. Total (summed over all players) one-period profit from the scheme is:

$$x_{Total} = \left(1 - F^M(c)\right)E_{v_M^{(1)}}\left[\left(v_M^{(1)} - Prob\left(v_M^{(2)} > c\right)E_{v_M^{(2)}}\left[v_M^{(2)} - c\middle|v_M^{(2)} > c\right]\middle|v_M^{(1)} > c\right)\right] + F^M(c)E[v|v \leq c]$$

The above expression is derived from the fact that with probability $\left(1 - F^M(c)\right)$ at least one player has value above c. In that case the winner is the one with the highest value $v_M^{(1)}$. His payment is 0 when the second highest valuation is below c and is $v_M^{(2)} - c$ when the second highest valuation is above c. Finally, with probability $F^M(c)$ all players have value below c and in that case the winner is the sure bidder. He pays 0 and has expected value $E[v|v \leq c]$.

Trivially, when $\rightarrow \infty$, x_{Total} converges to c, as $F^M(c)$ converges to 0 and the highest and the second highest valuations converge to v^H. We now show that it converges from above. The expected current auction value of the winner is at least equal to the expected first order statistic given it is larger than c times the probability of $v_M^{(1)}$ being greater than c.

Expected payment is equal to:

$$M\int_c^{v^H} [(v - c)f(v) + F(v) - 1]F^{M-1}(v)dv$$

And the valuation of the highest bidder, conditioning on it being higher than c is:

$$\int_{c}^{v^H} vf(v)F^{M-1}(v)dv$$

This leads to:

$$x_{Total} \geq M\int_{c}^{v^H} \{vf(v)F^{M-1}(v) - [(v-c)f(v) + F(v) - 1]\}F^{M-1}(v)dv$$

$$= c\left(1 - F^M(c)\right) + M\int_{c}^{v^H} \left(1 - F(v)\right)F^{M-1}(v)dv$$

Therefore, to show that the convergence is from above, it is sufficient to show that for large M:

$$cF^M(c) < M\int_{c}^{v^H} \left(1 - F(v)\right)F^{M-1}(v)dv$$

To verify that this is the case, note that:

$$M\int_{c}^{v^H} \left(1 - F(v)\right)F^{M-1}(v)dv \geq MF^{M-1}(c)\int_{c}^{v^H} \left(1 - F(v)\right)dv$$

Because $\int_{c}^{v^H}\left(1 - F(v)\right)dv$ is positive and independent of M, there clearly exists M_3 large enough such that for all $M \geq M_3$, $x_{Total} > c$.

Step 4: Show that, for any $c < V^H$ there exists N_4 such that for all $N \geq N_4$ there exists δ_4 such that for all $\delta \geq \delta_4$ for some $\alpha \in (0,1]$ and $T < N$ these bidding functions are best responses for included players in every auction (and $M = N - T \geq \max\{M_2, M_3\}$).

Fix $c = v^H - \epsilon$ and $M \geq \max\{M_2, M_3\}$). Now we show that we can find N_4 large enough so that for all $N \geq N_4$ we can find δ_4 such that for all $\delta \geq \delta_4$ for some $\alpha \in (0,1]$ the proposed scheme indeed induces c.

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Due to the stationarity and symmetry of the scheme it is easy to find expected payoffs of all players given x_{Total}. We have M included players that are ex-ante (before choosing the sure bidder) symmetric. Denote their average per-period expected payoff by x_0. There are also T excluded players that have to wait different number of periods to become included. Denote their payoffs by $x_{e,r}$ where $r \in \{1, \dots, T\}$ is the rank of the excluded player.

Summing over all players, the expected per-period total surplus of the cartel is:

$$x_{Total} = Mx_0 + \sum_{r=1}^{T} x_{e,r}$$

The expected payoffs of the excluded players can be expressed as:

$$x_{e,r} = \delta\big(\alpha x_{e,r-1} + (1-\alpha)x_{e,r}\big) \text{ for } r > 1$$

$$x_{e,1} = \delta\big(\alpha x_0 + (1-\alpha)x_{e,1}\big)$$

Therefore:

$$x_{e,r} = d^r x_0$$

Equation 1

Where

$$d = \frac{\delta\alpha}{1-\delta(1-\alpha)}$$

Summing over all players we get:

$$x_{Total} = x_0\left(M + d\frac{1-d^T}{1-d}\right)$$

Equation 2

Now consider the total (not average) expected payoffs of an included player conditional on winning and losing:

$$\underline{w} = \frac{\alpha x_{e,T} + (1-\alpha)x_0}{1-\delta} = \frac{\alpha d^t + (1-\alpha)}{1-\delta}x_0$$

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$$\overline{w} = \frac{1}{1-\delta}x_0$$

That implies $c = \delta(\overline{w} - \underline{w})$, so:

$$c = \frac{\delta\alpha}{1-\delta}(1-d^T)x_0$$

Equation 3

Substituting Equation 2 for x_0:

$$c = \left(\frac{(1-\delta)M}{\delta\alpha(1-d^T)} + 1\right)^{-1} x_{Total} = (\mu(\delta,\alpha,M,T)+1)^{-1}x_{Total}$$

Equation 4

The function $\mu(\delta,\alpha,M,T)$ varies continuously in α from $M\frac{1-\delta}{(1-\delta^T)\delta}$ to infinity for any $\delta < 1$. Moreover:

$$\lim_{\delta\to1}\mu(\delta,\alpha,M,T) = \frac{M}{T}$$

Recall that x_{Total} - the total expected per-period surplus of the cartel - depends only on M and c (and not on T or δ or α) and that $x_{Total} > c$. Therefore for any c and M we can find N_4 such that:

$$c < \left(\frac{M}{N_4 - M} + 1\right)^{-1} x_{Total}$$

Now, for all $N \geq N_4$, there exists $\delta_4 < 1$ such that for all $\delta \geq \delta_4$, there exists $\alpha \in (0,1]$ such that $c = v^H - \epsilon$. Note that for small ϵ it is needed that the share of the included players gets negligible and the cartel member with the highest realized value will almost surely not win the object. That does not destroy asymptotic efficiency, because the winner will have almost surely a very similar value.

Step 5: Show that, for large N and sufficiently high δ no player has ever incentives to deviate in his bid.

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The previous step has shown that given the derived N, M, δ and α, the included players have no incentives to deviate from the prescribed bidding functions. It is only left to show that the excluded players do not have profitable deviations as well. It is sufficient to consider a player that is excluded with rank T and got a draw v^H at an auction.

From Equation 1 and Equation 3 we get:

$$x_{e,T} = \frac{1-\delta}{\delta\alpha}\frac{d^T}{(1-d^T)}c$$

On the other hand, if the participant deviates when his valuation for the good is v^H, he has a payoff of, at most, v^H in the current period and his mean expected continuation payoff will be of (as in a myopic Nash Equilibrium):

$$\delta\frac{E\left[v_N^{(1)} - v_N^{(2)}\right]}{N}$$

Therefore, deviating will not be profitable if:

$$\frac{1-\delta}{\delta\alpha}\frac{d^T}{(1-d^T)}c < (1-\delta)v^H + \delta\frac{E\left[v_N^{(1)} - v_N^{(2)}\right]}{N}$$

When $\delta \to 1$, the left hand side of the equation goes to c/T and the right hand side to $\frac{E\left[v_N^{(1)} - v_N^{(2)}\right]}{N}$. On the other hand, for N sufficiently high, $\frac{E\left[v_N^{(1)} - v_N^{(2)}\right]}{N} < c$. Naming N_5 the lowest N such that the last inequality becomes true, for all $N \geq N_5$ it is possible to find δ_5 such that for all $\delta \geq \delta_5$ the above restriction is satisfied, and no excluded participant will prefer to offer a bid.

Step 6: Show that the parameters above are sufficient (even though not necessary) to make truth telling optimal for all participants in all situations.

It is necessary to check that: (1) an included participant who won will have no gain in saying he lost the auction; (2) an included participant who lost the auction will have no interest in saying he actually won; (3) an excluded

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participant who wins will gain nothing from saying he lost; and (4) an excluded participant who did not win will not wish to say he won.

Bellow follows the justifications:

1. An included participant who won will have no gain in saying he lost the auction.

First, notice that in case a participant lies, from the next period onwards there will be a reversal to the myopic Nash Equilibrium. Second, notice that for the parameters of Step 5, even an excluded participant of index T is better off by following the collusion scheme than by deviating. Therefore, we can conclude that an included participant who won would be better off telling the truth (and thus be in a lottery between being an included participant and an excluded one) than lying (and thus being in the myopic Nash Equilibrium).

2. An included participant who lost the auction will have no interest in saying he actually won.

By doing this deviation, this participant will forgo the situation of being an included participant in the next period, for being in a myopic Nash Equilibrium. However, we have already checked that every participant is better off with the collusion than without it (Step 5), so this participant has no incentive of deviating in his announcement.

3. An excluded participant who wins will gain nothing from saying he lost.

Whether this participant lies or tells the truth, the cartel will break at the next period, therefore he is indifferent on his announcement, whatever the parameters of the model.

4. An excluded participant who did not win will not wish to say he won.

Notice that, in case an excluded participant announces he wins, he would have been better off if he had in fact won (as in both cases there will be an end to the collusion scheme). However, at step 5 it has been proven that he would rather not bid than bid and win the auction, for the parameters above.

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Step 7: If there is no possibility of communication punishment, set $N^* = \max\{N_4, N_5\}$. So for all $\geq N^*$, set $\delta \geq \max\{\delta_4, \delta_5\}$. With this, it is possible to choose α, T and τ that satisfy the conditions obtained through the past steps and make this scheme a Subgame Perfect Equilibrium and asymptotically efficient.

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9. Appendix B

The proof of Proposition 2 is similar to the one described in Appendix A for Proposition 1, with a few changes and added steps.

Steps 1 through 4 are identical, as the fine does not affect the incentives behind choosing the bids.

Step 5B: Show that, for large N and sufficiently high δ, no player has ever incentives to deviate in his bid.

The previous step has shown that given the derived N, M, δ and α the included players have no incentives to deviate from the prescribed bidding functions. It is only left to show that the excluded players do not have profitable deviations as well. It is sufficient to consider a player that is excluded with rank T and got a draw v^H in an auction at the beggining of the cycle (thus announcements will be made only after τ periods).

From Equation 1 and Equation 3 we get:

$$x_{e,T} = \frac{1-\delta}{\delta\alpha}\frac{d^T}{(1-d^T)}c$$

On the other hand, if the participant deviates when his valuation for the good is v^H, he has a payoff of, at most, v^H in the current period and

$$E_{v,v_{M-1}^{(1)}}\left[\max\left\{v-\left(v_{M-1}^{(1)}-c\right),0\right\}\right]$$

during the next $\tau-1$ periods and, after τ periods, his mean expected continuation payoff will be of (as in a myopic Nash Equilibrium):

$$\delta^{\tau}\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$$

Therefore, deviating will not be profitable if:

$$\frac{1-\delta}{\delta\alpha}\frac{d^T}{(1-d^T)}c$$
$$\geq v^H(1-\delta)+\delta E_{v,v_{M-1}^{(1)}}\left[\max\left\{v-\left(v_{M-1}^{(1)}-c\right),0\right\}\right](1-\delta^\tau)$$
$$+\delta^\tau\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$$

When $\delta \to 1$, the left hand side of the equation goes to $\frac{c}{T}$ and the right hand side goes to $\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$. On the other hand, for N sufficiently high, $E\left[v_N^{(1)} - v_N^{(2)}\right] < c$. Naming N_5 the lowest N such that the last inequality becomes true, for all $N \geq N_5$ it is possible to find δ_5 such that for all $\delta \geq \delta_5$ the above restriction is satisfied, and no excluded participant will prefer to offer a bid.

Step 6B: Show that the parameters above are sufficient (even though not necessary) to make truth telling optimal for all participants in all situations.

It is necessary to check that: (1) an included participant who won will have no gain in saying he lost the auction; (2) an included participant who lost the auction will have no interest in saying he actually won; (3) an excluded participant who wins will gain nothing from saying he lost; and (4) an excluded participant who did not win will not wish to say he won.

Bellow follows the justifications:

1. An included participant who won will have no gain in saying he lost the auction.

Notice that it will be necessary and sufficient that this restriction be satisfied for the first period of the cycle. First, notice that in case a participant lies, from the next cycle on there will be a reversal to the myopic Nash Equilibrium, so that in case the participant wishes to lie, he would be better off by also deviating from the suggested bids during the next $\tau - 1$ periods of the current cycle. Second, notice that this participant will be more prone to lie (and deviate from the suggested bids) in case he becomes excluded of index T on the next period. In this case, however, it

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is an individual of index T planning to deviate which, for the parameters of step 5, is not profitable.

2. An included participant who lost the auction will have no interest in saying he actually won.

Notice that this participant would have been better off if he had won the auction (offering a higher bid than he in fact offered), as in this case there would be no reversal to the myopic Nash Equilibrium. However, in step 4 it was demonstrated that this deviation is not optimal.

3. An excluded participant who wins will gain nothing from saying he lost.

Whether this participant lies or tells the truth, the cartel will break at the end of the cycle, therefore he is indifferent on his announcement, whatever the parameters of the model.

4. An excluded participant who did not win will not wish to say he won.

Notice that, in case an excluded participant announces he wins, he would have been better off if he had in fact won (as in both cases there will be an end to the collusion scheme). However, at step 5 it has been proven that he would rather not bid than bid and win the auction, for the parameters above.

Step 7B: Show that for ρ and κ such that $\rho\kappa < v^H$ there are N and δ such that no participants will prefer not to send an announcement.

Here we should separate two cases: (1) if only the parties caught in communication will be punished; and (2) if punishment will occur to all participants whenever the auctioneer catches at least one of them communicating.

On the first case, we have:

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Notice that the non-realization of an announcement will mean the reversal to the myopic Nash Equilibrium, in other words, the continuation payoff will be of $\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$. Sending a true announcement, however, will mean at the worst case scenario (that is, for the winner), an expected payoff of $(1-\alpha)x_0+\alpha x_{e,T}-\rho\kappa=(1-\alpha)x_0+\alpha d^T x_0-\rho\kappa$. It therefore suffices to check whether:

$$(1-\alpha)x_0+\alpha d^T x_0-\rho\kappa \geq \frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$$

$$\frac{1-\delta}{\delta\alpha}\frac{c}{(1-d^T)}\left[(1-\alpha)+\alpha d^T\right]-\rho\kappa \geq \frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$$

When δ converges to 1, the left hand side converges to $\frac{c}{T}-\rho\kappa$, while the right hand side is constant. For large enough N, $\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}\to 0$, independently of δ. Also $\frac{c}{T}\to 0$, as T gets closer to N when N grows. Therefore, the inequality will not be satisfied for too large an N, and there can not be asymptotic efficiency as the collusion cannot be sustained asymptotically.

However, on the second case, we have that the non-realization of an announcement will mean the reversal to the myopic Nash Equilibrium minus the expected payment on the following period, in other words, the continuation payoff will be of $\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}-\rho\kappa$. Therefore, it suffices to check whether:

$$(1-\alpha)x_0+\alpha d^T x_0-\rho\kappa \geq \frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}-\rho\kappa$$

$$\frac{1-\delta}{\delta\alpha}\frac{c}{(1-d^T)}\left[(1-\alpha)+\alpha d^T\right] \geq \frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$$

Again, the left hand side converges to $\frac{c}{T}$ as δ goes to 1, and as N grows, $\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}\to 0$. Therefore, for large enough N, c becomes greater than

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$\frac{E\left[v_N^{(1)}-v_N^{(2)}\right]}{N}$. As $\leq N \; \forall \; N$, it is clear that it must be true that for large enough N_7, there exists δ_7 such that the above inequality holds.

Step 8B: If there is a fine for communication, and it applies to only those caught communicating, the collusion scheme proposed will break for large enough N.

If there is a fine for communication, and it is applied against all auction participants whenever at least one of them is caught communicating, then set $N^* = \max\{N_4, N_5, N_7\}$. So for all $\geq q \; N^*$, set $\delta \geq \max\{\delta_4, \delta_5, \delta_7\}$. With this, it is possible to choose α, T and τ that satisfy the conditions obtained through the past steps and make this scheme a Subgame Perfect Equilibrium, and asymptotically efficient.

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10. Appendix C

Below is the proof to Proposition 3.

Step 1: To show that the above strategy is dominant for the included participants, even though the good is not sent with probability $1 - \psi$.

Notice that, in the original model, an included player would solve:

$$\max_{b \geq 0} (v - c)Q(b) - P(b) + \delta\overline{w}$$

c represents the "punishment" for winning the auction, which guarantees that all included participants would offer less than their valuation for the good (in a second price auction). Such a punishment also appears in this setting.

Step 2: To show that excluded participants have no profit in offering bids.

Here, both the expected profit and the expected punishment are multiplied by $(1 - \psi)$ when compared to the original model, therefore with the same parameters the equilibrium is kept.

Step 3: To show that no participant would prefer to send lies as their announcements.

It suffices to check for the winner: in case he admits victory, he has expected utility for the next periods equal to:

$$A = \alpha x_{e,T} + (1 - \alpha)x_0$$

Where the first term refers to the case of the winner receiving the good and being punished for winning, and the second term for he receiving the good but not being punished. Notice that the winner can only announce victory in case the good is received, therefore the new variables (ψ, ϕ) do not appear.

In case the winner does not admit victory, he will have an expected continuation payoff of:

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$$B = \phi \left[\frac{y_{NC} \frac{1-\delta^P}{1-\delta}}{N} + \delta^P x_0 \right] + (1-\phi) x_0$$

Where y_{NC} is the expected gain for a winner in a period of non-collusion. The first term refers to the case in which there is a period of punishment, with no collusion during P periods, and right after that the original state returns (and, as the winner did not announce his victory, he will return to being an included participant). The second term refers to the probability of there not being a punishment period, multiplied by the expected return of an included individual.

It remains to check if $A - B \geq 0$:

$$A - B = \{\alpha x_{e,T} + (1-\alpha)x_0\} - \left\{\phi \left[\frac{y_{NC}\frac{1-\delta^P}{1-\delta}}{N} + \delta^P x_0\right] + (1-\phi)x_0\right\}$$

Therefore, it is needed that:

$$\alpha x_{e,T} + (1-\alpha)x_0 \geq \phi \left[\frac{y_{NC}\frac{1-\delta^P}{1-\delta}}{N} + \delta^P x_0\right] + (1-\phi)x_0$$

For ϕ close to 1, the second term in the right hand side of the inequation is arbitrarily small. The left hand side is strictly larger than $x_{e,T}$ because $x_0 > x_{e,T}$. Setting $P \geq \frac{1}{\alpha}T$ (that is, the punishment for not announcing the victory to be at least as long as the expected punishment for victory, in case it occurs), the term in brackets will be strictly smaller than $x_{e,T}$ as participants of order T would rather keep the collusion than deviate and get the myopic Nash Equilibrium payoffs and, after $\frac{1}{\alpha}T$ periods, on average, an excluded participant becomes included. Notice that in this case the inequality is strict, therefore the conditions above are sufficient but not necessary.

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