4
The Dynamics of Decision Taking

Having shown that the introduction of Participation Constraints bound equilibrium payoffs away from the ex-ante efficient payoffs, we turn to the characterization of the dynamics of decision taking. We consider first how current decisions are taken, and then discuss the dynamics of continuation values (future decisions). Before that, however, we establish two results that will be of use later on.

Its is well known from the mechanism design literature that the incentive constraints imposed by (IC_1) and (IC_2) can be equivalently re-stated in terms of a 'first order condition' for an optimal truthful announcement and a monotonicity condition, which guarantees that if a local deviation from truthtelling is not optimal, the same will be true for a global deviation. For the following result, we assume that the function a has its two derivatives well defined. This is not implied by the expected monotonicity condition (see below); but is weaker than assuming a monotonic in both entries.

Lemma 1 *A pair* $(a(\cdot,\cdot), w(\cdot,\cdot))_{\theta,x}$ *is Incentive Compatible if, and only if, they satisfy*

$$E_{\theta_2,x}\left[(1-\delta)\frac{\partial u}{\partial a}(a(\theta,x),\theta_1)\frac{\partial a}{\partial\theta_1}(\theta,x)+\delta V'(w(\theta,x))\frac{\partial w}{\partial\theta_1}(\theta,x)\right]=0, \tag{Local IC$_1$}$$

$$E_{\theta_1,x}\left[(1-\delta)\frac{\partial u}{\partial a}(a(\theta,x),\theta_2)\frac{\partial a}{\partial\theta_2}(\theta,x)+\delta\frac{\partial w}{\partial\theta_2}(\theta,x)\right]=0, \tag{Local IC$_2$}$$

and

$$E_{\theta_{-i},x}\left[\frac{\partial u}{\partial\theta}(a(\tau,\theta_{-i},x),\theta_i)\right] \text{ is non-decreasing in } \tau \text{ for } i=1,2. \tag{Expected Monotonicity}$$

The second result states that the value function $V(\cdot)$ is strictly concave when players are patient. Hence, we can use Lagrangian methods to solve for the optimal contract (Luenberger [9]).

Lemma 2 *The value function $V(\cdot)$ is strictly concave.*

With these two results in hands, we are able to turn to the analysis of the optimal actions and continuation values picked by an optimal mechanism.

4.1 Actions

Assigning multipliers $\{\lambda_i(\theta_i)\}_{\theta_i\in[0,1]}$, $i=1,2$, to, respectively, the first order condition counterparts of (IC_1), and (IC_2), and multiplier γ to the (PK) constraint, the first order necessary condition (this is shown in the Appendix) for an optimal action $a(\theta)$ is:

$$\left[\left(f(\theta_1)-\dot{\lambda}_1(\theta_1)\right)\frac{\partial u}{\partial a}(a(\theta,x),\theta_1)-\lambda_1(\theta_1)\frac{\partial^2 u}{\partial\theta\partial a}(a(\theta,x),\theta_1)\right]f(\theta_2)+$$
$$\left[\left(\gamma f(\theta_2)-\dot{\lambda}_2(\theta_2)\right)\frac{\partial u}{\partial a}(a(\theta,x),\theta_2)-\lambda_2(\theta_2)\frac{\partial^2 u}{\partial\theta\partial a}(a(\theta,x),\theta_2)\right]f(\theta_1)=0 \tag{4.1}$$

As suggested by Myerson [11], it is convenient to think about the Lagrangian that yields this first order condition as representing the weighted sum of the agents' virtual utilities.[1] Indeed, defining new multipliers

$$\tilde{\lambda}_1(\theta_1)=\lambda_1(\theta_1), \tilde{\lambda}_2(\theta_2)=\frac{\lambda_2(\theta_2)}{\gamma},$$

and, letting agent i's instantaneous virtual utility be

$$\tilde{u}(a(\theta,x),\theta_i)=\left(1-\frac{1}{f(\theta_i)}\dot{\tilde{\lambda}}_i(\theta_i)\right)u(a(\theta,x),\theta_i)-\frac{\tilde{\lambda}_i(\theta_i)}{f(\theta_i)}\frac{\partial u}{\partial\theta}(a(\theta,x),\theta_i),$$

it can be seen from the first order condition for $a(\cdot,\cdot)$, that the optimal mechanism maximizes the weighted sum of the agents' virtual instantaneous utilities, with the weight given to agent 1 being equal to one, and the weight given to agent two being equal to γ.[2]

Analogously, by assigning multipliers $\{\zeta(\theta,x)\}_{\theta,x}$ to inequality

$$w(\theta,x)\leq\overline{w}_2$$

[1] See also Myerson's notes on virtual utility at:
http://home.uchicago.edu/ ˜rmyerson/research/virtual.pdf

[2] In comparison to an agent's real utility, the virtual utility function incorporates two terms related to the effects an action schedule has on incentives. First, the term $\left(1-\frac{1}{f(\theta_i)}\dot{\tilde{\lambda}}_i(\theta_i)\right)u(a(\theta,x),\theta_i)$ captures how tempting it is, for a given agent i, to deviate locally when his preference shock is θ_i. Second, the term $-\frac{\tilde{\lambda}_i(\theta_i)}{f(\theta_i)}\frac{\partial u}{\partial\theta}(a(\theta,x),\theta_i)$ captures how tempting it is for types *other* than θ_i to report that their preference shock is θ_i.

and $\{\xi(\theta,x)\}_{\theta,x}$ to inequality

$$w(\theta,x) \geq \underline{w}_2,$$

which together are the Participation Constraints in (IR'), we can write the first order condition for $w(\theta,x)$ as

$$\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] V'(w(\theta,x))f(\theta_2)+\left[\gamma f(\theta_2) - \dot{\lambda}_2(\theta_2)\right] f(\theta_1)+\xi(\theta,x)-\zeta(\theta,x) = 0 \quad (4.2)$$

Using the above conditions, we can divide the analysis of how current decisions are taken according to two sets of states:

1. **States for which neither of the Participation Constraints bind**: The action chosen is the same as the one derived in Carrasco and Fuchs [3]. In particular, as argued by them, compared to a one-shot setting (or to the case in which $\delta = 0$), more weight is given to a relatively extreme player, who forgoes future decision power.

2. **States for which one Participation Constraint binds:** This is a more interesting case. Note, first, that whenever neither of the IRs bind (or, alternatively, when participation is forced), the weight given to player two when $a(\cdot,\cdot)$ is chosen, γ, can be written as

$$\gamma = -V'(w(\theta,x)) + \frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} + \frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)}V'(w(\theta,x))$$

Consider the case in which agent two's participation constraint is binding: $w(\theta,x) = \underline{w}_2$ and $\xi(\theta,x) > 0$ (the analysis for the other case is analogous). The first order condition for $w(\theta)$ reads

$$(V'(\underline{w}_2) + \gamma) f(\theta_1) f(\theta_2) - \dot{\lambda}_2(\theta_2) f(\theta_1) - \dot{\lambda}_1(\theta_1) V'(\underline{w}_2) f(\theta_2) + \xi(\theta,x) = 0,$$

so that

$$\gamma = -V'(\underline{w}_2) + \frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} + \frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)}V'(\underline{w}_2) - \frac{\xi(\theta,x)}{f(\theta_1)f(\theta_2)}$$

Since $\frac{\xi(\theta,x)}{f(\theta_1)f(\theta_2)} > 0$, whenever agent two's participation constraint is binding, the optimal mechanism incorporates an additional, negative, term to the weight given to agent two on the current decision. The reason why this is the case is simple. In states for which the participation constraint binds, it is not feasible to give an agent more weight on current decisions in exchange for less weight on future ones. This intertemporal exchange of decision rights would have been implemented in a forced participation setting, however. Therefore, the agent is given less weight relatively to what would prevail in a forced participation environment.

Whenever none of the agents are tempted to exercise their outside options, the allocation will be such that a player with extreme preferences – as measured by its distance to $\frac{1}{2}$ – will trade more weight on the *current* allocation for less weight in *future* allocation (Carrasco and Fuchs [3]). In states for which one of the participation constraints is binding, the intertemporal exchange of decision power has to take an additional factor into account: it is not feasible anymore for the player whose constraint binds to forgo future decision power in exchange for a higher weight on current decisions.

4.2 Continuation Values

Equation (4.2) implicitly defines promised continuation values $w(\theta, x)$ as a function of current value, w. Let the relationship between these values be given by

$$w' = g(w, \theta, x). \tag{4.3}$$

At an optimum, continuation values must vary from period to period to reflect the agents' weights in the allocation rule. Continuation values tend to increase (higher future decision power) for an agent with less weight on the current decision, and to decrease (lower future decision power) for a player that is given more weight on the current decision. The next result states this precisely

Proposition 1 (Spreading Values) *If* $w \in (\underline{w}_2, \overline{w}_2)$, *there is a strictly positive probability of both* $w' > w$ *and* $w' < w$.

The variation in continuation values allows agents to get more weight in the current decision in exchange for forgoing decision rights in the future. This is the mean by which *ex-ante* efficiency gains are attained. We seek to derive the *ex-post* implications of such variation in values on the agents' relative bargain power.

In a setting without participation constraints, the variation of values implied by Proposition 1 is a force toward a degenerate limiting distribution of power: whenever participation is forced, the continuing variation in values necessarily leads to a dictatorship in the limit (Carrasco and Fuchs [3]).

In the current setting, however, outside options induce mean reversion: whenever one of the extreme values, $\{\underline{w}_2, \overline{w}_2\}$, is hit, a force toward intermediate values kicks in. When one of the agents is taken to his outside option payoff at a period t, there will be strictly positive probability of him being promised strictly higher continuation values for period $t+1$.

Proposition 2 (Mean Reversion) *Whenever* $w = \overline{w}_2$ *(respectively,* $w = \underline{w}_2$*), there is a strictly positive probability of* $w' < \overline{w}_2$ *(respectively,* $w' > \underline{w}_2$*).*

When participation is forced, from the moment a player is promised dictatorship values, it is not feasible anymore to implement any exchange of decision rights. In other words, dictatorship is an absorbing state. In the current setting, in contrast, even when taken to his outside option, a player can exchange some current decision rights for more stake in future decisions. Implementing such exchange is optimal, as it allows both players to continue trading decisions rights over time.

A joint implication of Propositions 1 and 2 is that, whenever ex-post participation constraints have to be satisfied for both players, there are no absorbing states. So, if a limiting (invariant) distribution exists, it will necessarily be non-degenerate.

4.3 The Limiting Distribution of Power

We now move on to show that an unique invariant distribution of power exists. Toward that, define, for any set $A \subset [\underline{w}_2, \overline{w}_2]$ and a fixed w, the inverse of $g(\cdot)$ as

$$\Gamma(A|\, w) = \left\{\theta \in \Theta^2, x \in [0,1] : g(w, \theta, x) \in A\right\}$$

Then,

$$Q(w, A) = \Pr_{\theta, x}[\Gamma(A|w)]$$

is a transition function (see Stokey et al. [13], p.212).

For an arbitrary distribution φ over $[\underline{w}_2, \overline{w}_2]$, define the operator T^* as follows

$$(T^*\varphi)(A) = \int Q(w, A)\, \varphi(dw).$$

This operator gives the probability of agent two's next period promised values lying in A given that the current period promised value is drawn according to φ.

In the appendix, we show that T^* is a contraction map in the total variation norm. Hence, starting with any initial distribution φ^0, the sequence defined by

$$\varphi^n = \left(T^*\varphi^{n-1}\right)$$

converges to a *unique* invariant φ^*. This leads to the following result.

Theorem 2 *The Markov Process that governs w has an unique invariant distribution φ^* over $[\underline{w}_2, \overline{w}_2]$. This distribution is non-degenerate and assign positive likelihood to every w in $[\underline{w}_2, \overline{w}_2]$.*

In the limit, the distribution φ^* fully determines the probability of agent two being promised any feasible value. Using the Envelope Theorem, one has

$$-V'(w) = \gamma.$$

Therefore, the time varying weight of agent two on the current allocation is equal to the negative of agent one's marginal value at an optimal. Hence, as $V'(\cdot)$ is continuous (this is shown in the Appendix), an invariant distribution φ^* for continuation values will imply an invariant distribution γ^* for the weights. This distribution is itself non-degenerate. It follows that, when both agents have outside options, each will always have stake on current decisions, with their relative weights being fully determined by γ^*.

Two properties of the dynamics of decision power are worth mentioning. First off, the limiting distribution of power is memoryless: even if the partnership starts with, say, agent 1 having all the bargain power (meaning, the initial promised value to agent two is $\underline{w}_2$), in the far future, the relative bargain power will have no dependence whatsoever on this fact. This holds because the sequence $\{\varphi^n\}_n$ converges to φ^* for any φ^0.

Second, power continually changes hands in the limit, meaning that the weight agents have on decisions varies from period to period. This last property is a consequence of Propositions 1 and 2.

We summarize the above discussion in the following result:

Theorem 3 *There exists a unique limiting distribution of power. This distribution is non-degenerate, memoryless and such that the weights agents have on decisions continually vary from period to period.*