3
Transverse Section

In this section, we show that for a C^1 open and dense subset of $\mathfrak{X}^1(M)$, we can construct a transverse section and a return map with some properties (Lemma 3.0.6) that will permit us to use, in Section 6, a non-invariant Rokhlin lemma (Lemma 6.0.2) to obtain a disjoint finite union of tubular neighborhoods that cover M, except for a set of negligible Lebesgue measure.

Recall that a *cross-section* for a flow is a codimension 1 closed submanifold with boundary that is transverse to the vector filed.

Lemma 3.0.6 *Let $X \in \mathfrak{X}^1(M)$ be a vector field with only hyperbolic singularities. Then there exists a cross-section $\Sigma \subset M$ such that:*

1. *if $x \in M$ does not belong to a stable manifold of a sink or saddle singularity then the future orbit of x hits Σ;*

2. *if $x \in M$ does not belong to an unstable manifold of a source or saddle singularity then the past orbit of x hits Σ.*

Before showing how to construct the cross-section Σ, we will prove an intermediate step, which gives the appropriate cross-sections in the neighborhood of a saddle-type singularity.

Lemma 3.0.7 (singular flow-box) *Let p be a hyperbolic singularity of $X \in \mathfrak{X}^1(M)$ of saddle type. Then there exist compatible cross-sections Σ^u and Σ^s with the following properties:*

1. *If $f : \tilde{\Sigma}^u \to \tilde{\Sigma}^s$ is the Poincaré map given by Proposition 2.1.3, then*

$$\tilde{\Sigma}^u = \Sigma^u \setminus \partial\Sigma^u \setminus W^s_{\mathrm{loc}}(p),$$
$$\tilde{\Sigma}^s = \Sigma^s \setminus \partial\Sigma^s \setminus W^u_{\mathrm{loc}}(p).$$

2. *Letting τ be the associated hitting-time, the set*

$$V = \overline{\bigcup_{x \in \tilde{\Sigma}^u} \bigcup_{t \in [0,\tau(x)]} \varphi^t(x)}, \tag{3.1}$$

is a closed neighborhood of the saddle p.

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3. *For any point $x \in M \setminus V$, if the future (resp. past) orbit of x hits V then the first hit is in Σ^u (resp. Σ^s).*

See Figure 3.1.

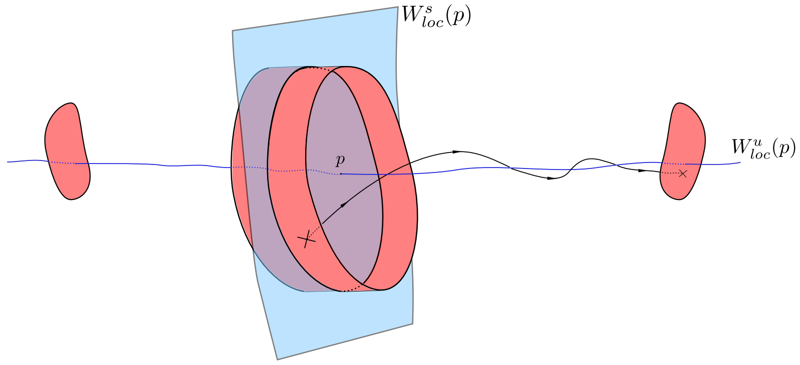

Figure 3.1: A saddle p with $\dim W^s_{loc}(p) = 2$ and $\dim W^u_{loc}(p) = 1$; the cross-sections Σ^u and Σ^s are respectively a cylinder and a union of two disks.

Proof: Let (F, U) be the adapted chart given by Lemma 2.1.7 and r_1, $r_2 > 0$ such that $\overline{B_{r_1}(0)} \times \overline{B_{r_2}(0)} \subset F(U)$. For simplicity of notation, we will work with the adapted coordinates without mentioning the chart F; with abuse of notation, $\{\varphi^t\}$ will denote the flow of the vector field $F_*(X|U)$ on the domain $F(U)$ (therefore not defined for all $t \in \mathbb{R}$).

For $\rho > 0$ sufficiently small, define the following subsets of $\mathbb{R}^d \equiv \mathbb{R}^s \times \mathbb{R}^u$:

$$C^u_\rho \equiv \{x = (x^s, x^u) : \|x^s\| = r_1,\ \|x^u\| \le \rho\}, \qquad \hat{C}^u_\rho \equiv \{x = (x^s, x^u) \in C^u_\rho : x^u \neq 0\},$$

$$C^s_\rho \equiv \{x = (x^s, x^u) : \|x^s\| \le \rho,\ \|x^u\| = r_2\}, \qquad \hat{C}^s_\rho \equiv \{x = (x^s, x^u) \in C^s_\rho : x^s \neq 0\}.$$

Claim 3.0.8 *For any $\epsilon \in (0, r_1)$, if $\delta \in (0, r_2)$ is sufficiently small then the future orbit of every point $x \in \hat{C}^u_\delta$ leaves the chart neighborhood without returning to C^u_δ, hitting C^s_ϵ along the way.*

Proof of the Claim: Let $\Lambda > \lambda > 0$ be the constants given by Lemma 2.1.7. Given $\epsilon \in (0, r_1)$, take any $\delta \in (0, r_2)$ such that

$$\left(\frac{\delta}{r_2}\right)^{\lambda/\Lambda} < \frac{\epsilon}{r_1}.$$

Fix a point $x = (x^s, x^u) \in \hat{C}^u_\delta$ and denote its trajectory under the flow by $(x^s(t), x^u(t))$. The norm inequalities from Lemma 2.1.7 hold until the orbit leaves the neighborhood $\overline{B_{r_1}(0)} \times \overline{B_{r_2}(0)}$, i.e., either $\|x^s(t)\| = r_1$ or $\|x^u(t)\| = r_2$. Since $\|x^s(t)\|$ decreases and $\|x^u(t)\|$ exponentially increases with

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t, there exists $T > 0$ such that $\|x^u(T)\| = r_2$. Using (2.5) in Lemma 2.1.7, we have

$$r_2 = \|x^u(T)\| \leq e^{\Lambda T} \cdot \|x^u(0)\| = e^{\Lambda T}\delta,$$

which leads us to

$$T > \frac{1}{\Lambda} \log\left(\frac{r_2}{\delta}\right).$$

From the choice of δ, we obtain

$$T > \frac{1}{\Lambda} \log\left(\frac{r_2}{\delta}\right) > \frac{1}{\lambda} \log\left(\frac{r_1}{\epsilon}\right).$$

So, using (2.4) in Lemma 2.1.7, we have

$$\|x^s(T)\| \leq e^{-\lambda T}\|x^s(0)\| = e^{-\lambda T} r_1 < \epsilon$$

Therefore $\varphi^T(x) \in C^s_\epsilon$. This proves the claim. ■

We now continue with the proof of the lemma. Fix any $\epsilon \in (0, r_1)$ and let $\delta \in (0, r_2)$ be given by the claim. By Proposition 2.1.3, there is a Poincaré map $f_+ : \hat{C}^u_\delta \to C^s_\epsilon$ which is a diffeomorphism onto its image.

By symmetry, the claim above also applies to the inverse flow. Therefore we can find some $\epsilon' \in (0, \epsilon)$ (depending on δ) such that the past orbit of every point in $\hat{C}^s_{\epsilon'}$ leaves the chart neighborhood without returning to $C^s_{\epsilon'}$, hitting C^u_δ along the way. By Proposition 2.1.3, there is a Poincaré map $f_- : \hat{C}^s_{\epsilon'} \to C^u_\delta$ which is a diffeomorphism onto its image. Clearly, f_- is a restriction of $(f_+)^{-1}$.

Define $\Sigma^s = C^s_{\epsilon'}$ and $\Sigma_u = \overline{f_-(\hat{C}^s_{\epsilon'})}$. Then Σ_u and Σ_s are compatible cross-sections with the required properties. ■

Remark 3.0.9 *If one assumes that the flow to be smoothly linearizable in a neighborhood of the saddle, then one can slightly simplify the proof of Lemma 3.0.7. By Sternberg Linearization Theorem, that assumption holds for a dense subset of vector fields. However, we preferred to keep things more elementary and avoid linearizations.*

Proof of Lemma 3.0.6: For each point $p \in M$, we define a closed neighborhood $V(p)$ of p and a closed codimension 1 submanifold $\Sigma(p)$ contained in $V(p)$ as follows:

- If p is a saddle-type singularity of X, then apply Lemma 3.0.7 and let $V(p) = V$ and $\Sigma(p) = \Sigma^u \cup \Sigma^s$.
- If p is a sink (resp. source) singularity, let $V(p)$ be a closed ball inside the stable (resp. unstable) manifold of p, whose boundary is a sphere $\Sigma(p)$ transverse to X.

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– If $p \in M$ is a non-singular point, let $V(p)$ be a flow-box around p (i.e., a domain given by the flow-box theorem). Let $\Sigma(p)$ be the union of the two “lids” of the flow-box.

Cover the manifold by a finite number of sets $\operatorname{int} V(p)$, and let Σ be the union of the corresponding $\Sigma(p)$. We can arrange that this union is disjoint, and therefore a manifold with boundary. Then Σ is a cross-section with the desired properties. This proves the lemma. ∎

Let Σ be the cross-section given by Lemma 3.0.6. Once we have constructed this transverse section, we need to know how to reduce the study of the dynamics on the manifold to the study of the discrete dynamics on the Poincaré section. Some remarks and propositions in this Section will help answering this question, but it will be totally clear only in Section 6, with Lemma 6.0.2.

Applying Proposition 2.1.3, we obtain subsets $\tilde{\Sigma}_1, \tilde{\Sigma}_2 \subset \Sigma$ and a Poincaré map $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$. Let σ be the $(d-1)$-dimensional Riemannian volume on Σ.

Let us introduce some notation that will be used not only in the proof of the following remark but also in Section 6. If $A \subset \Sigma$ is a set for which $f(A)$, $f^2(A)$, ..., $f^{J-1}(A)$ are defined, then we denote

$$\mathcal{T}_J(A) \equiv \bigcup_{j=0}^{J-1} \bigcup_{p\in f^j(A)} \bigcup_{t\in[0,\tau(p)]} \varphi^t(p).$$

Remark 3.0.10 *For all $\epsilon > 0$ and for all $n \in \mathbb{N}$, there exists $\delta > 0$ such that if $A \subset \Sigma$ is a measurable set with $\sigma(A) < \delta$ and $f(A)$, $f^2(A)$, ..., $f^{n-1}(A)$ are defined then*

$$m(\mathcal{T}_J(A)) < \epsilon.$$

Proof: Recall that σ-a.e. point in Σ that returns to Σ belongs to $\tilde{\Sigma}_1$. By Corollary 2.1.4, there exists $\delta^* > 0$ such that $B \subset \tilde{\Sigma}_1$, $\sigma(B) < \delta^*$ implies $m(\mathcal{T}_1(A)) < \epsilon/n$.

The Poincaré map $f : \tilde{\Sigma}_1 \to \tilde{\Sigma}_2$ is a C^1 diffeomorphism. Therefore if A_j is the subset of Σ where f^j is defined, the push-forward $f^j_*(\sigma|A_j)$ is absolutely continuous with respect to σ. So there exists $\delta > 0$ such that if $A \subset A_n$ and $\sigma(A) < \delta$ then $\sigma(f^j(A)) < \delta^*$ for all integer $0 \le j \le n-1$. Thus we conclude that $m(\mathcal{T}_1(f^jA)) < \epsilon/n$ for each such j, which yields $m(\mathcal{T}_{n-1}(A)) < \epsilon$. ∎

Let Λ be the set of points $x \in \Sigma$ such that $f^n(x)$ is well defined for all $n \in \mathbb{Z}$. Notice that this is a measurable set.

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The set of points in the manifold that hit Σ infinitely many times will be denoted by M_R, that is:

$$M_R = \bigcup_{t\in\mathbb{R}} \varphi^t(\Lambda).$$

Remark 3.0.11 *The set M_R is the complement of the union of stable and unstable manifolds of the singularities of X.*

Proof: Assume that the point x is in a stable manifold of a singularity, i.e. $\varphi^t(x)$ converges to a singularity q as $t \to +\infty$. Since Σ is compact and does not contain q, the future orbit of x hits Σ at most finitely many times, showing that $x \notin M_R$.

Conversely, if a point x is in no stable or unstable manifolds of singularities then it follows from Lemma 3.0.6 that its orbit $\{\varphi^t(x)\}$ hits Σ in the future and in the past. By invariance of stable and unstable manifolds, infinitely many such hits occur. This shows that $x \in M_R$. ■

In the following remark we will show that the crushing property is already satisfied on $M\backslash M_R$, so we do not need to perturb the vector field on that set.

Remark 3.0.12 *For all $\epsilon > 0$ there exist $\bar{t} > 0$ and a compact set $K \subset M\backslash M_R$ such that*

$$m(K) > m(M\backslash M_R) - \epsilon \quad \textit{and} \quad m(\varphi^t(K)) < \epsilon \textit{ for all } t > \bar{t}.$$

Proof: Recall Remark 3.0.11. Since the stable and unstable manifolds of saddles have zero m-measure, the set $M \setminus M_R$ coincides m-mod. 0 with the union M_S of stable manifolds of sinks and unstable manifolds of saddles. We have seen in Remark 2.5.2 that this is a “self-crushing” set. ■

From now on, (f, Λ, σ) will denote the dynamical system defined by the return Poincaré map $f : \Lambda \to \Lambda$, together with the (non necessarily invariant) measure σ.

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