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Intertemporal Discounting and Uniform Impatience

The uniform impatience assumption – see Magill & Quinzii (1994, Hypotheses A2 and A4), Hernandez & Santos (1996, Assumption C.3) or Magill & Quinzii (1996, Assumptions B2 and B4) – is a usual requirement for the existence of equilibrium in economies with infinite-lived debt-constrained agents. In this note, we fully characterize uniform impatience in terms of intertemporal discount factors. As an interesting side result, we obtain that the uniform impatience assumption does not hold for agents with hyperbolic intertemporal discounting.[1]

We follow the notation of Magill & Quinzzi (1994). Consider a framework where an infinite-lived price-taker agent demands L different commodities at any node ξ of an infinite countable event-tree D. This agent may trade financial assets to implement intertemporal transfers of wealth. At any $\xi \in D$, she receives a physical endowment $w(\xi) \in \mathbb{R}^L_+$ and makes contingent consumption plans, $x(\xi)$, to maximize her preferences, which are represented by a function $U : \mathbb{R}^{L\times D}_+ \to \mathbb{R}_+ \cup \{+\infty\}$. Aggregated physical endowments in the economy at node ξ are given by $W_\xi \in \mathbb{R}^L_{++}$. The date associated with a node $\xi \in D$ is denoted by $t(\xi)$.

Assumption A. Let $U^h(x) := \sum_{\xi\in D} u^h(\xi, x(\xi))$, where for any $\xi \in D$, $u^h(\xi,\cdot) : \mathbb{R}^L_+ \to \mathbb{R}_+$ is a continuous, concave and strictly increasing function. Also, $\sum_{\xi\in D} u^h(\xi, W_\xi)$ is finite.

Uniform impatience. *There are $\pi \in [0, 1)$ and $(v(\mu); \mu \in D) \in \mathbb{R}^{D\times L}_+$ such that, given a consumption plan $(x(\mu); \mu \in D)$, with $0 \le x(\mu) \le W_\mu$, for any $h \in H$, we have*

$$u^h(\xi, x(\xi) + v(\xi)) + \sum_{\mu>\xi} u^h(\mu, \pi' x(\mu)) > \sum_{\mu\ge\xi} u^h(\mu, x(\mu)), \qquad \forall \xi \in D,\ \forall \pi' \ge \pi.$$

Moreover, there is $\delta^h > 0$ such that, $w^h(\xi) \ge \delta^h v(\xi)$, $\forall \xi \in D$.

[1] Models of intertemporal choice in which agents have hyperbolic preferences have been widely studied recently (for some of these models, see, for example, Laibson (1998)).

The requirements of impatience above depend on both preferences and physical endowments. The assumptions imposed by Hernandez & Santos (1996) and Magill & Quinzii (1994, 1996) are particular instances of such requirements. Indeed, in Hernandez & Santos (1996), for any $\mu \in D$, $v(\mu) = W_\mu$. Also, since in Magill & Quinzii (1994, 1996) initial endowments are uniformly bounded away from zero by an interior bundle $\underline{w} \in \mathbb{R}^L_+$, they assume that $v(\mu) = (1, 0, \ldots, 0)$, $\forall \mu \in D$.

Our main result is,

Proposition 1. *Suppose that Assumption A holds, that* $(W_\xi; \xi \in D)$ *is a bounded plan and that there is* $\underline{w}^h \in \mathbb{R}^L_+ \setminus \{0\}$ *such that,* $w^h(\xi) \geq \underline{w}^h$, $\forall \xi \in D$. *Moreover, there exists a function* $u^h : \mathbb{R}^L_+ \to \mathbb{R}_+$ *such that, for any* $\xi \in D$, $u^h(\xi, \cdot) \equiv \beta^h_{t(\xi)} \rho^h(\xi) u^h(\cdot)$, *where* $\beta^h_{t(\xi)} > 0$, $\rho^h(\xi) = \sum_{\mu \in \xi^+} \rho^h(\mu)$ *and* $\rho^h(\xi_0) = 1$. *For each* $t \geq 0$, *let* $s^h_t = \frac{1}{\beta^h_t} \sum_{r=t+1}^{+\infty} \beta^h_r$. *Then, the function* U^h *satisfies uniform impatience if and only if* $(s^h_t)_{t \geq 0}$ *is bounded.*

Proof. Assume that $(W_\xi; \xi \in D)$ is a bounded plan. That is, there is $\overline{W} \in \mathbb{R}^L_+$ such that, $W_\xi \leq \overline{W}$, $\forall \xi \in D$. If $(s^h_t)_{t \geq 0}$ is bounded, then there exists $\overline{s}^h > 0$ such that, $s^h_t \leq \overline{s}^h$, for each $t \geq 0$. Also, since $\mathbb{F} := \{x \in \mathbb{R}^L_+ : x \leq \overline{W}\}$ is compact, the continuity of u^h assures that there is $\pi \in (0,1)$ such that $u^h(x) - u^h(\pi' x) \leq \frac{u^h(\overline{W} + \underline{w}^h) - u^h(\overline{W})}{2\overline{s}^h}$, $\forall x \in \mathbb{F}$, $\forall \pi' \geq \pi$. Thus, uniform impatience follows by choosing $\delta = 1$ and $v(\xi) = \underline{w}^h$, $\forall \xi \in D$. Indeed, given a plan $(x(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ such that, $x(\mu) \leq W_\mu$ $\forall \mu \in D$, the concavity of u^h assures that, for any $\xi \in D$ and $\pi' \geq \pi$,

$$\begin{aligned} \sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(x(\mu)) \quad &- \quad \sum_{\mu > \xi} \beta^h_{t(\mu)} \rho^h(\mu) u^h(\pi' x(\mu)) \\ &\leq \quad \frac{\beta^h_{t(\xi)} s_t}{2\overline{s}^h} \rho^h(\xi) \left(u^h(\overline{W} + \underline{w}^h) - u^h(\overline{W}) \right) \\ &< \quad \beta^h_{t(\xi)} \rho^h(\xi) u^h(x(\xi) + v(\xi)) - \beta^h_{t(\xi)} \rho^h(\xi) u^h(x(\xi)). \end{aligned}$$

Reciprocally, suppose that uniform impatience property holds. Then, given $(x(\mu); \mu \in D) \in \mathbb{R}^{L \times D}_+$ such that, $x(\mu) \leq W_\mu$, for all $\mu \in D$, there are $(\pi, \delta^h) \in [0,1) \times \mathbb{R}_{++}$ and $(v(\mu); \mu \in D)$ $\mathbb{R}^{D \times L}_+$ satisfying, for any $\xi \in D$, $w^h(\xi) \geq \delta^h v(\xi)$, such that, for any $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}\rho^h(\xi)}\left[\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(x(\mu)) - \sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(\pi x(\mu))\right]$$

$$< u^h(x(\xi)+v(\xi)) - u^h(x(\xi)).$$

It follows that, for any node ξ,

$$\frac{1}{\beta^h_{t(\xi)}\rho^h(\xi)}\left[\sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(\underline{w}) - \sum_{\mu>\xi}\beta^h_{t(\mu)}\rho^h(\mu)u^h(\pi\underline{w})\right] < u^h\left(\left(1+\frac{1}{\delta^h}\right)\overline{W}\right).$$

Therefore, we conclude that, for any $\xi \in D$,

$$\frac{1}{\beta^h_{t(\xi)}}\left(u^h(\underline{w}^h) - u^h(\pi\underline{w}^h)\right)\sum_{t=t(\xi)+1}^{+\infty}\beta^h_t \;<\; u^h\left(\left(1+\frac{1}{\delta^h}\right)\overline{W}\right),$$

which implies that the sequence $(s^h_t)_{t\geq 0}$ is bounded. □

Under the conditions of Proposition 1, if intertemporal discount factors are constant, i.e. $\exists c^h \in \mathbb{R}_{++} : \frac{\beta^h_{t(\xi)+1}}{\beta^h_{t(\xi)}} = c^h, \forall \xi \in D$, then $c^h < 1$ and $s^h_t = \frac{c^h}{1-c^h}$, for each $t \geq 0$. In this case, the utility function U^h satisfies the uniform impatience condition.

However, even with bounded plans of endowments, uniform impatience is a restrictive condition when intertemporal discount factors are time varying. For instance, if we consider *hyperbolic intertemporal discount factors*, that is, $\beta^h_t = (1+at)^{-\frac{b}{a}}$, where $b > a > 0$, then the function U^h, as defined in the statement of Proposition 1, satisfies Assumption A and the sequence s^h_t goes to infinity as t increases. Therefore, in this case, uniform impatience does not hold.