7
Appendix

7.1 The Lagrangian Representation

We first prove that the function $V(\cdot)$ is strictly concave. This will allow us to make use of Lagrangian methods. In order to do so, we start by using a standard result from the Mechanism Design literature. Since the agents' preferences satisfy a single crossing condition, Incentive Compatibility can be replaced by a first order condition for truthtelling and a monotonicity condition.

Proof on Lemma 1: The proof is standard, and, therefore, omitted.

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Lemma 3 *The constraint set in program (P1) is convex.*

Proof: The set R is clearly convex. Since all players' payoffs are concave in elements of R, and all the constraints are either equalities or weak inequalities, the result follows.

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Lemma 4 *The constraint set of program (P1) is compact in the weak-*topology.*

Proof: The set of all distributions is compact in the weak-*topology. R is a subset of the set of all distributions over elements in D. Moreover, since all the constraints are either equalities or weak inequalities, the subset of elements of R that satisfies the constraints is (weak-*) closed. Hence, it is compact.

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Lemma 5 *There exists a solution to the problem above.*

Proof: This follows because the objective functional is continuous (note that it is continuous in elements of R and is bounded) and the constraint set is weak-* compact.

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Proof on Lemma 2:

First define recursively:

$$V_n(v) = \sup_{\left\{\{a(\theta,x),w(\theta,x)\}_{\theta,x}\right\}\in R} E_{\theta,x}\left[(1-\delta)\,u\left(a\left(\theta,x\right),\theta_n\right) + \delta V_{n-1}\left(w\left(\theta,x\right)\right)\right]$$

subject to

$$E_{\theta,x}\left[(1-\delta)\,u\left(a\left(\theta,x\right),\theta_2\right) + \delta w\left(\theta,x\right)\right] = v$$

$$E_{\theta_2,x}\left[(1-\delta)\,u\left(a\left(\theta,x\right),\theta_1\right) + \delta V\left(w\left(\theta,x\right)\right)\right] \geq \\ E_{\theta_2,x}\left[(1-\delta)\,u\left(a\left(\hat{\theta}_1,\theta_2,x\right),\theta_1\right) + \delta V\left(w\left(\hat{\theta}_1,\theta_2,x\right)\right)\right] \forall\theta_1,\hat{\theta}_1\in\Theta$$

$$E_{\theta_1,x}\left[(1-\delta)\,u\left(a\left(\theta,x\right),\theta_2\right) + \delta w\left(\theta,x\right)\right] \geq \\ E_{\theta_1,x}\left[(1-\delta)\,u\left(a\left(\theta_1,\hat{\theta}_2,x\right),\theta_2\right) + \delta w\left(\theta_1,\hat{\theta}_2,x\right)\right] \forall\theta_2,\hat{\theta}_2\in\Theta$$

$$w\left(\theta,x\right)\in\left[\underline{w}_2,\overline{w}_2\right]\forall\theta\in\Theta^2, x\in[0,1]$$

Where $\overline{w}_2$ is just a number; that is, at this point we are simply interested on the existence of an upper bound for $w(\cdot,\cdot)$ rather than in the exact value of $V^{-1}(\underline{w}_1)$.

Taking V_0 to be (strictly) concave, each V_n will be (strictly) concave; because the choice set satisfying all the restrictions is convex and the objective function is a convex combination of two concave functions: $u(\cdot,\cdot)$ and $V_{n-1}(\cdot)$. Moreover, from the Theorem of the Maximum, we know that in the limit, the sequence of functions $V_n(\cdot)$ converge to the desired function $V(\cdot)$ and is (weakly) concave. However, from the definition of the function defined in problem (P1), the limit value function has to be strictly concave.

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A property of $V(\cdot)$ that we will be of use latter on is:

Lemma 6 *$V(\cdot)$ is continuously differentiable over $(\underline{v},\overline{v})$.*

Proof: Since $V(\cdot)$ is concave, this follows from Corollary 2 in Milgrom and Segal [10].

■

Assigning multipliers $\{\lambda_i(\theta_i)\}_{\theta_i}, i = 1, 2$, to, respectively, the first order condition counterparts of (IC_1), and (IC_2), multiplier γ to (PK), and multipliers $\{\xi(\theta, x)\}_\theta$ and $\{\zeta(\theta, x)\}_\theta$ to the participation constraints satisfying $w(\theta, x) \in [\underline{w}_2, \overline{w}_2]$ for all θ, one can write $V(v)$ as

$$\begin{aligned} V(v) = \max_{\left\{\{a(\theta,x),w(\theta,x)\}_{\theta,x}\right\}\in R} & E_{\theta,x}\left[(1-\delta)u(a(\theta,x),\theta_1) + \delta V(w(\theta,x))\right] \\ & +\gamma\left(E_{\theta,x}\left[(1-\delta)u(a(\theta,x),\theta_2) + \delta w(\theta,x)\right] - v\right) \\ & + \int_0^1 \lambda_1(\theta_1)\int_0^1 E_{\theta_2}\Bigg[(1-\delta)\frac{\partial u}{\partial a}(a(\theta,x),\theta_1)\frac{\partial a}{\partial \theta_1}(\theta,x) \\ & \qquad + \delta V'(w(\theta,x))\frac{\partial w}{\partial \theta_1}(\theta,x)\Bigg] d\theta_1 dx \\ + \int_0^1 \lambda_2(\theta_2)\int_0^1 E_{\theta_1} & \left[(1-\delta)\frac{\partial u}{\partial a}(a(\theta,x),\theta_2)\frac{\partial a}{\partial \theta_2}(\theta,x) + \delta\frac{\partial w}{\partial \theta_2}(\theta,x)\right] d\theta_2 dx \\ + \int_0^1\int_0^1\int_0^1 & \Big(\xi(\theta,x)\left[w(\theta,x) - \underline{w}_2\right] - \zeta(\theta,x)\left[w(\theta,x) - \overline{w}_2\right]\Big) d\theta_1 d\theta_2 dx \end{aligned}$$

Notice that the symmetry of the problem and the single crossing property guarantees the almost everywhere differentiability of the Lagrangian multipliers associated to the incentive compatibility restrictions. That is, player one's best guess for player two's type is $\frac{1}{2}$. Therefore, if player one has a shock equal to $\frac{1}{2}$ he will not have any incentive to report another type. Thus, $\lambda_1(\frac{1}{2}) = 0$. Now, assume that player one's type is $\frac{1}{2} - \epsilon$. Then this player would like to under-report his type; because, from the expected monotonicity condition, he knows that under-reporting his type will lead to an action that will be closer to his type.[1] Moreover, a higher ϵ, implies a higher incentive to under-report his type. Therefore, for any type lower than $\frac{1}{2}$, $\lambda_1(\cdot)$ is monotonic, thus almost everywhere differentiable on $(0, \frac{1}{2})$. Analogously, the same happens for types higher than $\frac{1}{2}$. Moreover, since types $0, \frac{1}{2}, 1$ have measure zero, this function is a.e. differentiable.

Some rounds of integration by parts allow us to re-write $V(v)$ as

[1]The optimal action has to be some number between the two reports; otherwise there would be a loose of efficiency.

$$
\begin{aligned}
V(v) = \max_{\left\{\{a(\theta,x),w(\theta,x)\}_{\theta,x}\right\}\in R} & E_{\theta,x}\left[(1-\delta)\,u\left(a(\theta,x),\theta_1\right)+\delta V\left(w(\theta,x)\right)\right] \\
& +\gamma\left(E_{\theta,x}\left[(1-\delta)\,u\left(a(\theta,x),\theta_2\right)+\delta w(\theta,x)\right]-v\right) \\
& -(1-\delta)\int_0^1 \dot{\lambda}_1(\theta_1)\left[E_{\theta_2,x}\left[u\left(a(\theta,x),\theta_1\right)\right]\right]d\theta_1 \\
& -(1-\delta)\int_0^1 \lambda_1(\theta_1)\,E_{\theta_2,x}\left[\frac{\partial u}{\partial\theta}\left(a(\theta,x),\theta_1\right)\right]d\theta_1 \\
& -\delta\int_0^1 \dot{\lambda}_1(\theta_1)E_{\theta_2,x}\left[V\left(w(\theta,x)\right)\right]d\theta_1 \\
+\lambda_1(\theta_1)(1-\delta)\,E_{\theta_2,x}&\left[u\left(a(\theta,x),\theta_1\right)\right]\Big|_{\theta_1=0}^{\theta_1=1}+\delta E_{\theta_2,x}\left[w(\theta,x)\right]\Big|_{\theta_1=0}^{\theta_1=1} \\
& -(1-\delta)\int_0^1 \dot{\lambda}_2(\theta_2)\left(E_{\theta_1,x}\left[u\left(a(\theta,x),\theta_2\right)\right]\right)d\theta_2 \\
& -(1-\delta)\int_0^1 \lambda_2(\theta_2)\,E_{\theta_1,x}\left[\frac{\partial u}{\partial\theta}\left(a(\theta,x),\theta_2\right)\right]d\theta_2 \\
& -\delta\int_0^1 \dot{\lambda}_2(\theta_2)E_{\theta_1,x}\left[w(\theta,x)\right]d\theta_2 \\
+\lambda_2(\theta_2)(1-\delta)\,E_{\theta_1,x}&\left[u\left(a(\theta,x),\theta_2\right)\right]\Big|_{\theta_2=0}^{\theta_2=1}+\delta E_{\theta_1,x}\left[w(\theta,x)\right]\Big|_{\theta_2=0}^{\theta_2=1} \\
+\int_0^1\int_0^1 E_x&\left[\xi(\theta,x)\left(w(\theta,x)-\underline{w}_2\right)-\zeta(\theta,x)\left(w(\theta,x)-\overline{w}_2\right)\right]d\theta_1 d\theta_2
\end{aligned}
$$

As it is standard (see Theorems 1 and 2 in sections 8.3-8.4 of Luenberger [9]), $\{a^*(\theta,x), w^*(\theta,x)\}_{\theta,x}$ – with $a^*(\cdot,\cdot)$ satisfying strictly the expected monotonicity property – is optimal if, and only if, there are multipliers $\{\lambda_i(\theta_i)\}_{i=1,2}$, $\{\xi(\theta,x), \zeta(\theta,x)\}$ and γ for which $\{a^*(\theta,x), w^*(\theta,x)\}_{\theta,x}$ maximizes the above Lagrangian. It is easy to see that the First Order Conditions for such problem are the ones in the text.

7.2 The Inefficiency Result

Proof of Theorem 1: Define $\hat{V}(w;\delta)$ as the value function for the problem without participation constrains when the discount factor is δ; call this problem unrestricted. Let $V(w;\delta)$ be the value function for the problem with participation constraints; call this problem restricted. From Carrasco and Fuchs [3], we know that, for all $w<\overline{v}$, there is strictly positive probability of next period's continuation value for player two being both higher and lower than the current value value w. Suppose that the current value for player two is $\overline{w}_2 < \overline{v}$. Consider the subset of $[0,1]^2$ for which (IR_1) is binding. In the unrestricted program, it would be optimal to make next period's promised value higher with positive probability. However, in the restricted problem this is not possible. Thus, there exists $\epsilon' > 0$ such that $\hat{V}(\overline{w}_2;\delta) - \epsilon' > V(\overline{w}_2;\delta)$ for all $\delta \in [0,1)$. Furthermore, as will be shown in the proof of Theorem 2, in the restricted problem, starting from any value w, there is positive probability of $\overline{w}_2$ being hit in a finite number of steps. Therefore, for any w, there exists $\epsilon > 0$ – that may depend on w – such that $\hat{V}(w;\delta) - \epsilon > V(w;\delta)$ for all $\delta \in [0,1)$. Let w^* be the value promised to agent two by the mechanism in Carrasco and Fuchs [3] that approximates efficiency. It then follows that, for all δ,

$$w^* + V(w^*,\delta) < w^* + \hat{V}(w^*;\delta) - \epsilon \leq v^{FB} - \epsilon.$$

■

7.3 The Assymptotic Distribution

Lemma 7 *There exists a measure $\mathcal{Q}$ such that*

$$E^{\mathcal{Q}}\left[V'(w(\theta,x))\right] = V'(w) - E^{\mathcal{Q}}\left[\frac{(\xi(\theta,x) - \zeta(\theta,x))}{f(\theta_1)\,f(\theta_2) - \dot{\tilde{\lambda}}_1(\theta_1) f(\theta_2)}\right] \tag{7.1}$$

Proof: From the Lagrangian representation, the First Order Condition with respect to $w(\theta,x)$ is

$$\left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] V'(w(\theta,x)) f(\theta_2) + \left[\gamma f(\theta_2) - \dot{\lambda}_2(\theta_2)\right] f(\theta_1) + \xi(\theta,x) - \zeta(\theta,x) = 0$$

In terms of the multipliers, defining

$$\tilde{\lambda}_1(\theta_1) = \lambda_1(\theta_1),\ \tilde{\lambda}_2(\theta_2) = \frac{\lambda_2(\theta_2)}{\gamma},$$

the First Order Condition reads

$$\left[f(\theta_1) - \dot{\tilde{\lambda}}_1(\theta_1)\right] V'(w(\theta,x)) f(\theta_2) + \gamma\left[f(\theta_2) - \dot{\tilde{\lambda}}_2(\theta_2)\right] f(\theta_1) + \xi(\theta,x) - \zeta(\theta,x) = 0$$

Using the Envelope Theorem, one has $\gamma = -V'(w)$. Plugging this in the above equation, we have that

$$V'(w(\theta,x))\left[f(\theta_1)\,f(\theta_2)-\dot{\tilde{\lambda}}_1(\theta_1)f(\theta_2)\right]= \\ V'(w)\left[f(\theta_1)\,f(\theta_2)-\dot{\tilde{\lambda}}_2(\theta_2)f(\theta_1)\right]-\left[\xi(\theta,x)-\zeta(\theta,x)\right]$$

If one divides both sides by

$$\left[f(\theta_1)\,f(\theta_2)-\dot{\tilde{\lambda}}_2(\theta_2)f(\theta_1)\right]$$

we have

$$V'(w(\theta))\frac{\left[f(\theta_1)\,f(\theta_2)-\dot{\tilde{\lambda}}_1(\theta_1)f(\theta_2)\right]}{\left[f(\theta_1)\,f(\theta_2)-\dot{\tilde{\lambda}}_2(\theta_2)f(\theta_1)\right]}=V'(w)-\frac{\left[\xi(\theta)-\zeta(\theta)\right]}{\left[f(\theta_1)\,f(\theta_2)-\dot{\tilde{\lambda}}_2(\theta_2)f(\theta_1)\right]}$$

Note that we can do this, because $f(\theta_2)$ cannot be equal to $\dot{\tilde{\lambda}}_2(\theta_2)$ in a positive measure subset. This is so, because for a given w such that neither IR constraint is active; it must be true that:

$$\left[f(\theta_1)-\dot{\tilde{\lambda}}_1(\theta_1)\right]V'(w(\theta,x))f(\theta_2)=-\gamma\left[f(\theta_2)-\dot{\tilde{\lambda}}_2(\theta_2)\right]f(\theta_1)$$

Therefore, if we assume that $f(\theta_2)=\dot{\tilde{\lambda}}_2(\theta_2)$, it would be true that $\left[f(\theta_1)-\dot{\tilde{\lambda}}_1(\theta_1)\right]V'(w(\theta,x))f(\theta_2)=0$; implying $f(\theta_1)=\dot{\tilde{\lambda}}_1(\theta_1)$ for every θ_1. This, however, says that the program is linear and contradicts the strict concavity of $V(\cdot)$.

Moreover, if w were such that one of the IRs is active, we would have that, for instance, $\zeta(w,\theta)>0$. Thus, assuming again that $f(\theta_2)=\dot{\tilde{\lambda}}_2(\theta_2)$:

$$\left[f(\theta_1)-\dot{\tilde{\lambda}}_1(\theta_1)\right]V'(w(\theta,x))f(\theta_2)=\zeta(\theta,x)$$

However, this identity will not depend on the promised value w. Therefore, this should hold also for different promised values such that neither IR is active. This yields again to a contradiction.

Hence,

$$E_\theta \left[V'\left(w\left(\theta\right)\right) \frac{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \tilde{\dot{\lambda}}_1(\theta_1) f\left(\theta_2\right) \right]}{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\tilde{\lambda}}_2(\theta_2) f\left(\theta_1\right) \right]} \right] =$$

$$V'\left(w\right) - E_\theta \left[\frac{\left[\xi\left(\theta\right) - \zeta\left(\theta\right)\right]}{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\tilde{\lambda}}_2(\theta_2) f\left(\theta_1\right) \right]} \right]$$

or

$$E^{\mathcal{Q}}\left[V'\left(w\left(\theta\right)\right)\right] = V'\left(w\right) - E^{\mathcal{Q}} \left[\frac{\left[\xi\left(\theta\right) - \zeta\left(\theta\right)\right]}{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\tilde{\lambda}}_1(\theta_1) f\left(\theta_2\right) \right]} \right]$$

where, as suggested by the notation, $\mathcal{Q}$ is the distribution associated with the measure[2]

$$\frac{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\tilde{\lambda}}_1(\theta_1) f\left(\theta_2\right) \right]}{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\tilde{\lambda}}_2(\theta_2) f\left(\theta_1\right) \right]} f\left(\theta_1\right) f\left(\theta_2\right)$$

The result then follows.

■

Proof of Proposition 1: Suppose, toward a contradiction, that $w\left(\theta, x\right) \geq w > \underline{w}_2$ for almost all θ and x. Since $V\left(\cdot\right)$ is strictly concave, it must be the case that $V'\left(w\left(\theta\right)\right) \leq V'\left(w\right)$. Moreover, if there is a positive probability set for which $w\left(\theta, x\right) > w$, it must be true that $E^{\mathcal{Q}}\left[V'\left(w\left(\theta, x\right)\right)\right] < V'\left(w\right)$. Then, from (7.1),

$$E^{\mathcal{Q}} \left[\frac{\left(\xi\left(\theta, x\right) - \zeta\left(\theta, x\right)\right)}{\left[f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\tilde{\lambda}}_1(\theta_1) f\left(\theta_2\right) \right]} \right] > 0,$$

implying that $\xi\left(\theta, x\right) > 0$ for some θ, so that $w\left(\theta, x\right) = \underline{w}_2$ for those θ; which is a contradiction, because it was supposed that neither IR binds. Similar arguments can be used to show that one cannot have $w\left(\theta, x\right) \leq w < \overline{w}_2$ for all θ, with strict inequality with positive probability.

The only remaining possibility is that $w\left(\theta, x\right) = w \in \left(\underline{w}_2, \overline{w}_2\right)$ for almost all θ. Plugging this into the first order condition for $w\left(\cdot, \cdot\right)$, we get:

$$V'\left(w\right) f\left(\theta_1\right) f\left(\theta_2\right) + \gamma f\left(\theta_1\right) f\left(\theta_2\right) - \dot{\lambda}_2\left(\theta_2\right) f\left(\theta_1\right) - \dot{\lambda}_1\left(\theta_1\right) V'\left(w\right) f\left(\theta_2\right) = 0$$

[2]Dividing both sides by the constant $\int\int \frac{\left[f(\theta_1)f(\theta_2) - \dot{\tilde{\lambda}}_1(\theta_1)f(\theta_2)\right]}{\left[f(\theta_1)f(\theta_2) - \dot{\tilde{\lambda}}_2(\theta_2)f(\theta_1)\right]} f\left(\theta_1\right) f\left(\theta_2\right) d\theta_1 d\theta_2$ can make this a probability measure. This, however, is irrelevant since we are only interested in the sign of $E^{\mathcal{Q}}\left[\frac{\left(\xi(\theta,x) - \zeta(\theta,x)\right)}{f(\theta_1)f(\theta_2) - \dot{\tilde{\lambda}}_1(\theta_1)f(\theta_2)}\right]$. And dividing this by a positive constant does not affect the analysis.

or using, by the Envelope Theorem, $V'(w) = -\gamma$:

$$\dot{\lambda}_2(\theta_2) f(\theta_1) + \dot{\lambda}_1(\theta_1) V'(w) f(\theta_2) = 0,$$

Dividing both sides by $f(\theta_1) f(\theta_2) > 0$,

$$V'(w) - \frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} - V'(w) \frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)} = V'(w) \text{ for almost all } (\theta_1, \theta_2)$$

Therefore, one must have

$$\frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} = -\frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)} V'(w) \text{ for almost all } (\theta_1, \theta_2)$$

Since the left hand side depends only on θ_2, and the right hand side on θ_1, the above equality can hold for almost all (θ_1, θ_2) only if $\dot{\lambda}_1(\theta_1) = \dot{\lambda}_2(\theta_2) = 0$ for almost all (θ_1, θ_2).

Furthermore, since for all $s \in \left[0, \frac{1}{2}\right]$, $\lambda_i\left(\frac{1}{2} - s\right) = -\lambda_i\left(\frac{1}{2} + s\right)$ [3], one must have $\lambda_i(\theta_i) = 0$ for all i, and θ_i. Plugging this in the FOC for $a(\cdot, \cdot)$, one gets:

$$\left[\frac{\partial u}{\partial a}(a(\theta, x), \theta_1) + \gamma \frac{\partial u}{\partial a}(a(\theta, x), \theta_2)\right] = 0$$

It is easy to see that the policy $a(\cdot, \cdot)$ implicitly defined by the above equation is not incentive compatible when continuation values are constant, unless $\gamma = 0$, or $\gamma = \infty$ which cannot hold as $\gamma \in [-V'(\underline{w}_2), -V'(\overline{w}_2)]$.

■

Proof of Proposition 2: Assume, toward a contradiction, that, once $w(\theta, x)$ hits $\underline{w}_2$, it stays there forever. The FOC for $w(\theta, x)$ evaluated at $\underline{w}_2$ is given by

$$V'(\underline{w}_2) f(\theta_2) \left[f(\theta_1) - \dot{\lambda}_1(\theta_1)\right] + f(\theta_1) \left[\gamma f(\theta_2) - \dot{\lambda}_1(\theta_2)\right] = -\xi(\theta, x)$$

where $\xi(\theta, x) \geq 0$ is the multiplier associated to the (IR_2) constraint.

Using the Envelope Condition, $\gamma = -V'(\underline{w}_2)$, we have

$$V'(\underline{w}_2) \left(\left(f(\theta_2) \left[f(\theta_1) - \dot{\tilde{\lambda}}_1(\theta_1)\right] - f(\theta_1) \left[f(\theta_2) - \dot{\tilde{\lambda}}_2(\theta_2)\right]\right)\right) = -\xi(\theta, x)$$

Since $\xi(\theta, x) \geq 0$ and $V'(\underline{w}_2) < 0$

$$f(\theta_2) \left[f(\theta_1) - \dot{\tilde{\lambda}}_1(\theta_1)\right] - f(\theta_1) \left[f(\theta_2) - \dot{\tilde{\lambda}}_2(\theta_2)\right] \geq 0 \; for \; all \; \theta$$

[3]this follows from the symmetry of the problem around $\frac{1}{2}$

In particular this must hold for θ' of the form (θ_2, θ_1), so

$$f(\theta_1)\left[f(\theta_2) - \dot{\tilde{\lambda}}_1(\theta_2)\right] - f(\theta_2)\left[f(\theta_1) - \dot{\tilde{\lambda}}_2(\theta_1)\right] \geq 0$$

Therefore,

$$f(\theta_2)\left[f(\theta_1) - \dot{\tilde{\lambda}}_1(\theta_1)\right] - f(\theta_1)\left[f(\theta_2) - \dot{\tilde{\lambda}}_2(\theta_2)\right] = 0, \text{ for all } \theta_1,\ \theta_2$$

so that $\xi(\theta, x) = 0$, for all θ. The proof now follows exactly the same steps as in the previous one. Indeed, plugging $\xi(\theta, x) = 0$ for all θ back in the FOC for $w(\cdot)$ evaluated at $w(\theta, x) = \underline{w}_2$ for all θ, we get

$$\frac{\dot{\lambda}_2(\theta_2)}{f(\theta_2)} = -\frac{\dot{\lambda}_1(\theta_1)}{f(\theta_1)} V'(w) \text{ for almost all } (\theta_1, \theta_2)$$

which calls for

$$\dot{\lambda}_1(\theta_1) = \dot{\lambda}_1(\theta_2) = 0$$

And we get the same contradiction of the proof in Proposition 1.

■

Proof of Theorem 2: Proposition 2 implies that, if $w = \underline{w}_2$, there is a strictly positive probability of $w' > \underline{w}_2$. Moreover, Proposition 1 implies that, for all $w \in (\underline{w}_2, \overline{w}_2)$, there is a strictly positive probability of $w' > w$.

For any $w \in [\underline{w}_2, \overline{w}_2]$ and set A, define $Q^n(w, A)$ as the probability of, starting at w, getting to the set A in n periods. Since $[\underline{w}_2, \overline{w}_2]$ is compact, it follows from Propositions 1 and 2 that there is a finite $M > 0$ and a $\gamma(M) > 0$ so that, for all $w \in [\underline{w}_2, \overline{w}_2]$, $Q\left(w, w' \geq w + \frac{1}{M}\right) \geq \gamma(M) > 0$. Define $w^1 = \underline{w}_2$ and $w^n = w^{n-1} + \frac{1}{M}$.

Define $A_n = [w^n, \overline{w}_2]$. Note that $Q(w^1, A_2) \geq \gamma(M) > 0$. Furthermore, we have that

$$Q^2(w^1, A_3) = \int\limits_{z \in W} Q(z, A_3) Q(w^1, dz) \geq$$

$$\int\limits_{z \in A_2} Q(z, A_3) Q(w^1, dz) \geq \int\limits_{z \in A_2} \gamma(M) Q(w^1, dz) \equiv q_2 > 0$$

The first equality is the definition of $Q^2(\cdot, \cdot)$. The first inequality follows because $A_2 \subset W$. The second inequality follows because, for all z in A_2, $Q(z, A_3) \geq \gamma(z, M)$. Finally, q_2 being strictly positive follows because, since $Q(w^1, A_2) \geq \gamma(M) > 0$, for a positive probability subset $B_2 \subset A_2$, $Q(w^1, B_2)$ is positive and bounded away from zero.

Proceeding inductively, assume that $Q^{n-1}(w^1, A_n) > 0$

$$\begin{aligned} Q^n(w^1, A_{n+1}) &= \int\limits_{z\in W} Q(z, A_{n+1})Q^{n-1}(w^1, dz) \geq \int\limits_{z\in A_n} Q(z, A_{n+1})Q^{n-1}(w^1, dz) \\ &> \int\limits_{z\in A_n} \gamma(M)Q^{n-1}(w^1, dz) \equiv q_n > 0 \end{aligned}$$

where, again, the equality is the definition of $Q^n(w^1, A_{n+1})$, the first inequality follows because $A_n \subset W$, the second inequality because, for all $z \in A_n$, $Q(z, A_{n+1}) > \gamma(M)$. As before, q_n is strictly positive because for a positive probability subset $B_n \subset A_n$, $Q^{n-1}(w^1, B_n)$ is positive and bounded away from zero.

Pick $\overline{N}$ such that $w^{\overline{N}} + \frac{1}{M} \geq \overline{w}_2$. Clearly, this $\overline{N}$ is finite. Since $\overline{N}$ is finite, $Q^{\overline{N}}(\underline{w}_2, \{\overline{w}_2\}) \equiv q_{\overline{N}} > 0$. Setting $\epsilon = q_{\overline{N}}$, condition M of Stokey et al. ([13], p. 348) holds. Theorem 11.12 of Stokey et al. [13] then applies, implying that the operator T^*, defined in the text, is a contraction in the total variation norm. Hence, starting from any distribution φ_0, the sequence φ_n converges to a unique distribution φ^*, which is the unique fixed point of T^*.

■

Proof of Theorem 3: Immediately, since $V(\cdot)$ strictly concave implies that $V'(\cdot)$ is a one to one map from $[\underline{w}_2, \overline{w}_2]$ onto some bounded subset of $\mathbb{R}$.

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7.4 Dictatorship Under a Single Participation Constraint

Proof of Proposition 3: Immediate from Lemma 7 and the fact that, when only agent two has an outside option, $\zeta(\theta, x) = 0$ for all θ.

■

Proof of Proposition 4: Immediate from Lemma 7 and the fact that, when only agent one has an outside option, $\xi(\theta, x) = 0$ for all θ. ■

Proof of Theorem 4: We prove the result for the case in which only agent two has an outside option; the other case is analogous. From Proposition 3, when only agent two has an outside option, $V'(w)$ is a supermartingale. By Doob's Convergence Theorem (Dobb, [5]), the stochastic process $V'(w)$ converges to a random variable, R . Suppose there was a positive probability of finding a path $V'(w_t)$ with the property that $lim_{t\to\infty} V'(w_t) = C$, where $-\infty < C \leq V'(\underline{w}_2)$. Since $V'(w)$ is continuous for any $v \in (\underline{w}_2, \overline{v})$, the sequence v_t must converge. Let $lim_{t\to\infty} v_t = v' \in [\underline{v}_2, \overline{v})$, be the limit of agent two's continuation values.

Let $g(w, \theta, x)$ denote the next period's continuation value given the current promised value w and reported state θ. For w_t to converge to w', it must be that $g(w', \theta, x) = w'$ for all θ. This however contradicts Propositions 1 and 2. ■

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