1
Introduction

Throughout the text, let M be a smooth compact Riemannian manifold without boundary, and let m be the normalized Riemannian volume. The space of all C^r vector fields on M endowed with the C^r topology will be denoted by $\mathfrak{X}^r(M)$. The flow induced by a vector field $X \in \mathfrak{X}^1(M)$ will be denoted by $\{\varphi_X^t\}_{t\in\mathbb{R}}$ or simply $\{\varphi^t\}_{t\in\mathbb{R}}$ if the generating vector field is clear from the context. Let *acip* stand for absolutely continuous invariant probability, where absolute continuity is understood with respect to the volume measure m.

1.1
Absolutely continuous invariant probabilities

The main aspect of invariant probabilities is that they reflect the asymptotical behavior of almost every point with respect to those measures. Although the Krylov-Bogolubov Theorem guarantees the existence of invariant probabilities for compact metrizable spaces, it does not give any other information about the measure. The invariant measure on a Riemannian manifold could be singular to respect to the Riemannian volume. On the other hand, if an invariant probability is absolutely continuous with respect to a volume measure, then it is guaranteed that it reflects the asymptotical behavior of points in a set with positive volume.

The problem of dealing with acips is that, except for the case of $C^{1+\alpha}$ expanding maps (which always admit an acip), it is not known of any other system for which the existence of acips is open (in any topology). Even in the context of expanding maps, it was shown by A. Quas ([Q]) in dimension one and generalized by Avila and Boch in any dimension ([AB1]), that C^1-generic invariant probabilities are singular. Avila and Bochi also generalized Quas result (in the one dimensional case) for σ-finite measures ([AB2]). In the space of C^k Anosov Systems, the absence of acips is an open and dense property. This follows from the Livsic periodic orbit criterion (See [L]).

1.2 Main Theorem

In this work we extend the result of [AB1] for C^1 flows. Let us state precisely the theorem we prove.

Theorem 1.2.1 *There exists a C^1-residual subset $\mathcal{R} \subset \mathfrak{X}^1(M)$ such that if $X \in \mathcal{R}$, then X has no acip.*

Notice that we are not assuming any regularity on the density of the acip, other then integrability. If we ask the acip to be smooth or even holder-continuous, the proof might be much simpler. Our strategy (like in [AB1]) does not need to use these stronger hypotheses.

We assume that M has dimension $d \geq 3$. There is no loss of generality to do so, since the 2-dimensional case is a consequence of the fact that Morse-Smale systems cannot admit an acip (Remark 2.5.3) and the following celebrated result:

Theorem 1.2.2 (Peixoto, Pugh) *Let M be a compact surface. The set of all Morse-Smale systems is (open and) dense in $\mathfrak{X}^1(M)$.*

This theorem was proved for orientable surfaces (and a few non-orientable ones) by M. Peixoto (actually in any C^r topology), and then for every surface (in the C^1 topology) by C. Pugh, using the Closing Lemma. See [PdM, Chapter IV].

Peixoto's original result points to the possibility that the lack of acips might be generic even in higher topologies, since it implies that this is true at least for orientable surfaces.

1.3 Remarks about the proof

The idea of the proof is similar to [AB1]. We consider for each $\delta \in (0,1)$, the set

$$\mathcal{V}_\delta = \big\{X \in \mathfrak{X}^1(M) : \text{there exist a Borel set } K \subset M \text{ and } T \in \mathbb{R} \text{ such that} \\ m(K) > 1-\delta \;\text{ and }\; m(\varphi_X^T(K)) < \delta\big\}$$

These sets are clearly open (as shown in Remark 2.5.4); thus if we prove that they are C^1 dense, then the set

$$\mathcal{R} \equiv \bigcap_{\delta \in \mathbb{Q}\cap(0,1)} \mathcal{V}_\delta$$

will be a residual set. The fact that a vector field in $\mathcal{R}$ does not admit an acip is a direct consequence of Lemma 2.5.1. We say that a vector field is δ-*crushing* if $X \in \mathcal{V}_\delta$. All our effort in this work is to prove that δ-crushing is a dense property. Thus we begin with an arbitrary $X \in \mathfrak{X}^1(M)$ and a constant $\delta \in (0,1)$ and show how to construct a perturbation of X with the δ-crushing property.

The strategy to prove denseness of $\mathcal{V}_\delta$ has two main parts. First, we show how to construct a perturbation of X supported on a tubular neighborhood of a very long segment of orbit in a way that the δ-crushing property with respect to the normalized volume can be verified inside this neighborhood. This is the content of the Fettuccine's Lemma (Lemma 5.0.22).

The next step is to show that we can cover the manifold (except for a negligible measure set) with "crushable" sets, permitting us to construct the perturbation globally and, consequently, to obtain the δ-crushing property with respect to the volume of the whole manifold. This is done by a combination of Lemma 3.0.6, where we construct a transverse section and a first return map with some nice properties, and Lemma 6.0.2, which gives us a Rokhlin-like tower with respect to that first return map.

Although the general idea of the proof follows [AB1], there are some difficulties in adapting the proof to the continuous-time case. In both cases, the crushing is done in one dimension only, making d-dimensional objects essentially $(d-1)$-dimensional. In the continuous case, the choice of the crushing direction and the construction of the perturbation is done with the help of a tubular chart with several technical properties (Theorem 4.0.15), while in the discrete setting, an atlas is fixed with the only requirement that charts on the atlas take the volume in M to the Lebesgue measure in $\mathbb{R}^d$.

In [AB1], the crushable sets are contained in a discrete open tower and it is possible, in that case, to make 'a priori' adjustments, like a linearizing perturbation of the map in each level of the tower or a rotation of coordinates that makes $\mathbb{R}^{d-1} \times \{0\}$ invariant by the linear perturbed map. Moreover, these adjustments make the discrete version of Fettuccine's Lemma ([AB1, Lemma 3]) much simpler, since the lemma needs only to give a crushing perturbation of a sequence of linear isomorphisms.

1.4 Structure of the work

In Section 2, we present some basic background which will be used throughout the text. In §2.1, we give a slightly more general definition of Poincaré maps and present a change of coordinates that straightens the local stable and unstable manifolds around a hyperbolic saddle, with the additional property that the Euclidean norm in this coordinate system is adapted, that is, the flow presents immediate hyperbolic contraction (resp., expansion) in the stable (resp., unstable) coordinate. These adapted coordinates are used in the proof of the existence of singular flow boxes around hyperbolic saddles (Lemma 3.0.7). In §2.3, we make some remarks about linear cocycles, specially about one specific cocycle that plays a major role in the proof of our result - the linear Poincaré flow. We give also an example of a nonlinear cocycle - the orthonormal frame flow - which is a main tool in the construction of the tubular chart in Section 4. We have already mentioned that the non-existence of acips is equivalent to a volume crushing property. In §2.5, we state and prove this criterion, with some important observations about volume crushing. In §2.6, we prove a lemma about integrals of functions with bounded logarithmic derivative which is used to proof that, for long tubular neighborhoods, the volume concentrated on the edges are relatively small. As in [AB1], we need to use the Vitali covering theorem to guarantee that, except for a small set, we can cover the manifold with crushable sets. In §2.7, we make a precise definition of Vitali Coverings and state this theorem.

In Section 3, as mentioned above, we prove the existence of a singular flow box around a hyperbolic saddle (Lemma 3.0.7) and use this lemma to construct a transverse section with the property that every point in the manifold not contained in the stable manifold of a sink (resp. the unstable manifold of a source) must hit the section for the future (resp. for the past).

Section 4 is devoted to prove the existence of a C^2 tubular chart which enable the construction of the perturbation in $\mathbb{R}^d$. The chart has several natural properties and some technical ones. Before the proof of Theorem 4.0.15 we give some informal explanation about this properties and how they help us in the construction of the perturbation.

We have already made some remarks about the Fettuccine's Lemma, which gives us a perturbation of the vector field inside a long tubular neighborhood. Besides the tubular chart, the proof of this lemma needs several other ingredients and Section 5 is all devoted to present those tools and proving the Lemma (the proof is given only in § 5.4). In §5.1 we give, in Lemma 5.1.2, an explicit formula for the time $t_0 = t_0(\epsilon, \delta)$ that must elapse for an ϵ-perturbation generate a δ-crushing property. We call this amount of time the *crushing-time*. As in [AB1], the "size" $T > 0$ of the tubular neighborhood which supports the perturbation (in that case, the height n of the open tower) must be much bigger than the crushing-time. The reason is that the end of the crushable set (with size less than t_0) cannot be crushed. However, taking $T \gg t_0$, we guarantee that the relative volume of the non-crushed part is sufficiently small. In §5.2 we define the *sliced tubes*, a type of tubular neighborhood saturated by orthogonal cross-sections which are related to the Linear Poincaré Flow. These sets are convenient to work with for many reasons. They are not bent in the direction of the flow, for example, and their volumes are easily computed. Proposition 5.2.7 shows that we can approximate a standard tubular neighborhood by sliced tubes and in §5.3 we show how to construct a bump function with bounded C^1-norm inside a sliced tube. Finally, in Section 6, we extend the local crushing property to the whole manifold, proving that it is possible to cover the manifold (except for a small set) with crushable sets given by Lemma 5.0.22.