# 3. The Literature Review

## 3.1 The Facility Location Problem

The facility location problem as described in the literature has the objective of locating one or more facilities in order to minimize or maximize an utility function, subject to some constraints, especially the demand ones. Revelle et al. (1970) show that the objectives for the private and public sectors are basically different: In the private sector, the aim is mostly the minimization of the costs or the maximization of profit while in the public sector the objective is normally to maximize the benefits offered to the society or the minimization of the costs of the services offered.

According to these authors, the public sector can be divided into ordinary and extraordinary services. Ordinary services may be schools, mail, water, gas services, and the aim is to minimize the average distance traveled between the customer and the facility, also known as a *minisum* problem, in order to minimize a distance summation. The extraordinary services are those related to firefighters, police and hospitals, with the focus on the maximum distance or time eventually traveled, also known as a *minimax* problem.

The location problems can also be classified according to two fundamental issues: the location in a plane and the location in a network. In a plane, the facility is free to be located anywhere in the given plane while in the network these points can be located just on the vertices or on the arcs of the network.

Pizzolato (2002) describes the study of Weber in 1909 and cites that the Weber Problem considers a set of weighted points N, where a central point P needs to be located, in accordance with the following function.

$$Min\ Z = \sum_{i \in I} W_i d_{ip} = \sum_{i \in I} W_i \sqrt{\left(X_i - X_p\right)^2 + \left(Y_i - Y_p\right)^2}, \quad (3.1)$$

Where:

$W_i$: The weight associated to the point *i*;

$X_i\, and\ Y_i$: Coordinates of point *i*;

$X_p and Y_p$: Unknown coordinates of central point $P$, to determine;

$d_{ip}$: Euclidean distance from point *i* to central point $P$;

N: Number of points to be served.

Location problems can be classified as being capacitated or uncapacitated. The capacitated problems are those in which the facilities have a maximum capacity to serve the demand points while the uncapacitated ones are those that have no capacity restrictions to serve the demand points. Usually, the cited author asserts that the capacitated problems tend to be easier to be solved optimally because the model has fewer restrictions.

Pizzolato et al. (2004) describe the p-median problem as follows: there is a graph $G = (N, A)$ with N vertices, $N = \{1, \ldots, n\}$. For every pair of vertices $i, j \in N$ is associated a weight $q_i$ and a distance $d_{ij}$ between them. The binary decision variables $x_{ij}$ determine the allocation of the demand nodes $i$ to the median facilities $j$. The objective function (3.2) minimizes the weighted distances from every vertex *i* to the nearest facility *j*. The constraints (3.3) and (3.5) impose that every vertex *i* must be allocated to only one facility *j*, which must be a facility. The constraints (3.4) define that the number of facilities to be located is *p* and the constraints (3.6) are the integrality conditions of the problem.

$$Min \sum_{i=1}^{n} \sum_{j=1}^{n} q_i d_{ij} x_{ij}, \qquad (3.2)$$

s.t.:

$$\sum_{j=1}^{n} x_{ij} = 1, \qquad \forall i \in N, \qquad (3.3)$$

$$\sum_{j=1}^{n} x_{jj} = p, \qquad (3.4)$$

$$x_{ij} \leq x_{jj}, \qquad \forall i, j \in N, \qquad (3.5)$$

$$x_{ij} \in \{0,1\}, \qquad \forall i, j \in N, \qquad (3.6)$$

Where:

N: the set of vertices in the network;

$D = [d_{ij}]_{nxm}$: the symmetric distance matrix;

p: the number of facilities to be located;

$q_i$: the weight of vertex $i$;

$x_{ij}$: the allocation decision variables, where 1 denotes an allocation of a node $i$ to a facility $j$ and 0, otherwise; And $x_{jj}$ is equal to 1 if a vertex j is a facility and 0, otherwise.

The uncapacitated p-median problem has a number of variants, one of which is the p-median problem with fixed costs. The formulation for the p-median problem with fixed costs is very close to the traditional p-median problem, with just the addition of a term in the objective function to assume the form (3.7) that follows and a modification of the constraint (3.4) in the previous model to an inequality, as shown below:

$$Min \sum_{i=1}^{n}\sum_{j=1}^{n} q_i d_{ij} x_{ij} + \sum_{j \in I} f_j x_{jj} \qquad (3.7)$$

s.t.:

$$\sum_{j \in J} x_{jj} \leq p, \qquad (3.8)$$

And (3.3), (3.5), (3.6).

For the formulation of the capacitated p-median problem, a parameter $Q_j$ that states the capacity of a facility $j$ is inserted, resulting in the addition of the constraint (3.9) into the formulation for the uncapacitated p-median problem.

$$\sum_{i \in N} q_i x_{ij} \leq Q_j x_{jj,} \qquad \forall j \in N, \quad (3.9)$$

Three other useful location problems are shown in the study of Pizzolato (2002): the Simple Plant Location Problem (SPLP), the Set Covering Location Problem (SCLP) and the Maximal Covering Location Problem (MCLP). The SPLP aims to locate, out of a given set of potential locations, one or more facilities to serve a set of demand points. The SCLP aims the minimization of the number of facilities to be located required to serve all demand points, and the MCLP objectives the maximization of the covered demand, subject to a pre-determined *p* number of facilities.

## 3.2 Hub-and-spoke Networks

### 3.2.1 Concepts in Hub-and-Spoke Networks

With the increasing level of competitiveness observed in international markets, a very successful and useful strategy is the well-known hub-and-spoke network configuration. This configuration allows the decrease in the number of total links in a network. For instance, in a network with 9 completely interconnected nodes, the number of links is 36. The same network, configured in a hub-and-spoke strategy, with 3 hubs might have only 9 links, with the flows being channeled through the hubs, which are shown in Figure 3.1.



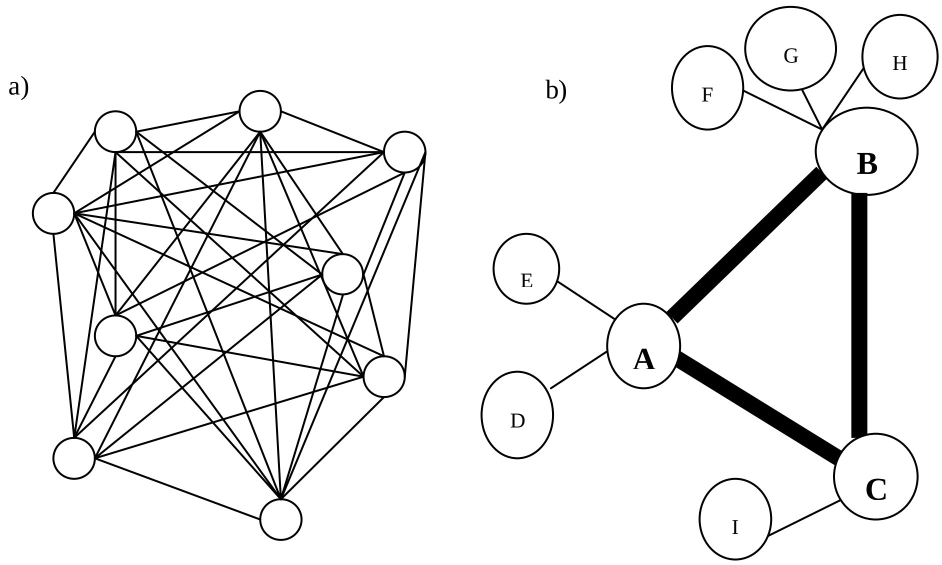


**Figure 3.1: Network completely interconnected (a) and network configured in hub-and-spoke model (b) - O'Kelly and Bryan (1999) adapted**

O`Kelly and Miller (1994) state that hubs are central transshipment facilities that allow the construction of a network where a large numbers of direct connections can be replaced with fewer, indirect ones. This also reduces and simplifies the network construction costs, centralizing commodity handling and sorting, allowing carriers to take advantage of scale economies through consolidation of flows.

The authors mention that for air passenger transportation, the US Federal Aviation Administration (FAA) defines the term hub as a geographical area and classifies it on the basis of its percentage of total passengers enplaned in that area. In their study, the authors define the term hub as a major sorting or a switching center in a many-to-many distribution system. According to the authors, the design of a hub network is a complex mixture of locational analysis and spatial interaction theory, involving in its most general form issues like finding the optimal locations for hub facilities, assigning spoke points to the hubs, determining linkages between hubs and the routing of flows through the network.

The authors define as Protocol A the standard hub network product of three simplifying restrictions: all hubs are fully interconnected; all nodes are connected to only one hub; and there are no direct non-hub to non-hub connections. Figure 3.2 shows this type of network.

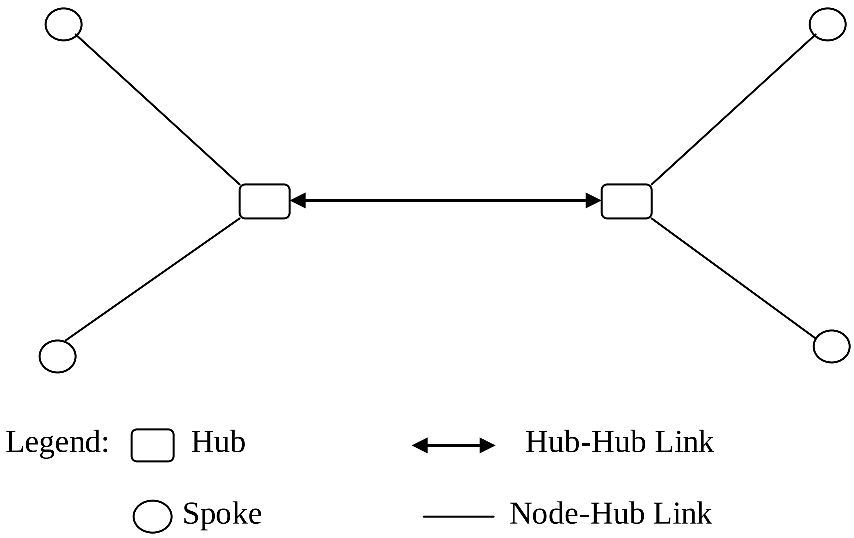


**Figure 3.2: Example of Protocol A (O`Kelly and Miller – 1994) – Adapted**

The authors mention that the Protocol A has two important properties: the first one is the *deterministic routeing*. This is based on the fact that, given fixed hub locations and the allocations of non-hub origins and destinations to hubs, there is only one shortest path between any origin-destination pair in the network because of the triangle inequality property. A second property is the *p-median problem constraint set*. It means that the Protocol A network characteristics allow the hub network design problem to be stated in similar format to a traditional optimal location problem. These two properties, according to the authors, allow the hub network design problem to be stated as analogues to traditional location problems.

The authors also say that a hub network consists of three major components: *service nodes*, *hubs* and *arcs*. *Service nodes*, also known as *spoke nodes*, are points, in which or for which, flows can be originated or destined. A *hub*, which can be characterized as a transshipment point, has the characteristic of a service node. All through flow that enters a hub must also exit that hub. The *arcs* that connect *service nodes* and the *hubs* have two properties: every *service node* must be connected to at least one hub and a valid path must exist between *hubs*. Table 3.1 summarizes the hub network classification system.



**Table 3.1: Hub Network Classification System (O`Kelly and Miller-1994)**

| Design Class | Node-hub assignment | Internodal Connetions | Interhub |
|---|---|---|---|
| Protocol A | Single hub only | Not allowed | Full |
| Protocol B | Single hub only | Not allowed | Partial |
| Protocol C | Single hub only | Allowed | Full |
| Protocol D | Single hub only | Allowed | Partial |
| Protocol E | Multiple hubs allowed | Not allowed | Full |
| Protocol F | Multiple hubs allowed | Not allowed | Partial |
| Protocol G | Multiple hubs allowed | Allowed | Full |
| Protocol H | Multiple hubs allowed | Allowed | Partial |

**Source: O`Kelly and Miller (1994)**

O`Kelly (1998) mentions that hubs are special nodes in a network and are located in a way to facilitate the connection between its interacting locations.

From the spatial organization viewpoint, the concept of hub-and-spoke networks is associated with linkages, hinterlands and hierarchies. A comparison between air passenger and air express freight is made. It is mentioned that the hub network design involves a decision about where to place the hub and how to route the traffic between origins and destinations. In the referred study, the author says that hub-and-spoke networks are a tangible physical manifestation of a process. Hubs are geographical because they serve a specific regional area, providing benefits to the region in which they are located.

The author outlines some features that pose difficulties in analyzing and modeling hub-and-spoke networks. They are: location of nodes, the linkages (routes and rates) and geographic issues (*hinterlands* and *hierarchies*). Regarding the location of nodes, the author says that it is important to differentiate the kind of application between the concept of *delivery systems* and *user attracting systems*. In the former one, the decision maker determines the location of the facilities and decides the rules of allocation to the facilities. In the later one, the facility is located by one agent, but the allocation decisions are decentralized.

The author (O'Kelly, 1998) makes a comparison about the application of these two concepts in air freight transportation and air passenger. For air freight hub-and-spoke case, it is mentioned that the concept best suited is a *delivery system*, because the operator decides where to place the sorting centers and has complete control over the rules for routing packages between these centers. In this way, the author says that it is fair to assume that the attraction of the service for end user is not a function of the routes.

On the other way, according to the same author, the air passenger transportation can be considered a case of *attracting system* because the role of consumer behavior and the inconvenience of making intermediate stops cannot be ignored. In this case, the location of hubs and the routing of a plane is under the control of a single decision maker. The author emphasizes that the critical difference between the air passenger system and the air freight system is the tension between price, demand and routing.

In relation to the linkages (routes and rates), this study shows that in hub-and-spoke systems the routing problem is not just a matter of solving shortest path problems. The determination of the optimal routing for any particular origin-destination pair is a complex question. As it was mentioned before, there are a lot of difficulties in modeling it. And most of these issues are about the allocation of the spoke points to the hubs (if they can choose among the hubs or if they face unique choice of hubs, the capacity limitations on the preferred shortest routes and the question involving economies of scale, density or scope, giving the incentives to the operator to route the flows differently from their myopically preferred alternative.

In *delivery systems,* based on air freight transportation, the decision maker can choose to direct flows in such a way that the lowest cost for the entire network is achieved. This does not happen for user *attracting systems* because a high efficient flow system may represent a decrease in the level of service provided, forcing the passengers to travel greater distances to achieve their final destinations. Regarding the tariffs, the economies of scale in hub networks can appear from the efficient bundling of flows, the centralized ground support and the maintenance shops.

The author cites that the conventional definition of agglomeration economies reflects the reduction in the costs of operation (or increase in revenues) that accrue to a business as a result of its location near to firms in similar or dissimilar industries. The former case is known as *localization economies* and the latter case is known as *urbanization economies*. For instance, in air passenger transportation the kinds of agglomeration effects that accrue to a hub are related to conferences, meetings and convention business. Another fact is that when a city becomes a hub for a major carrier, it is much harder for another non-hub carrier to operate in that airport, caused by the level of service achieved by the hub-carrier.

The strategic configuration of a network is strongly dependent of the market in which it is inserted. Point–to–point systems have an advantage in short-haul market pairs with a dense level of demand and hub-and-spoke systems look to be an ideal solution when it is possible to channel the flows through some switching points. One interesting thing to note is that air passenger and air freight transportation belong to different types of allocation rules. In the case of air passenger transportation, it is much better for a passenger to have an option to pick the convenient hub through which to make transfer. For air freight, the opportunity to maximize load factors, regardless of routing, gives the carrier every incentive to divert flows towards major hubs. In this way, it is clear to state that multiple allocation is better suited for passengers and single allocation is better suited for freight (O`Kelly – 1998).

O'Kelly and Bryan (1998) divide the hub-and-spoke configuration in two types: the first one, called single allocation, the spoke nodes are allowed to be allocated to more than one hub, and in the second one these nodes are allowed to be allocated to only one hub. In both cases, hubs are completely interconnected. Figure 3.3 illustrates this.

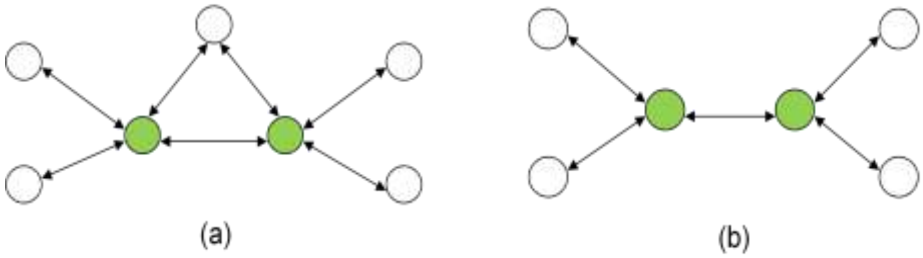


**Figure 3.3: Single Allocation (a) and Multiple Allocation (b) Networks**

O`Kelly et al. (1996) analyze the patterns for the single and multiple assignment allocation models. It is stated that the overall costs for the multiple allocation are smaller than those for the single allocation, except for very low values of the discount factor. When the value of the discount factor increases, the number of multiple allocations also increases, making the hubs closer to each other.

The authors recall the existing relation between the number of hubs and the value of the discount factor. Figure 3.4 illustrates an application for the single and multiple assignment p-hub location problems in a network with 20 nodes, using a different number of hubs for these two applications.

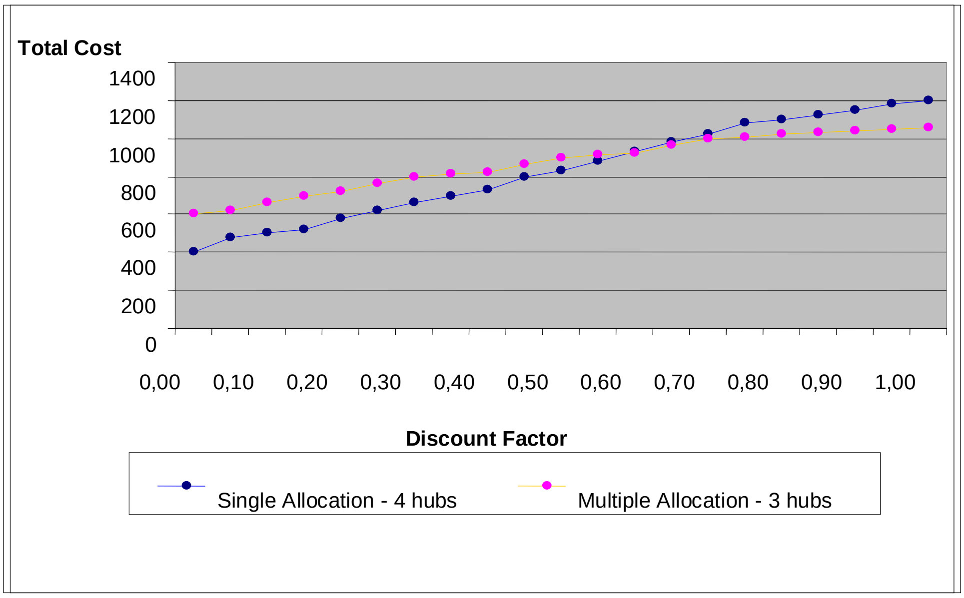


**Figure 3.4: The Relation Between Single Allocation (4 hubs) and Multiple Allocation (3 hubs)**



It is possible to notice that for small values of the discount factor (alpha), the single allocation model with four hubs is more economic than the multiple allocation models with three hubs. With the increasing value of alpha (decrease in the discount factor), the links amongst hubs become less attractive, making the allocation of a spoke node to more than one hub more advantageous.

O`Kelly and Bryan (2002) say that the "providers of transportation services may reduce their average unit costs by bundling flows and channeling them between hubs". These points are also called *transshipment points* or *distribution centers*. An important feature is the focusing on intensive levels of traffic into these points and one consequence is the creation of demand for labor and other specialized services at these locations.

In this study, the authors make a description and outline the differences between the single allocation and the multiple allocation models. According to the authors, "these interhub connectors are assumed to be some kind of efficient high capacity link because they are supposed to carry the entire concentrated flow between regional groups of nodes". The multiple allocation model allows more flexibility in the connections, being more useful to the passenger transportation, while the single assignment model is more appropriate for freight.

O`Kelly (1998b) introduces the mini-hub concept in a study for express package delivery systems: the mini-hub is mainly used to catch regional flows within a region. The author mentions that for a package delivery network, "there is tension between the need to provide timely connectivity between dispersed places, and at the same time, a crucial need to centralize sorting of packages. As a result, time constraints are of primary concern in network design".

The author outlines the difference between an express package network and a passenger network for a commercial airline. For the passengers, it is not recommended that they make large detours on their routes while packages do not face this problem, as long as they arrive at their destinations on time. According to the author, "congestion at hubs is another major concern. An easy way to avoid that is to locate a hub in a small city. For passenger airlines, this is not feasible. However, it is common for express package delivery systems".

In the cited study the author deals with the question "whether a group of nodes in a region is better off sending their interactions through a mini-hub, rather than through a central facility, or major-hub. The basic idea is that the sorting operation skims off some local interactions within a region instead of sending all the shipments through a central mid-continent hub". The author considered three distinct features: a) just one hub stop is allowed; b) the interhub links have no connection; and c) a distinction is made between different types of hubs, in this case major and mini hubs. The mini-hubs can only serve the nodes within a pre-determined region, filtering out the regional flow and not acting as a major hub.

To model such features, the author suggested the following notation: $Z_{ij}$ is one of the decisions variables and assumes the value of 1 if a (*i*,*j*) pair is connected to a mini-hub or 0, otherwise. $A_{ij}$ is the cost per unit of flow of interacting between *i* and *j* through the mini-hub and $B_{ij}$ has the same description, but using the major-hub as a central point. The costs are measured in this way: $A_{ij} = f(d_{ih} + d_{hj})$ and $B_{ij} = f(d_{iH} + d_{Hj})$, where $h$ is the location of the mini-hub and $H$ is the location of the major hub. It is assumed that *f(a)* > *g(a)*, to reflect the higher rates applied to smaller volume shipments that are processed by the mini-hub.

The author mentions that “the major/mini-hub model is a special case of more general multiple assignment model. This is because, with no interhub discount, the general multiple assignment model, by connecting every node pair (*i,j*), via the cheapest route, will not utilize the interhub link”. However, according to the author, “there is still a distinction between the two models in that the major/mini-hub model differentiates between two different types of hubs – major and mini – while the multiple assignment does not”.

Aykin (1995) mentions that the development of the hub-and-spoke network is one of the most important innovations in the aviation industry since deregulation. For the author, in a hub-and-spoke network flows from a set of outlying hubs, non-hub nodes arrive at hubs and, after regrouping, all leave the hub facilities bound either to other hubs or to the their ultimate destinations. The centralization and broader scope of operations let the system take advantage of economies of scale. In this study two networking policies are considered: *non-strict hubbing*, in which channeling flows through hubs is not required but chosen if cost efficiency is found and *strict and restrictive hubbing*, in which all flows to and from a node are channeled through the same hub.

The author formulates an integer programming model for the *non-strict* policy followed by an enumeration procedure and solves the problem optimally under this policy while an efficient heuristic is proposed. For the *strict and restrictive hubbing policy* a quadratic integer program is developed, followed by an application of a branch-and-bound algorithm and a simulated annealing based heuristic, achieving optimal results for the network under this policy.

In respect to the *non-strict hubbing policy,* the author suggests three types of services to be provided: *nonstop*, *one-hub stop* and *two-hub stop*. It is emphasized that the cost per unit of flow per mile for the hub connected (*one-hub stop* and *two-hub stop*) services is generally lower than that for the *nonstop* service on the same route segment due to higher traffic volumes and centralization of the operations, although the distance travelled in the hub connected services is longer. Under this policy, *nonstop* services are permitted between all nodes and many factors including demand and availability of resources affect the decision to provide *nonstop* service between two nodes.

Martin and Roman (2003) studied the viability of a hub-and-spoke network between South America and Europe. According to these authors, regarding passengers, some issues are related to the increase in the total travel time, including the connection time at a hub, extra operations of landing and taking-offs, and extra distance traveled in links among spoke nodes. But most of the time, these are offset by the decrease in flight delays and the increase in the frequency of flights. This configuration increases the accessibility of cities and regions, and creates new opportunities in the job market. The authors cite the market between Europe and USA, that have experienced a deregulation at the end of the 80`s decade, with the air fares having faced a reduction between 35% and 45% and the boarding rates in the US airports having increased around 55%.

In Europe, according to Cento et al. (2005): "The development of the hub-and-spoke network started some time ago in the long history of European aviation. Before liberalization, the Hub-and-Spoke network in Europe has developed out of the former national flag carriers and took advantage of operating in a regulated industry: bilateral agreements, protected markets, and administered prices. Indeed, the former bilateral regime of air service agreements has already led to the development of hubs".

O`Kelly and Bryan (1998) state that the majority of the hub location models in Geography, Operations Research and Transportation, do not treat adequately the economies of scale experienced in the links between hubs. Usually, according to the authors, it is assumed that costs do not depend on the flow transported. Sasaki et al. (1999) recall that the hub-and-spoke strategy generates benefits for both air companies and passengers. It is mentioned that the most usual model is the one using two hubs although in their study they consider a model with only one hub.

Kara and Tansel (2000) consider the hub location problem taking into consideration the *minimax* criteria. For instance, in their study, they consider the minimization of the maximum delivery time of a product.

Huston and Butler (2001) suggest that the decision to locate a hub airport is very important in the operation context of every kind of industry, especially because of the economic activity associated to a hub. Fundamental issues that ought to be considered, according to these authors, are: climate, demographic and climate features of a place.

Aykin (1995) remembers that the economy of scale attained by the increase in the flow volume between hubs is one of the main advantages of this network configuration. In his paper, the author introduces the hub location and routing problem on the plane. In this modeling, for a pair of demand nodes ($i$,$j$) it is allowed to ship the flows either through a hub or directly (without stopping at a hub). Two cases are considered: the one that allows 100% of nonstop links amongst demand points and the one that does not allow any. The set of routes that can be made directly is an input to the problem. A formulation of the problem and a solution algorithm with four methods for finding starting solutions is also proposed.

Kara and Alumur (2008) classify and make a survey of the models for the hub location subject, having reviewed more than 100 papers. They state that the hub location problem is concerned with the issue of locating the hub facilities and the allocation problem of the demand nodes to the hubs for routing the traffic between origin and destination pairs. In this subject, according to the authors, often there are three assumptions: there exists a complete hub network, with a link between every hub pair; economies of scale are applied to the inter-hub connections and no direct services amongst any spoke nodes are allowed.

The authors mention that almost all of the hub location models defined in the literature have analogous location versions: the p-hub median problem, the hub location problem with fixed costs, the p-hub center problem and hub covering problems. For the p-hub median problem, the authors divide into two sub-sections: the single and the multiple allocation models. For the single allocation p-hub median problem, according to the authors: “in terms of required number of variables and constraints, Ebery (2001) provides the best mathematical formulation while in terms of computing time requirement the best mathematical formulation is the study of Ernst and Krishnamoorthy (1996). The most efficient exact solution procedure is the shortest-path based branch-and-bound algorithm presented in Ernst and Krishnamoorthy (1998b). Up to now, the largest set of problems solved to optimality had 100 nodes”.

Table 3.2 was extracted from Kara and Alumur`s study and overviews the literature on single allocation p-hub median.

**Table 3.2: Literature for the Single Allocation P-Hub Median Problem**

Single allocation *p*-hub median literature

| Year | Authors | Notes |
|---|---|---|
| 1987 | O'Kelly | Quadratic integer program, HEUR1, HEUR2 |
| 1990 | Aykin | Procedure to find optimal allocations |
| 1991 | Klincewicz | Exchange heuristic |
| 1992 | Klincewicz | Tabu search and GRASP heuristics |
| 1994b | Campbell | First linear integer formulation |
| 1994 | Skorin-Kapov and Skorin-Kapov | Tabu search heuristic |
| 1995 | O'Kelly, Skorin-Kapov and Skorin-Kapov | Lower bounding technique |
| 1996 | Campbell | MAXFLO and ALLFLO heuristics |
| 1996 | Ernst and Krishnamoorthy | New formulation, simulated annealing heuristic, B&B method |
| 1996 | O'Kelly, Bryan, Skorin-Kapov and Skorin-Kapov | New formulation for symmetric flow data |
| 1996 | Skorin-Kapov, Skorin-Kapov and O'Kelly | New mathematical formulation leading to tight LP relaxation |
| 1996 | Smith, Krishnamoorthy and Palaniswami | Modified Hopfield neural network heuristic |
| 1997 | Sohn and Park | Two-hub location problem |
| 1998b | Ernst and Krishnamoorthy | Shortest path based B&B algorithm |
| 1998 | Pirkul and Schilling | Lagrangean relaxation heuristic |
| 1998 | Sohn and Park | New formulation for symmetric cost, and allocation problem |
| 2000 | Sohn and Park | Three-hub allocation problem |
| 2001 | Abdinnour-Helm | Simulated annealing heuristic |
| 2001 | Ebery | New formulations for $p = 2$ and 3 |
| 2005 | Elhedhli and Hu | Minimized congestion at hubs |

**Table 3.3: Literature for the Mutliple Allocation P-Hub Median Problem**

Multiple allocation *p*-hub median literature

| Year | Authors | Notes |
|---|---|---|
| 1992 | Campbell | First linear integer program |
| 1994b | Campbell | New formulations, flow thresholds, fixed costs |
| 1996 | Campbell | Greedy-interchange heuristic |
| 1996 | Skorin-Kapov, Skorin-Kapov and O'Kelly | New mathematical formulation leading to tight LP relaxation |
| 1998a | Ernst and Krishnamoorthy | New formulation, B&B method, two heuristics |
| 1998b | Ernst and Krishnamoorthy | Shortest path based B&B algorithm |
| 1999 | Sasaki, Suzuki and Drezner | 1-stop problem |
| 2004 | Boland, Krishnamoorthy, Ernst and Ebery | Preprocessing and tightening constraints |

**Source: Kara and Alumur (2008)**

Tables 3.4, 3.5 and 3.6 show the literature listed by the authors for the hub location problem with fixed costs, for the p-hub center problem and for the hub covering problem, respectively.

**Table 3.4: Literature for the Hub Location Problem with Fixed Costs**

The literature on the hub location problem with fixed costs

| | | | |
|---|---|---|---|
| Uncapacitated hub location problem | Single Allocation | O'Kelly (1992a) | Quadratic integer program |
| | | Campbell (1994b) | First linear integer formulation |
| | | Abdinnour-Helm and Venkataramanan (1998) | New quadratic formulation, genetic algorithm |
| | | Abdinnour-Helm (1998) | Hybrid heuristic algorithm |
| | | Labbé and Yaman (2004) | Valid and facet defining inequalities |
| | | Topcuoglu et al. (2005) | Genetic algorithm |
| | | Cunha and Silva (2007) | Hybrid genetic algorithm |
| | | Chen (2007) | Hybrid heuristic algorithm |
| | Multiple Allocation | Campbell (1994b) | First linear integer formulation |
| | | Klincewicz (1996) | B&B algorithm |
| | | Mayer and Wagner (2002) | B&B algorithm (HubLocater) |
| | | Boland et al. (2004) | Preprocessing procedures, tightening constraints |
| | | Hamacher et al. (2004) | Polyhedral study, new formulation |
| | | Marín (2005b) | Valid inequalities, relax-and-cut algorithm |
| | | Marín et al. (2006) | New formulation |
| | | Cánovas et al. (2007) | Heuristic based on dual-ascent technique, best B&B algorithm |
| Capacitated hub location problem | Single Allocation | Campbell (1994b) | First linear integer formulation |
| | | Aykin (1994) | New formulation allowing direct connections |
| | | Aykin (1995a) | Formulation with given number of hubs to locate, allows direct connections |
| | | Ernst and Krishnamoorthy (1999) | New formulation, two heuristics, B&B algorithm |
| | | Labbé et al. (2005) | B&B algorithm |
| | | Costa et al. (2007) | New bi-criteria problems minimizing total cost and service time |
| | Multiple Allocation | Campbell (1994b) | First linear integer formulation |
| | | Ebery et al. (2000) | New formulation, a heuristic, B&B algorithm |
| | | Sasaki and Fukushima (2003) | Formulation for the 1-stop hub location problem, B&B algorithm |
| | | Boland et al. (2004) | Preprocessing procedures, tightening constraints |
| | | Marín (2005a) | New formulation |

**Source: Kara and Alumur (2008)**

**Table 3.5: Literature for the P-Hub Center Problem**

*p*-Hub center literature

| Year | Authors | Notes |
|---|---|---|
| 1994b | Campbell | Different types of *p*-hub center formulations |
| 2000 | Kara and Tansel | Various linear formulations for single allocation |
| 2001 | Pamuk and Sepil | Heuristic for the single allocation problem |
| 2002a | Ernst et al. | New formulations for both single and multiple allocation, heuristic and a B&B algorithm |
| 2002b | Ernst et al. | Heuristic algorithms for the allocation subproblem |
| 2003 | Baumgartner | Polyhedral properties, valid inequalities and branch-and-cut algorithm |
| 2006 | Hamacher and Meyer | Solving hub covering problems combined with binary search |
| 2006 | Gavriliouk and Hamacher | Applied aggregation and proposed error measurements |
| 2007 | Campbell, Lowe and Zhang | Complexity results and formulations for the allocation subproblem |

**Source: Kara and Alumur (2008)**

**Table 3.6: Literature for the Hub Covering Problem**

Hub covering literature

| Year | Authors | Notes |
|---|---|---|
| 1994b | Campbell | Different types of hub-covering formulations |
| 2003 | Kara and Tansel | Various linear formulations for the single allocation hub set covering problem |
| 2004b | Wagner | Improved model formulations for both single and multiple allocation hub covering problems |
| 2005 | Ernst et al. | New formulations for both single and multiple allocation, implicit enumerative solution method |
| 2006 | Hamacher and Meyer | Compared formulations, identified facet defining valid inequalities |

**Source: Kara and Alumur (2008)**

Bryan and O`Kelly (1999) make an analytical review for hub-and-spoke networks in air transportation. The authors found that "research needs to be devoted to developing more reliable heuristics for the multiple assignment model and its extensions and that additional research is needed to understand the conditions under which the model will tend to have integer solutions".

Alvarez et al. (2007) present a hub location model for a cargo transportation network, considering capacity issues and introducing costs due to congestion in hubs. The authors developed a Simulated Annealing algorithm to solve this problem. Lin and Chen (2008) present the constrained generalized hub-and-spoke network design problem which consists in the determination of the smallest fleet size with their routes and freight paths to minimize operating costs and "an implicit enumeration algorithm with embedded integral constrained multi-commodity minimization of costs is presented".

Tan and Kara (2007) focus their paper on cargo delivery systems making an application of the hub covering model, presenting integer programming formulations and large-scale implementations of the models in Turkey. Takano and Arai (2009) make an application of the meta-heuristic Genetic Algorithms for the hub-and-spoke problem in a containerized cargo transport and present a case study with 18 ports.

Bookbinder et al. (2007) make a study for the hub-and-spoke network in the railroad freight transportation considering the decision to find transport routes, frequency of service, length of trains to be used and volume carried. The authors outline that hub-and-spoke networks, found to be very useful in air freight transportation, were not in the past considered useful for railways systems. The authors develop a linear integer programming model whose objective function includes also the costs due to the transit time spent by freight in the network. An application in rail freight systems in Europe is made through the developing of a heuristic algorithm that solves large instances followed by a sensitivity analysis.

Kara and Tansel (2001) analyze the latest arrival hub location problem emphasizing that the traditional hub location problem does not take into consideration issues like "transient times spent at hubs for unloading, loading and sorting operations". The focus of this study is on the minimization of the arrival time of the last arrived item in cargo delivery systems considering a "model that correctly computes the arrival times by taking into account the flight times and the transient times". Linear and nonlinear integer formulations are provided, followed by computational experiments.

Yaman et al. (2007) focus their study on the latest arrival hub location problem for cargo delivery systems with stopovers, taking into consideration the service structure of ground transportation based cargo delivery companies. According to the authors, this type of problem is a "new *minimax* model that focuses on the minimization of the arrival time of the last item to arrive, taking into account journey times as well as transient times at hubs". In this paper, the authors propose a "generic mathematical model that allows stopovers for the latest arrival hub location problem".

Martin and Roman (2004) analyze the hub location problem under competition in intercontinental aviation markets through the developing of a game theoretical model. The results are applied to intercontinental routes between Europe and South-America considering duopoly.

Adler and Smilowitz (2007) analyze the alliances and mergers in hub-and-spoke networks under competition. This study combines "profit-maximizing objectives to cost-based network design formulations within a game theoretic framework" and the results can enable the "merging airlines to choose appropriate international hubs for their integrated network based on their own and their competitor`s costs and revenues in the form of best response functions".

Elhedhli and Hu (2005) consider the design of a hub-and-spoke network under congestion, taking related issues into consideration. This is done through the addition to the objective function of a non-linear cost term. Then, the model is linearized and a Lagrangean heuristic that finds high-quality solutions in reasonable time is provided.

Campbell and Krishnamoorthy (2005) provide an integer programming formulation for the hub arc location model and describe two optimal solution approaches and compare their performance using standard hub location data. Campbell et al. (2007) discuss the p-hub center allocation problem and assume that it is a sub-problem of the location problem, where hub locations are known. Integer Programming formulations and complexity results are presented.

Campbell (1992) analyses the problem of location and allocation for distribution systems considering transshipments and economies of scale. Campbell et al. (2003) study the hub arc location problem on a cluster of workstations using regular data for hub location models.

Wagner (2007) proposes an exact solution procedure for a cluster hub location problem presenting a new mixed integer programming formulation under a non-restrictive policy system introduced by Sung and Jin (2001). The author shows computational results that demonstrate that the new formulation solves much larger instances.

Ernst and Krishnamoorthy (1998) consider the p-hub median problem aiming to an exact solution approach based on shortest-paths. The authors describe “a novel exact solution approach for solving the multiple allocation case of the p-hub median problem and show how a similar method can be adapted for solving the more difficult single allocation case” with “numerical results showing the superiority of this new approach over traditional LP-based methods”.

Ebery (2001) presents an efficient approach for solving large single allocation p-hub problems using two or three hubs with a new mixed integer linear programming formulations requiring fewer variables. Gonzales and Martin (2008) study the solution of a capacitated hub location problem, consisting of the determination of the routes and the hubs to be used in a given capacitated network. According to the authors, the capacities and costs of the arcs and hubs are given and the arcs connecting the hubs are not assumed to create a complete graph. A mixed integer linear programming formulation is presented and two branch-and-cut algorithms based on decomposition techniques are shown.

Abdinnour-Helm and Venkataramanan (1998) present solution approaches for hub location problems with the presentation of a quadratic integer formulation for the Uncapacitated Hub Location Problem (UHP), based on the idea of multi-commodity flows in networks and use a branch-and-bound procedure to find optimal solutions. The authors also used in this study the artificial intelligence-based technique – Genetic Search – to find solutions efficiently.

Kara and Tansel (2003) present models and linearizations for the single-assignment hub covering problem, which is mentioned to be the unstudied hub location problems in the literature. The authors present a combinatorial and a new integer programming formulation, giving three linearizations for the old model and one linearization for the new model.

Krishnamoorthy et al. (2000) present formulations and algorithms for the capacitated multiple allocation hub location problems, showing a “new mixed integer linear programming formulation” and constructing “an efficient heuristic algorithm using shortest-paths”. Skorin-Kapov et al. (1996) develop a mixed binary linear formulation with tight linear programming relaxations.

Landete et al. (2006) “review the integer linear formulations of the Uncapacitated Multiple Allocation Hub Location Problem (UMAHLP)” and study the “scope of validity of these formulations and give new ones that generalize the older formulations”. Yaman (2005) studies the problem of locating hubs in a telecommunication network aiming to the minimization of the costs of installing hubs and capacity units on arcs and make a polyhedral analysis for it.

Sohn and Park (2000) “consider the single allocation problem in the interacting three-hub network with fixed hub locations”, showing that this problem is considered to be NP-hard as long as the number of hubs is at least three, while in a two-hub system this model assumes a characteristic of a polynomial time algorithm. In this study, the authors propose a mixed integer formulation for this problem and take into consideration its polyhedral properties.

Aykin and Brown (1992) make an examination of the problem of location-allocation on a plane and on a sphere. The authors consider a situation in “which flows between the existing facilities are channeled through the new facilities and the level of interaction between them is determined by the flows between the existing facilities they serve. Aykin (1988) formulates a multihub location problem and develops a condition that a destination is optimally assigned to a hub facility.

Abdinnour-Helm (1998) presents a hybrid heuristic for the uncapacitated hub location problem based on Genetic Algorithms (GA) and Tabu Search (TS). In this study, the number of hubs, the location of the hubs and the assignment of spoke points to the hubs are determined by the model. Camargo et al. (2008) make an application of the Benders Decomposition concept for Uncapacitated Multiple

Allocation Hub Location Problem (UMAHLP) and outline that the UMAHLP arises when commodities must be transported between several origin-destination pairs. The authors mention that in this study they were able to solve large instances in a reasonable time.

Chen (2007) also proposes a hybrid heuristic for the Uncapacitated Single Allocation Hub Location Problem (USAHLP) through the consideration of “two approaches to determine the upper bound for the number of hubs and along with a hybrid heuristic based on the simulated annealing method, tabu list and improvement procedures” to solve. According to the authors, computational results showed that the hybrid heuristic proposed outperformed the simulated annealing and the genetic algorithms methods in solving the USAHLP.

Captivo et al. (2008) provide a bi-criteria approach for the capacitated single allocation hub location problem. The authors introduce a second objective function to the model that tries to minimize the time to process the flow entering the hubs, despite the capacity constraints to limit the amount of flow that can be received by the hubs. In this study, two bi-criteria single allocation hub location problems are presented, with the first model considering total time as the second criteria and in the second model the maximum service time for the hubs is minimized.

Contreras et al. (2009) consider the single assignment capacitated hub location problem and propose a Lagrangean relaxation to obtain tight upper and lower bounds. According to the authors, the decomposing into smaller sub-problems that can be solved efficiently is explored and according to the authors the results obtained using benchmark instances are impressive.

Garcia et al. (2007) use the dual-ascent technique to solve the uncapacitated multiple allocation hub location problem and instances with up to 120 nodes are solved. Pamuk and Sepil (2001) address the p-hub center problem via a single-relocation algorithm using Tabu Search.

Krishnamoorthy et al. (2004) consider formulations and solution approaches using the cutting and preprocessing technique for multiple allocation hub location problems. The authors also employ “flow cover constraints for capacitated problems to improve computation times”.

Klincewicz (2002) presents an enumeration and search procedures for a hub location problem with economies of scale. The author shows that the FLOWLOC model proposed by O`Kelly and Bryan (1998) can be solved using the classic Uncapacitated Facility Location Problem for a fixed set of hubs, motivating an optimal enumeration procedure for this model, “as well as some search heuristics that are based upon Tabu Search and GRASP”.

Labbe et al. (2005) present a branch-and-cut algorithm for hub location problems with single assignment, investigating polyhedral properties of this type of problems and developing a branch and cut algorithm based on these results. Labbe and Yaman (2007) “consider the problem of locating hubs and assigning terminals to hubs for a telecommunication network”, in a star-star fashion and “present two formulations and show that the constraints are facet-defining inequalities in both cases”.

Marin (2005) formulates and solves splittable capacitated multiple allocation hub location problems. The concern in this study is the interpretation of the capacity and “Tight linear programming formulations for the problem are presented, along with some useful properties of the optimal solutions which can be used to speed up the resolution”.

Pirkul and Schilling (1998) outline the importance of hub-and-spoke networks in logistics, communication and mass transportation and present an efficient procedure for designing the single allocation for hub-and-spoke systems, providing “a method that delivers both high quality solutions and firm measures of that quality and allowing them to be solved in a reasonable time”.

Sung and Jin (2001) analyze the hub network design problem where the node clusters are given and non-stop service is allowed. The objective is to determine the location for the hubs in the node clusters such as the total cost is minimized. A dual-based solution approach is proposed with an application of numerical examples.

Yaman and Carello (2005) study the capacitated hub location problem with modular link capacities which differs from the classical hub location problem in two ways: “the cost of using an edge is not linear but stepwise and the capacity of a hub restricts the amount of traffic transiting through the hub rather than the incoming traffic". Exact and heuristic methods are provided.

Mayer and Wagner (2002) apply a different branch-and-bound procedure called *hub locator* for the multiple allocation hub location problem. This technique, *hub locator*, considers an aggregated model formulation that enables the tightening of the lower bounds. The authors made computational experiments with problems with up to 40 nodes in a reasonable time. Topcuoglu et al. (2005) present a "new and robust solution based on a genetic search framework for the Uncapacitated Single Allocation Hub Location Problem (USAHLP)" and compare the solutions achieved by this technique with the best solutions presented in the literature.

Button (2002) emphasizes that "in order to the airlines achieve the highest possible load factor, minimize costs and keep the fares down, it is needed to keep aircraft in the air for the longest possible time. To achieve that, many airlines operate hub-and-spoke networks which consist of consolidating traffic from a diverse range of origins and are destined to a diverse range of final destinations at large airports called hubs". By the author, an important fact to make the fare lower was the US domestic deregulation, where also a greater variety of service choices became available to the passengers. Taking into consideration the period between 1978 and 1987, passengers enplanements were up 55%, employment had risen from 340,000 to 450,000, while scheduled passenger revenue miles were up 62% and seat availability up 65%.

In terms of fare, still by the author (Button-2002), deregulation allowed discount fares and 90% of travelers were using them by 1986, enjoying an average discount of 61%. In this study, the author makes an explanation of 10 issues for conceptualizing the term hub, which he calls myths. They are defined as: "all hubs are created equal"; "hubs are unique to US air transportation"; "smaller communities are disadvantage by hub-and-spoke networks"; "hubs create excessive airport congestion"; "premium fares at hubs drive discourage economic developments"; "hubs are environmentally harmful"; "carriers have an incentive to restrict capacity expansions at hubs"; "hubs are deterrent to new airline entry"; "hubs limit competition"; and "hub based networks generate large profit for airlines".

An interesting analysis made by the author is that, according to him, “there is a no unique or widely used definition of what exactly constitutes a hub airport and a number of definitions coexist, which goes from an air traffic management perspective to academic ones. Within the air traffic management`s point of view, “hubs are airports that have a large preponderance of flights operated as part of an essentially radial network by one carrier”, with minor exceptions for just a few major airports that operate as hubs and have two main carriers. According to the author, a carrier operating in a hub airport feeds three or more banks of traffic daily, from some 40 or more cities.

Button (2002) also mentions that “the rationale for hub-and-spoke operations rest on both the cost and demand sides. On the cost side, economies of scope exist when one airline can produce two or more services more cheaply than if these services were produced by separate airlines. The equation (3.10) denotes how the economy of scope is assessed.

$$S = \{[C(Q_1) + C(Q_2)] - C(Q_1 + Q_2)\}/\{C(Q_1 + Q_2)\} \qquad (3.10)$$

Where:

$C(Q_1)$: Cost of producing $Q_1$ of output one alone;

$C(Q_2)$: Cost of producing $Q_2$ of output two alone;

$C(Q_1 + Q_2)$: Cost of producing $Q_1$ plus $Q_2$;

The main difference between economies of scope and economies of density is that when $S > 0$, economies of scope exist. When $C/Q$ falls as $Q$ expands, there are economies of scale. According to the author, “the economies of traffic density occur when the average unit cost of production declines as the amount of traffic increases between any given set of points served”.

### 3.2.2 Models in Hub-and-Spoke Networks

O`Kelly (1986) characterizes hubs as central points in a network and proposes a consolidation function at these central points. In his study, the author considers two scenarios: the utilization of one and two hubs in a network. Regarding the passengers, the author outlines that the connections need to be configured in a way that the inconvenience for the passengers is minimized. For the utilization of one hub, the author considers that every flow must go through the hub. The location of this hub in a plane is denoted by $Q = (x, y)$. The location of the n origins $i$, $i$ = 1, 2, …, n, and destinations $j$, $j$ = 1, 2, …, $n$, are denoted by $P_i = (x_i, y_i)$ and $P_j = (x_j, y_j)$. Denoting the distance between nodes $P_i \; and \; P_j$ by $C = (P_i, P_j)$ and the flow between then by $W_{ij}$, passing through the hub $Q_i$ the objective function would be as follows:

$$Min\, Z = \sum_{i \in N} C(P_i, Q) \sum_{j \in N} W_{ij} + \sum_{j \in N} C(Q, P_j) \sum_{i \in N} W_{ij} \qquad (3.11)$$

In the absence of scale effects and ignoring costs for establishing the routes amongst the cities, there is no rational reason to utilize a network with only one hub, because:

$$\sum_{i \in N} \sum_{j \in N} W_{ij} C_{ij} < \sum_{i \in N} \sum_{j \in N} W_{ij} \left[C(P_i, Q) + C(P_j, Q)\right] \qquad (3.12)$$

Nevertheless, assuming that there is a cost k associated to each route $i,j$ and comparing it with a network using only one hub, concludes that it is useful to use a hub only if:

$$\sum_{i \in N} \sum_{j \in N} W_{ij} \left[C(P_i, Q) + C(P_j, Q)\right] + kn < \sum_{i \in N} \sum_{j \in N} W_{ij} C_{ij} + \left(k/_2\right) n(n-1) \qquad (3.13)$$

O`Kelly (1987) presents the hub location problem as a quadratic problem and applies two different heuristics. The first, called HEUR1, the strategy adopted is the complete enumeration using all the nodes in the network and allocating them to the nearest hub. The second, called HEUR2, analyzes the allocation of every spoke node to the nearest hub and second nearest hub.

The model is shown below. The objective function minimizes the total transportation costs, through hubs *k* and *m*, which is divided into three parts: the first one accounts the cost from an origin *i* to the hub *k*, while the second one takes into consideration the cost from hub *m* to the destination *j*. The last part considers the transportation costs between hubs *k* and *m*. The decision variable for the location of hubs is $x_{jj}$, assuming the value of 1 if the location *j* is a hub and 0, otherwise; to denote the allocation of a node *i* to a hub *k*, the decision variable $x_{ik}$ assumes the value of 1 if the node *i* is allocated to a hub *k* and 0, otherwise. The number of hubs to be located is defined by the parameter *p*.

$$Min\ Z = \sum_{i \in N} \sum_{j \in N} W_{ij} \left( \sum_{k \in N} x_{ik} C_{ik} + \sum_{m \in N} x_{jm} C_{jm} + \alpha \sum_{k \in N} \sum_{m \in N} x_{ik} x_{jm} C_{km} \right) \quad (3.14)$$

s.t.:

$$(n - p + 1) x_{jj} - \sum_{i \in N} x_{ij} \geq 0, \qquad \forall j \in N, \qquad (3.15)$$

$$\sum_{j \in N} x_{ij} = 1, \qquad \forall i \in N, \qquad (3.16)$$

$$\sum_{j \in N} x_{jj} = p, \qquad (3.17)$$

$$x_{ij} \in \{0,1\} \qquad (3.18)$$

The set of constraints (3.15) say that no node is assigned to any location, unless a hub is already opened at that site (in other words: $x_{jj}$ must be equal one before any other node can be allocated to *j*. Constraint (3.16) guarantees that each node is assigned to only one hub and constraint (3.17) imposes the number *p* of hubs to be located.

The multiple assignment p-hub location problem, in which the spoke points may be allocated to more than one hub, was originally proposed by Campbell (1995). In this case, the number *p* of hubs to be located is known and there are no capacity constraints and fixed costs. Before reaching the destination *j*, the flow from an origin *i* passes through hubs *k* and *m*.

$$Min \sum_i \sum_j \sum_k \sum_m W_{ij} (C_{ik} + \alpha C_{km} + C_{mj}) X_{ijkm} \qquad (3.19)$$

s.t.:

$$\sum_k Y_k = p \qquad (3.20)$$

$$\sum_k \sum_m X_{ijkm} = 1, \qquad \forall i, j \qquad (3.21)$$

$$\sum_m X_{ijkm} - Y_k \leq 0, \qquad \forall i, j, k \qquad (3.22)$$

$$\sum_k X_{ijkm} - Y_m \leq 0, \qquad \forall i, j, m \qquad (3.23)$$

$$Y_k \in \{0,1\}, \qquad \forall k \qquad (3.24)$$

The objective function (3.19) seeks the minimization of the total costs involved in the transportation between every origin and destination pair. The set of constraints (3.20) establishes that *p* hubs must be opened. The set of constraints (3.21) imposes that all the flow transported between every origin-destination pair is made using hubs *k* and *m*. Constraints (3.22) and (3.23) impose that every flow moved are made through hub locations. The decision variables $X_{ijkm}$ determine the allocation of the spoke nodes to the hubs. The decision variable $Y_k$ indicates the location of hubs, i. e., $Y_k$ = 1 means that one hub is located in location *k*, while $Y_k$ = 0 otherwise.

For the single assignment p-hub location problem (Skorin-Kapov et al. – 1995), every spoke node must be allocated to only one hub. The objective function is the same as in the multiple assignment allocation model.

$$Min \sum_i \sum_j \sum_k \sum_m W_{ij}(C_{ik} + \alpha C_{km} + C_{mj}) X_{ijkm} \tag{3.19}$$

s.t.:

$$\sum_k Z_{kk} = p \tag{3.25}$$

$$\sum_k Z_{ik} = 1, \quad \forall i \tag{3.26}$$

$$Z_{ik} - Z_{kk} \leq 0, \quad \forall i \neq k \tag{3.27}$$

$$\sum_m X_{ijkm} - Z_{ik} \leq 0, \quad \forall i, j, k \tag{3.28}$$

$$\sum_k X_{ijkm} - Z_{jm} \leq 0, \quad \forall i, j, m \tag{3.29}$$

$$Y_k \in \{0,1\}, \quad \forall k \tag{3.30}$$

Where:

$X_{ijkm}$ : Fraction of flow transported between origin *i* and destination *j*, through hubs located in nodes *k* and *m*;

$Y_k$ = 1, if a location *k* is a hub; 0, otherwise;

$W_{ij}$ : Flow originated at location *i* and destined to location *j*;

$C_{ij}$ : Unit cost from *i* to *j* (denoted by the distance);

$p$ : The number of hubs to be opened;

$\alpha$ : The interhub discount factor $(0 \leq \alpha \leq 1)$.

$Z_{ik}$ : 1, if a spoke node in *i* is allocated to a hub in *k*; 0, otherwise.

Constraints (3.25) indicate the number *p* of hubs to be located. Constraint (3.26) imposes that a spoke node can be allocated to only one hub. The set of constraints (3.27) indicate that a spoke node can be allocated to a hub only if this point is a hub. Constraints (3.28) and (3.29) impose that a valid path only exists if an origin and a destination node are allocated to hubs.

Campbell (1994) mentions that discrete hub location problems is characterized by the aim of "locating a set of fully interconnected hubs, which serve as transshipment and switching points for the traffic between specified origins and destinations". For every o-d pair, there is an attribute associated, such as distance, time or cost and a non-negative flow $W_{ij}$ is applied. The author emphasizes that "in hub systems, origin to destination movements are generally via one or two hubs and as long as the cost of movement is a non-decreasing function of distance, no origin to destination movements are via more than two hubs, since hubs are fully interconnected”. In his study, the author presents different mathematical programming formulations for discrete hub location problems, making an analogy with the four fundamental types of discrete facility location problems: the p-median problem, the uncapacitated facility location problem, the p-center problem and the covering problem.

In the referred study, the author considers discrete hub location problems and assumes that the following items are given:

a) *n* demand locations (origin/destinations);
b) *r* potential hub locations;
c) the flow amongst the o-d pairs;
d) the per unit cost amongst all o-d pairs;
e) the interhub discount factor $\alpha$.

The author defines the following parameters and decision variables to be used throughout the study. They are:

$X_{ijkm}$: Fraction of flow from origin *i* to the destination *j* through hubs *k* and *m*;

$Y_k$: 1, if location *k* is a hub; 0, otherwise;

$Z_{ik}$: 1, if location *i* is allocated to the hub in *k*; 0, otherwise;

$W_{ij}$: Flow from location *i* to location *j*;

$C_{ij}$: Standard cost per unit of flow from location *i* to *j*;

$C_{ijkm}$: $C_{ik} + C_{mj} + \alpha C_{km}$.

According to the author (Campbell-1994), the decision variables $X_{ijkm}$ and $Z_{ik}$ determine the allocation of the spoke nodes to the hubs. The parameter $C_{ijkm}$ denotes the per unit cost of flow from an origin *i* to a destination *j* via hubs *k* and *m*, in this order. The decision variable $Y_k$ is used to denote the location of hubs at site k. The author describes the p-hub median problem as follows:

$$Min \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{m\in N} W_{ij}X_{ijkm}C_{ijkm} \qquad (3.31)$$

s.t.:

$$\sum_{k\in N} Y_k = p, \qquad (3.32)$$

$$0 \le Y_k \le 1, \quad and\ integer\ for\ all\ k, \qquad (3.33)$$

$$0 \le X_{ijkm} \le 1, \quad for\ all\ i,j,k,m, \qquad (3.34)$$

$$\sum_{k\in N}\sum_{m\in N} X_{ijkm} = 1, \quad for\ all\ i,j, \qquad (3.35)$$

$$X_{ijkm} \le Y_k, \quad for\ all\ i,j,k,m, \qquad (3.36)$$

$$X_{ijkm} \le Y_m, \quad for\ all\ i,j,k,m, \qquad (3.37)$$



The objective function aims the minimization of the total transportation costs over all o-d pairs. The set of constraints (3.32) defines the number p of hubs to be located. Constraint (3.33) restricts the decision variable $Y_k$ to be binary. Constraint (3.34) limits the range of $X_{ijkm}$. The set of constraints (3.35) guarantees that every flow is routed via some hub pair. Constraints (3.36) and (3.37) assure that a flow is valid only if the locations *k* and *m* are hubs. The author also mention that “in the absence of capacity constraints on the links, an optimal solution will have all $X_{ijkm}$ variables equal to zero or one, since the total flow for each o-d pair should be routed via the least cost hub”.

The author (Campbell-1994) makes an analogy assuming that a demand point in a p-median problem is analogous to an o-d pair in a p-hub median problem. In the p-median problem, each demand point must be allocated to a facility in order to minimize the transportation cost while in the p-hub median problem each *o-d* pair must be allocated to a hub pair to minimize the transportation costs. The mathematical formulation for the p-median problem and its set of parameters and decision variables is shown and described as follows.

$d_{ij}$: distance between nodes *i* and *j*;

$h_i$: weight associated to every vertice $i$;

$y_{ij}$: decision variable for the allocation of nodes to the medians. 1, if node $i$ is allocated to a median at j; 0, otherwise;

$x_j$: decision of locating a median at site *j*. 1, if the node *j* is a median; 0, otherwise;

*p*: the number of medians to be located

$$Min \sum_{i=1}^{n}\sum_{j=1}^{n} h_i d_{ij} y_{ij}, \qquad (3.38)$$

s.t.:

$$\sum_{j=1}^{n} x_j = p, \qquad (3.39)$$

$$\sum_{j=1}^{n} y_{ij} = 1, \qquad \forall i \in N, \qquad (3.40)$$

$$y_{ij} - x_j \leq 0, \qquad \forall i, j \in N, \qquad (3.41)$$

$$x_j \in \{0,1\}, \qquad \forall j \in N, \qquad (3.42)$$

$$y_{ij} \in \{0,1\}, \qquad \forall i, j \in N, \qquad (3.43)$$

The objective function (3.38) minimizes the total transportation costs. The constraint (3.39) determines the number p of medians to be located. The constraint (3.40) guarantees that a demand node *i* is only allocated to one median *j*. Constraint (3.41) says that a demand *i* can only be allocated to a facility at *j* if this facility was already established. Constraints (3.42) and (3.43) are the binary variables.

The second model defined by the author is the uncapacitated hub location problem and its mathematical formulation is shown as follows.

$$Min \sum_{i \in N} \sum_{j \in N} \sum_{k \in N} \sum_{m \in N} W_{ij} X_{ijkm} C_{ijkm} + \sum_{k \in N} F_k Y_k \qquad (3.44)$$

s.t.:

$$0 \leq Y_k \leq 1, and\ integer\ for\ all\ k, \qquad (3.45)$$

$$0 \leq X_{ijkm} \leq 1, for\ all\ i, j, k, m, \qquad (3.46)$$

$$\sum_{k \in N} \sum_{m \in N} X_{ijkm} = 1, for\ all\ i, j, \qquad (3.47)$$

$$X_{ijkm} \leq Y_k, for\ all\ i, j, k, m \qquad (3.48)$$

$$X_{ijkm} \leq Y_m, for\ all\ i, j, k, m \qquad (3.49)$$

This formulation differs for the p-hub median problem mainly in the number of hubs to be located, which is not specified for this case and where a non-negative cost $F_k$ is associated for each potential hub location. This modeling is analogous to the uncapacitated facility location problem with fixed costs, in the classic location theory.

The *p-hub center* is another analogous model, in comparison with the *p center problem*. In this last model, the aim is the minimization of a function of the maximum distance between a demand node and the nearest facility, as shown in the model as follows. The definition of the parameters and the decision variables was already done in the description of the p-median problem.

$$Min\ W \qquad (3.50)$$

s.t.:

$$\sum_{j \in J} x_j = p, \qquad (3.51)$$

$$\sum_{j \in J} y_{ij} = 1, \qquad \forall i \in I, \qquad (3.52)$$

$$y_{ij} - x_j \leq 0, \qquad \forall i \in I, \forall j \in J, \qquad (3.53)$$

$$W - \sum_{j \in J} h_i d_{ij} y_{ij} \geq 0, \quad \forall i \in I, \qquad (3.54)$$

$$x_j \in \{0,1\}, \qquad (3.55)$$

$$y_{ij} \in \{0,1\}, \qquad (3.56)$$

The author emphasizes that “the minimax center problems are fundamentally different from the minisum median and uncapacitated facility location problems. For instance, center problems are important for locating emergency service facilities and vehicles, and because of the insight into worst case scenarios”.

According to the author, “if an o-d pair in a hub location problem is viewed as analogous to a demand point in a p-center problem, then the natural definition of a hub center is a set of hubs such that the maximum cost for any o-d pair is minimized. The modeling is shown as follows.

$$Min\ Max_{ijkm} \left\{ X_{ijkm} C_{ijkm} \right\} \qquad (3.57)$$

s.t.:

$$\sum_{k \in N} Y_k = p, \qquad (3.58)$$

$$0 \leq Y_k \leq 1, and\ integer\ for\ all\ k, \qquad (3.59)$$

$$\sum_{k \in N} \sum_{m \in N} X_{ijkm} = 1, and\ integer\ for\ all\ k, \qquad (3.60)$$

$$X_{ijkm} \leq Y_k, for\ all\ i,j,k,m \qquad (3.61)$$

$$X_{ijkm} \leq Y_m, for\ all\ i,j,k,m \qquad (3.62)$$

The fourth model that the author presents is called the hub set covering problem and the analogy was made with the set covering location problem in the classical facility location theory. The set covering location problem aims the minimization of the number of facilities to be located in a way that all the demand nodes are covered. The mathematical formulation for this problem (the set covering location problem) is shown as follows.

$$Min \sum_{j \in J} x_j \qquad (3.63)$$

s.t:

$$\sum_{j \in J} x_j \geq 1, \qquad \forall i \in I, \quad (3.64)$$

$$x_j \in \{0,1\}, \qquad \forall j \in J, \quad (3.65)$$

The objective function minimizes the total number of facilities to be located. The constraint (3.64) guarantees that every demand node is served only by at least one facility. The constraint (3.65) restricts the decision variable $x_j$ to be binary. The analogy made using the set covering location problem for the hub location theory is the hub set covering problem. For this problem, the hub set covering location problem, the objective is the minimization of the number of hubs to be located.

The decision variable $V_{ijkm}$ denotes the allocation of an o-d pair to the hubs $k$ and $m$. If an o-d pair $i$,$j$ is allocated to hubs $k$ and $m$, this variable will assume the value one. If it does not happen, the variable will assume the value of zero. The mathematical formulation for this problem is shown as follows.

$$Min \sum_{k \in N} F_k Y_k \qquad (3.66)$$

s.t.:

$$0 \leq Y_k \leq 1, and\ integer\ for\ all\ k, \qquad (3.67)$$

$$X_{ijkm} \leq Y_k, for\ all\ i, j, k, m, \qquad (3.68)$$

$$X_{ijkm} \leq Y_m, for\ all\ i, j, k, m, \qquad (3.69)$$

$$\sum_{k\in N}\sum_{m\in N} V_{ijkm}X_{ijkm} \geq 1, for\ all\ i,j, \qquad (3.70)$$

For the maximal covering location problem, in the classical facility location theory, the objective is the maximization of the number of nodes to be covered, subject to a predetermined number of facilities to be located. The parameter $h_i$ is the demand or weight associated to the node $i$ and the decision variable $z_i$ denotes the covering of the demand node $i$: 1, if it is covered; 0, otherwise. The mathematical formulation for this problem is shown as follows.

$$Max \sum_{i\in I} h_i z_i \qquad (3.71)$$

s.t.:

$$\sum_{j\in J} x_j - z_i \geq 0, \quad \forall i \in I, \qquad (3.72)$$

$$\sum_{j\in J} x_j = p, \qquad (3.73)$$

$$x_j \in \{0,1\}, \qquad (3.74)$$

$$z_i \in \{0,1\}, \qquad (3.75)$$

In the hub maximal covering location problem, the aim is the maximization of the demand nodes covered with a given number of hub facilities $p$. The mathematical formulation is shown as follows.

$$Max \sum_{i\in N}\sum_{j\in N}\sum_{k\in N}\sum_{m\in N} W_{ij}X_{ijkm}V_{ijkm} \qquad (3.76)$$

$$\sum_{k\in N} Y_k = p, \qquad (3.77)$$

$$0 \leq Y_k \leq 1, and\ integer\ for\ all\ k, \qquad (3.78)$$

$$0 \leq X_{ijkm} \leq 1, for\ all\ i, j, k, m, \qquad (3.79)$$

$$\sum_{k \in N} \sum_{m \in N} X_{ijkm} = 1, for\ all\ i, j \qquad (3.80)$$

$$X_{ijkm} \leq Y_k, for\ all\ i, j, k, m \qquad (3.81)$$

$$X_{ijkm} \leq Y_m, for\ all\ i, j, k, m \qquad (3.82)$$

Other contributors to the subject are Sasaki et al (1999). They outline that the hub-and-spoke system is beneficial not only for the airline companies but also for the passengers. Their study considers the 1-stop model and it is also mentioned that this type is better suited for relatively small countries. In this study, an integer programming model is formulated, which may be further transformed into the *p-median problem*. Two solutions for this model are proposed: a branch-and-bound method and a heuristic method. A comparison is made between these proposed methods using the *nested-dual* algorithm for the *p-median problem* through computational experiments. A description for the model is shown below.

A *1-stop multiple allocation p-hub median problem* as a 0-1 integer program is formulated, using the following notation.

$A = \{A_1, \dots, A_n\}$: $the\ set\ of\ all\ airports,$

$w_{ij}$: $the\ number\ of\ passengers\ from\ A_i\ to\ A_j,$

$d_{ij}$: $the\ distance\ between\ A_i\ and\ A_j,$

$y_j$: $0-1\ variable\ that\ is\ equal\ to\ 1\ if\ A_j\ is\ a\ hub\ and\ 0\ otherwise.$

To simplify the problem, it is assumed that both $w_{ij}\ and\ d_{ij}$ are symmetric for all $i$ and $j$. An OD-Hub Table is introduced, with each row in the table corresponding to an origin-destination (OD) pair. By symmetry, the number of rows is $M = n(n-1)/2$. Each column represents a hub candidate. The sum of elements in the column $l$ is denoted as $s_l$, as shown bellow.

$$s_l = \sum_{i,j} E_{ijl}, \quad l = 1, \dots, n. \qquad (3.83)$$

The columns are arranged in an increasing order of $s_l$ convenience. The authors suppose that columns 1, 2,…, n in the table correspond to hub candidates $l_1, l_2, \dots, l_n$ respectively. Selecting $p$ hubs is equivalent to selecting the corresponding $p$ columns in the table. The distance between each O-D pair is the minimum element in the corresponding row of the p selected columns. Selecting optimal $p$ hubs is therefore equivalent to selecting p columns in the table that minimize the sum, over all rows, of the minimum element in each row of the selected $p$ columns.

The authors also show that the model described can be formulated as a *p-median problem* by replacing the pair of indices (*i,j*) by a single indice $\pi$, with this index $\pi$ representing an O-D pair. The 0-1 variables are defined as $x_{\pi j} = 1, if\ OD\ pair\ \pi\ is\ routed\ via\ hub\ j$; 0, otherwise.

The authors define Π as the set of all OD pairs and $C_{\pi j}$ the product of the travel between the OD pair $\pi$ via the hub airport $j$ and the demand between OD pair $\pi$. The model, formulated as a *p-median problem*, is shown below:

$$Min \sum_{\pi \in \Pi} \sum_{j \in A} C_{\pi j} x_{\pi j} \qquad (3.84)$$

s.t.:

$$\sum_{j \in A} x_{\pi j} = 1, \qquad \forall \pi \in \Pi, \qquad (3.85)$$

$$-x_{\pi j} + y_j \geq 0, \qquad \forall \pi \in \Pi, \forall j \in A, \qquad (3.86)$$

$$\sum_{j \in A} y_j = p, \qquad (3.87)$$

$$x_{\pi j} \in \{0,1\}, \qquad \forall \pi \in \Pi, \forall j \in A, \qquad (3.88)$$

$$y_j \in \{0,1\}, \qquad \forall j \in A. \qquad (3.89)$$

The constraint set (3.85) ensures that each OD pair uses one and only one hub. Constraint set (3.86) prohibits a connection via a non-hub. Constraints (3.87) ensures that exactly p hubs are selected and constraints (3.88) and (3.89) restrict $x_{\pi j}$ and $y_j$ to be zero or one.

To solve this problem, the authors use two algorithms. The first one is a branch-and-bound type algorithm that uses Lagrangian relaxation by dualizing the constraint on the number of hubs. The second algorithm is a greedy type one. The authors concluded that the former algorithm proposed (branch-and-bound) was more effective when the number of hubs was relatively small.

Campbell et al. (2002) emphasize that in the last two decades the hub location research has become an important area for location theory. The authors mention that the hub location problems present different characteristics as compared to the classical facility location problem: for the classical discrete facility location problem, the demand occurs at discrete points and the facilities are located at these discrete points while in the hub location problems the demand is characterized by flows amongst origins and destinations.

The authors mention the vast range of applications that exist for the hub location models: air passenger and freight, express shipments, large trucking systems, postal operations, rapid transit systems and in telecommunication area, as well. Researchers from different areas have been studying this type of problem, as can be seen in studies made by scholars from geography, regional science, location theory, transportation, operations research, computer science and telecommunication.

In this study (Campbell et al. - 2002), the authors present a mixed integer linear programming formulation for this problem. They consider a complete graph $G = (V, E)$ with a set of nodes $V = \{v_1, v_2, \ldots, v_m\}$, corresponding to the origins, destinations and the potential locations for the hubs. The demand and the distance between a node $i$ and a node $j$ is denoted by $W_{ij}\ and\ d_{ij}$, respectively. The number of hubs to locate is defined by the parameter $p$ and each origin-destination path has three components: the collection from an origin to the first hub with an application of the collection parameter $\chi$, the transfer between two hubs (the first and the last) with an application of the discount factor $\alpha$ and the distribution from the last hub to the final destination, with the application of the parameter $\delta$. It is important to state that $\alpha < \chi\ and\ \alpha < \delta$.

Four sets of decision variables are defined:

$Z_{ik}$: Flow from an origin *i* to a hub *k*;

$Y_{kl}^{i}$: Flow from a hub *k* to a hub *l* that originates at origin $i$;

$X_{lj}^{i}$: Flow from a hub *l* to a destination *j* that originates at origin $i$;

$H_k$: Binary decision variables for the location of hubs. 1 if a node *k* is a hub and 0, otherwise.

The model is shown below:

$$Min\ Z = \sum_{i \in V} \left[ \sum_{k \in V} \chi d_{ik} Z_{ik} + \sum_{k \in V} \sum_{l \in V} \alpha d_{kl} Y_{kl}^{i} + \sum_{l \in V} \sum_{j \in V} \delta d_{lj} X_{lj}^{i} \right] \quad (3.90)$$

s.t.:

$$\sum_{k \in V} H_k = p, \quad (3.91)$$

$$\sum_{k \in V} Z_{ik} = O_i, \quad \forall i \in V, \quad (3.92)$$

$$\sum_{l \in V} X_{lj}^{i} = W_{ij}, \quad \forall i, j \in V, \quad (3.93)$$

$$Z_{ik} + \sum_{l \in V} Y_{lk}^{i} = \sum_{l \in V} Y_{kl}^{i} + \sum_{j \in V} X_{kj}^{i}, \quad \forall i, k \in V, \quad (3.94)$$

$$X_{lj}^{i} \leq W_{ij} H_l, \quad \forall i, j, l \in V, \quad (3.95)$$

$$Z_{ik} \leq O_i H_k, \quad \forall i, k \in V, \quad (3.96)$$

$$Z_{ik}, Y_{kl}^{i}, X_{lj}^{i} \geq 0, \quad \forall i, j, k, l \in V, \quad (3.97)$$

$$H_k \in \{0,1\}, \quad \forall k \in V, \quad (3.98)$$

$$O_i = \sum_{j \in V} W_{ij}, \quad \forall i \in V, \quad (3.99)$$

The objective function (3.90) takes into consideration the costs for collection, transfer and distribution. Constraint (3.91) determines the number $p$ of hubs to be opened. Constraint (3.92) says that all flow from each origin *i* leaves that origin in direction to a hub. Constraint (3.93) imposes that all flow for each origin-destination pair arrives at the proper destination. The set of constraints (3.94) states the conservation of flow and constraints (3.95) and (3.96) ensure that the hub nodes are established for every distribution and collection movement, respectively.

The authors also present the problem of Uncapacitated Single Allocation p-hub Median Problem (USApHMP), the Uncapacitated Multiple and Single Allocation Hub Location Problem, where the number *p* of hubs to be located is determined endogenously and the capacitated versions for these problems, by adding constraints to restrict the total flow at a hub or the flow collected by a hub.

In this study (Campbell et al. - 2002), the authors propose a taxonomy to classify this type of problems. This scheme has five positions: the first will always contain the word 'hub', to indicate that the facilities to be located are hubs. In addition, it may point out the number of hubs to be located; the second provides information that indicates if the problem is discrete or not. If it is, the letter 'D' is used; the third position gives an information about the type of allocation (multiple or single). If multiple, it uses 'MA' and if single, it uses 'SA'; the fourth position is related to the relationship between new and existing facilities. If a distance function is used (which is usually the case), the symbol '$\cdot$' is used; in the last position the type of the objective function is indicated. If it involves flow, this is represented by $\Sigma_{flow}$ and if denotes fixed costs of establishing hubs this is represented by $\Sigma_{hub}$.

The authors emphasize that only a limited amount of work has been done to analyze the complexity of hub location problems and mostly for the Single Allocation p-hub Median Problem (SApHMP), which in the taxonomy showed above is denoted by p-hub/D/SA/$\cdot$ / $\Sigma_{flow}$ , and is known to be NP-Hard. For solving this type of problems, the authors cite five different techniques: Complexity Results, Pre-Processing, Linear Programming Based Approaches, Enumerative Algorithms and Heuristic Algorithms.

Huston and Buttler (2001) suggest that the decision of locating a set of hubs may influence the economic activity associated with the operation of a hub. They note that the operation of an airport hub can be a great competitive resource to low density traffic routes. They mention that the decision to locate a hub is based upon geographical and demographical characteristics of a region. According to these authors, government authorities have been showing an increasing interest in the development of hubs.